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Dedicated to the memory of A.B. Shabat and R.I. Yamilov

GENERALIZED INVARIANT MANIFOLDS FOR INTEGRABLE EQUATIONS AND THEIR APPLICATIONS

I.T. HABIBULLIN, A.R. KHAKIMOVA, A.O. SMIRNOV

Abstract. In the article we discuss the notion of the generalized invariant manifold introduced in our previous study. In the literature, the method of the differential constraints is well known as a tool for constructing particular solutions for the nonlinear partial differential equations. Its essence is in adding to a given nonlinear PDE, another much simpler, as a rule ordinary, differential equation, consistent with the given one. Then any solution of the ODE is a particular solution of the PDE as well. However the main problem is to find this consistent ODE. Our generalization is that we look for an ordinary differential equation that is consistent not with the nonlinear partial differential equation itself, but with its linearization. Such generalized invariant manifold is effectively sought. Moreover, it allows one to construct such important attributes of integrability theory as Lax pairs and recursion operators for integrable nonlinear equations. In this paper, we show that they provide a way to construct particular solutions to the equation as well.

Keywords: invariant manifold, integrable system, recursion operator, Lax pair, algebro-geometric solutions, Dubrovin equations, spectral curves.

Mathematics Subject Classification: 35Q51, 35Q53, 35Q55

1. Introduction

In the article, a notion of the generalized invariant manifold for nonlinear integrable equation is discussed. Recently in our works [1]–[7] it was observed that the objects of such kind provide an effective tool for evaluating the Lax pairs and recursion operators.

The approach developed in [1]–[7] explains the essence of the Lax pair phenomenon. In fact, the Lax pair in $1 + 1$ dimension is naturally (internally) derived from the nonlinear equation under consideration. First we find the linearization (Fréchet derivative) of the nonlinear equation. The linearized equation obviously includes the dynamical variables of the original equation as well, which are here considered as functional parameters. Now we find an ordinary differential equation consistent with the linearized equation, which also depends on the dynamical variables of the original equation. We call this ordinary differential equation a generalized invariant manifold. For a given equation, there are many such manifolds, including nonlinear ones. In order to evaluate the generalized invariant manifold, we use the consistency with the linearized equation that allows us to derive a system of differential (difference) equations that is highly overdetermined due to the presence of the independent parameters, which are the dynamical variables of the original nonlinear equation. In all of the examples discussed in [1]–[7] (KdV, Kaup-Kupershmidt equation, Krichever-Novikov equation, Volterra type lattices from Yamilov list, two equations of KdV type found by

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Svinolupov and Sokolov, Garifullin-Mikhailov-Yamilov non-autonomous lattice, sine-Gordon equation and several hyperbolic type equations, etc.) the corresponding overdetermined systems are effectively solved and the desired non-trivial manifolds are found. Trivial generalized invariant manifolds are constructed quite elementary by using the classical or higher symmetries, see examples in [5]. A manifold, which is consistent with the linearized equation if and only if the original nonlinear equation is satisfied, is called non-trivial. Actually, this condition means that a pair consisting of the linearized equation and the generalized invariant manifold defines a Lax pair. It is curious that usual Lax pairs do not belong to this class, but they can be derived from properly chosen nonlinear generalized invariant manifolds by suitable transformations. Note that new Lax pairs are of an independent interest. For instance, a generalized invariant manifold generated by a consistent pair of linear invariant manifolds is easily transformed into the recursion operator. It was shown in [7] at the example of the Volterra lattice that a nonlinear Lax pair can be used for constructing particular solutions of the nonlinear equation.

Let us briefly describe the content of the article. In the second section we recall the definition of the invariant manifold and generalized invariant manifold for the differential equations in partial derivatives. We explain how to look for the generalized invariant manifold and why it can be effectively found. We conjecture that each integrable equation admits a consistent pair of linear invariant manifolds and give examples supporting such conjecture. We assert, based on our previous work, that consistent pairs of linear invariant manifolds can be used to construct both recursion operators and Lax pairs. We illustrate the algorithm by the examples of NLS system and mKdV equation in Sections 3–5. The consistent pair of the linear generalized invariant manifolds usually can be reduced to nonlinear one of smaller order. In this form, the invariant manifold provides an efficient way to derive the Dubrovin equations, from which finite-gap solutions are obtained; on method of finite-gap integration see [8]–[12]. The description of the spectral curve, the derivation and study of the Dubrovin equations for the NLS equation are presented in Sections 3.1–3.3. The corresponding solutions of the generalized invariant manifolds and their relation with the Novikov equation are considered in Section 3.4. Examples of one-phase and two-phase solutions of the NLS equation are given in Section 3.5. Derivation of the Dubrovin equations for mKdV equation is presented in Section 4.

2. Invariant manifolds and their generalization

The concept of an invariant manifold is well known in the theory of partial differential equations. It forms the basis of the method of differential constraints, widely used to construct particular solutions of nonlinear equations. We recall briefly the main points of the method of the invariant manifolds using the example of equations of evolutionary type

$$u_t = f(x,t,u,u_x,u_{xx},\dots,u_k), \quad u_j = \frac{\partial^j u}{\partial x^j}. \tag{2.1}$$

An ordinary differential equation of the order r

$$u_r = g(x,t,u,u_x,u_{xx},\dots,u_{r-1}) \tag{2.2}$$

is called an invariant manifold for the equation (2.1) if it is consistent with (2.1), or, in other words, if the following condition is obeyed:

$$D_x^r f - D_t g|_{(2.1),(2.2)} = 0. \tag{2.3}$$

Here D_x and D_t are operators of the total derivative with respect to x and to t.

It is clear that if a solution $u(x,t)$ of equation (2.1) satisfies equation (2.2) for some moment $t = t_0$, then it remains a solution of (2.2) at all values of time t. This is the invariance of equation (2.2).

Obviously relation (2.3) defines a PDE for the desired function g. Sometimes this equation can be solved explicitly, although in the general case the problem of finding the function g is rather complicated.

The situation changes essentially if we look for an ordinary differential equation that is consistent not with the nonlinear equation (2.1) itself, but with its linearization

$$U_t = \frac{\partial f}{\partial u}U + \frac{\partial f}{\partial u_x}U_x + \frac{\partial f}{\partial u_{xx}}U_{xx} + \cdots + \frac{\partial f}{\partial u_k}U_k. \tag{2.4}$$

Let give rigorous definitions. We consider an ordinary differential equation of the form

$$U_m = F(x, t, U, U_x, U_{xx}, \ldots, U_{m-1}; u, u_x, u_{xx}, \ldots, u_n), \tag{2.5}$$

where $U = U(x,t)$ is a sought function, while an arbitrary solution $u = u(x,t)$ of the original equation (2.1) is interpreted in (2.5) as a functional parameter. In fact, the variables x, t, U, U_x, U_{xx}, $\ldots$, U_{m-1}, u, u_x, u_{xx}, $\ldots$, u_n in (2.5) are regarded as independent.

Definition 2.1. *Equation* (2.5) *determines a generalized invariant manifold if the relation*

$$D_x^m U_t - D_t U_m|_{(2.1),(2.4),(2.5)} = 0 \tag{2.6}$$

is satisfied identically for all values of the variables $\{u_j\}$, x, t, U, U_x, $\ldots$, U_{m-1}.

Here the variables u_t, U_t as well as their derivatives with respect to x are expressed due to equations (2.1) and (2.4), the variables $U_m, U_{m+1}, \ldots$ are replaced by means of (2.5). To emphasize that the solution $u(x,t)$ is arbitrary, we consider the variables u, u_x, $u_{xx}, \ldots$ as independent ones. By virtue of this assumption, the problem of finding the function $F(x, t, U, U_x, U_{xx}, \ldots, U_{m-1}; u, u_x, u_{xx}, \ldots, u_n)$ is overdetermined but, as it is suggested by numerous examples, can be effectively solved.

Linear generalized invariant manifolds, that is, those of the form

$$LU = 0,$$

where L is a linear differential operator

$$L = \sum_{i=0}^{N} a_i(u, u_x, u_{xx}, \ldots) D_x^i$$

are of a special interest.

Definition 2.2. *Let equations* $L_1 U = 0$ *and* $L_2 U = 0$ *define linear generalized invariant manifolds for the equation* (2.1). *We call these two manifolds consistent if for all* $\lambda, \mu \in \mathbb{C}$ *the linear combination*

$$(\lambda L_1 + \mu L_2)\, U = 0$$

is a generalized invariant manifold for (2.1).

The following conjecture is supported by numerous examples, see [1]–[7].

Conjecture 2.1. *Equation* (2.1) *is integrable if and only if it admits a pair of the consistent linear generalized invariant manifolds such that the quotient*

$$R = L_1^{-1} L_2$$

is a pseudodifferential operator; in fact, it is the recursion operator for (2.1).

Examples can be found below in Section 3.2 and at the end of Section 5.

3. Invariant manifolds for NLS equation

In this section we find an invariant manifold of the first order (the simplest nontrivial!) for the nonlinear Schrödinger equation. It is determined by the system

$$\begin{aligned} iu_t &= u_{xx} + 2u^2 v, \\ iv_t &= -v_{xx} - 2v^2 u \end{aligned} \tag{3.1}$$

under appropriate additional condition. Let us first find the linearized equation for the system by rule (2.4):

$$\begin{aligned} iU_t &= U_{xx} + 4uvU + 2u^2 V, \\ iV_t &= -V_{xx} - 2v^2 U - 4uvV. \end{aligned} \tag{3.2}$$

According to Definition 2.1, the generalized invariant manifold is a system of the ordinary differential equations consistent with (3.2) for arbitrary solution $u = u(x,t)$, $v = v(x,t)$ of (3.1). We look for it in the form

$$\begin{aligned} U_x &= f(U, V, u, v), \\ V_x &= g(U, V, u, v). \end{aligned} \tag{3.3}$$

The comptability condition for the equations (3.2) and (3.3) gives an overdetermined system of equations for a pair of unknowns f and g that is effectively solved and defines a generalized invariant manifold given by a system of the form (for the details see Appendix below)

$$\begin{aligned} U_x &= \lambda U - 2u\sqrt{C - UV}, \\ V_x &= -\lambda V - 2v\sqrt{C - UV}, \end{aligned} \tag{3.4}$$

where λ and C are arbitrary constants. Due to the obtained equations, linearized equation (3.2) converts into a system of the ordinary differential equations:

$$\begin{aligned} iU_t &= (2uv + \lambda^2)U - 2(u_x + \lambda u)\sqrt{C - UV}, \\ iV_t &= -(2uv + \lambda^2)V + 2(v_x - \lambda v)\sqrt{C - UV}. \end{aligned} \tag{3.5}$$

The following statement can be easily proved by straightforward computations.

Theorem 3.1. *A pair of systems* (3.4) *and* (3.5) *is consistent if and only if the functions u and v solve equation* (3.1).

Therefore, the pair of equations (3.4) and (3.5) defines a Lax pair for the NLS equation. Unlike the usual Lax pair found by V.E. Zakharov and A.B. Shabat, this pair is nonlinear and contains two arbitrary constants, but with the help of a simple technique it is reduced to the usual one [13]. Indeed, by setting $C = 0$, $U = \varphi^2$, $V = \psi^2$ we reduce equations (3.4) and (3.5) to the form

$$\begin{aligned} \varphi_x &= \frac{1}{2}\lambda\varphi - iu\psi, \\ \psi_x &= -iv\varphi - \frac{1}{2}\lambda\psi, \end{aligned}$$

and, respectively,

$$\begin{aligned} \varphi_t &= \left(uv + \frac{1}{2}\lambda^2\right)\varphi - i(u_x + \lambda u)\psi, \\ \psi_t &= i(v_x - \lambda v)\varphi - \left(uv + \frac{1}{2}\lambda^2\right)\psi. \end{aligned}$$

3.1. Invariant manifolds and spectral curves. Let us show that the found nonlinear Lax pair is of an independent interest since it provides opportunities for building particular solutions to the NLS equation. We change the variables in the nonlinear Lax pair as $U = u\Phi$, $V = v\Psi$ and this casts the pair into the form

$$\begin{aligned} &\frac{u_x}{u}\Phi + \Phi_x - \lambda\Phi = -2\sqrt{C - \Phi\Psi uv}, \\ &\frac{v_x}{v}\Psi + \Psi_x + \lambda\Psi = -2\sqrt{C - \Phi\Psi uv} \end{aligned} \tag{3.6}$$

and

$$\begin{aligned} &i\frac{u_t}{u}\Phi + i\Phi_t = (2uv + \lambda^2)\Phi - 2\left(\frac{u_x}{u} + \lambda\right)\sqrt{C - \Phi\Psi uv}, \\ &i\frac{v_t}{v}\Psi + i\Psi_t = -(2uv + \lambda^2)\Psi + 2\left(\frac{v_x}{v} - \lambda\right)\sqrt{C - \Phi\Psi uv}. \end{aligned} \tag{3.7}$$

Assume that the parameters C and λ are related such that C is a polynomial of λ with constant coefficients:

$$C = \frac{1}{4}\prod_{k=1}^{2N+2}(\lambda - \lambda_k) = \frac{1}{4}\nu^2(\lambda). \tag{3.8}$$

We see solutions to the nonlinear Lax equations in the form

$$\Phi = \prod_{k=1}^{N}(\lambda - \gamma_k), \quad \Psi = -\prod_{k=1}^{N}(\lambda - \beta_k). \tag{3.9}$$

We note that identity (3.8) defines the equation for the spectral hyperelliptic curve of the N-gap solution of the NLS equation, see [10]–[12].

We substitute representations (3.8) and (3.9) into system (3.6) and compare the coefficients at the like powers λ^N. This gives the following relations being the well known trace formulae:

$$\begin{aligned} \frac{u_x}{u} &= -\sum_{k=1}^{N}\gamma_k + \frac{1}{2}\sum_{k=1}^{2N+2}\lambda_k, \\ \frac{v_x}{v} &= \sum_{k=1}^{N}\beta_k - \frac{1}{2}\sum_{k=1}^{2N+2}\lambda_k. \end{aligned} \tag{3.10}$$

We substitute polynomials (3.9) into system (3.6), take $\lambda = \gamma_j$ in the first equation and $\lambda = \beta_j$ in the second. Then we get the well known Dubrovin formulae [8]

$$\gamma_j' = \frac{\nu(\gamma_j)}{\prod_{k\neq j}(\gamma_j - \gamma_k)}, \qquad \beta_j' = -\frac{\nu(\beta_j)}{\prod_{k\neq j}(\beta_j - \beta_k)}, \tag{3.11}$$

where $\gamma_j' = \frac{d\gamma_j}{dx}$, $\beta_j' = \frac{d\beta_j}{dx}$. By applying the same manipulations to (3.7) we obtain

$$\begin{aligned} i\dot{\gamma}_j &= \frac{\left(-\sum_{k\neq j}\gamma_k + \frac{1}{2}\sum_{k=1}^{2N+2}\lambda_k\right)\nu(\gamma_j)}{\prod_{k\neq j}(\gamma_j - \gamma_k)}, \\ i\dot{\beta}_j &= -\frac{\left(-\sum_{k\neq j}\beta_k + \frac{1}{2}\sum_{k=1}^{2N+2}\lambda_k\right)\nu(\beta_j)}{\prod_{k\neq j}(\beta_j - \beta_k)}, \end{aligned} \tag{3.12}$$

where $\dot{\gamma}_j = \frac{d\gamma_j}{dt}$, $\dot{\beta}_j = \frac{d\beta_j}{dt}$. In order to get the focusing NLS equation

$$iu_t = u_{xx} + 2|u|^2 u,$$

to system (3.1) we add a constraint of the form $v=\bar{u}$, where the bar over a letter means the complex conjugation. Then solution (Φ,Ψ) to the nonlinear Lax pair (3.6), (3.7) can be chosen in such a way

$$\bar{\Phi}\left(-\bar{\lambda}\right)=(-1)^{N+1}\Psi(\lambda).$$

Function $C(\lambda)$ and parameters λ_j, β_j, γ_j satisfy the involution

$$C(\lambda)=\bar{C}\left(-\bar{\lambda}\right),\qquad \bar{\lambda}_j=-\lambda_j,\qquad \beta_j=-\bar{\gamma}_j.$$

Evolution of γ_j in x and t is determined by a pair of the systems of ordinary differential equations

$$\gamma_j'=\frac{\nu(\gamma_j)}{\prod_{k\neq j}(\gamma_j-\gamma_k)},\tag{3.13}$$

$$i\dot{\gamma}_j=\frac{\left(-\sum_{k\neq j}\gamma_k+\frac{1}{2}\sum_{k=1}^{2N+2}\lambda_k\right)\nu(\gamma_j)}{\prod_{k\neq j}(\gamma_j-\gamma_k)}.\tag{3.14}$$

Thus, we arrive at systems of ODE describing the well-known algebro-geometric solutions for the NLS equations, see [10].

Theorem 3.2. *A pair of systems* (3.13) *and* (3.14) *is consistent.*

Proof. Let us show that a pair of systems (3.13), (3.14) is consistent. To do this, we differentiate system (3.13) with respect to t, multiply by i and subtract then system (3.14) differentiated first with respect to x. We get

$$\begin{aligned}
i\frac{d}{dt}\left(\gamma_j'\right)-\frac{d}{dx}\left(i\dot{\gamma}_j\right)=&\frac{i\frac{d}{dt}\nu(\gamma_j)}{\prod_{k\neq j}(\gamma_j-\gamma_k)}-\frac{i\nu(\gamma_j)\frac{d}{dt}\prod_{k\neq j}(\gamma_j-\gamma_k)}{\prod_{k\neq j}(\gamma_j-\gamma_k)^2}\\
&+\sum_{s\neq j}\left(\gamma_s'\right)\frac{\nu(\gamma_j)}{\prod_{k\neq j}(\gamma_j-\gamma_k)}\\
&-\left(-\sum_{k\neq j}\gamma_k+\frac{1}{2}\sum_{k=1}^{2N+2}\lambda_k\right)\frac{\frac{d}{dx}\nu(\gamma_j)}{\prod_{k\neq j}(\gamma_j-\gamma_k)}\\
&+\left(-\sum_{k\neq j}\gamma_k+\frac{1}{2}\sum_{k=1}^{2N+2}\lambda_k\right)\frac{\nu(\gamma_j)\frac{d}{dx}\prod_{k\neq j}(\gamma_j-\gamma_k)}{\prod_{k\neq j}(\gamma_j-\gamma_k)^2}.
\end{aligned}\tag{3.15}$$

We find the derivatives

$$i\frac{d}{dt}\nu(\gamma_j),\qquad \frac{d}{dx}\nu(\gamma_j),\qquad i\frac{d}{dt}\prod_{k\neq j}(\gamma_j-\gamma_k),\qquad \frac{d}{dx}\prod_{k\neq j}(\gamma_j-\gamma_k)$$

separately. We have:

$$\begin{aligned}
i\frac{d}{dt}\nu(\gamma_j)=&\frac{i\dot{\gamma}_j\nu(\gamma_j)}{2}\sum_{k=1}^{2N+2}\frac{1}{\gamma_j-\lambda_k}\\
=&\frac{1}{2}\left(-\sum_{k\neq j}\gamma_k+\frac{1}{2}\sum_{k=1}^{2N+2}\lambda_k\right)\frac{\nu^2(\gamma_j)}{\prod_{k\neq j}(\gamma_j-\gamma_k)}\sum_{k=1}^{2N+2}\frac{1}{\gamma_j-\lambda_k},
\end{aligned}$$

and

$$\frac{d}{dx}\nu(\gamma_j)=\frac{\gamma_j'\nu(\gamma_j)}{2}\sum_{k=1}^{2N+2}\frac{1}{\gamma_j-\lambda_k}=\frac{1}{2}\frac{\nu^2(\gamma_j)}{\prod_{k\neq j}(\gamma_j-\gamma_k)}\sum_{k=1}^{2N+2}\frac{1}{\gamma_j-\lambda_k}.$$

In the same way we find:

$$
\begin{aligned}
i\frac{d}{dt}\prod_{k\neq j}(\gamma_j-\gamma_k) =& \prod_{k\neq j}(\gamma_j-\gamma_k)\sum_{s\neq j}\frac{i\dot{\gamma}_j-i\dot{\gamma}_s}{\gamma_j-\gamma_s}\\
=&\left(-\sum_{k\neq j}\gamma_k+\frac{1}{2}\sum_{k=1}^{2N+2}\lambda_k\right)\nu(\gamma_j)\sum_{s\neq j}\frac{1}{\gamma_j-\gamma_s}\\
&+\prod_{k'\neq j}(\gamma_j-\gamma_{k'})\sum_{s\neq j}\left(\sum_{k\neq s}\gamma_k\right)\frac{\nu(\gamma_s)}{(\gamma_j-\gamma_s)\prod_{k\neq s}(\gamma_s-\gamma_k)}\\
&-\frac{1}{2}\sum_{k=1}^{2N+2}\lambda_k\prod_{k'\neq j}(\gamma_j-\gamma_{k'})\sum_{s\neq j}\frac{\nu(\gamma_s)}{(\gamma_j-\gamma_s)\prod_{k\neq s}(\gamma_s-\gamma_k)},
\end{aligned}
$$

and

$$
\begin{aligned}
\frac{d}{dx}\prod_{k\neq j}(\gamma_j-\gamma_k) =& \prod_{k\neq j}(\gamma_j-\gamma_k)\sum_{s\neq j}\frac{\gamma_j'-\gamma_s'}{\gamma_j-\gamma_s}\\
=&\nu(\gamma_j)\sum_{s\neq j}\frac{1}{\gamma_j-\gamma_s}-\prod_{k\neq j}(\gamma_j-\gamma_k)\sum_{s\neq j}\frac{\nu(\gamma_s)}{(\gamma_j-\gamma_s)\prod_{k\neq s}(\gamma_s-\gamma_k)}.
\end{aligned}
$$

We substitute the obtained identities into (3.15) and after simple transformations we arrive at

$$
\begin{aligned}
i\frac{d}{dt}\left(\gamma_j'\right)-\frac{d}{dx}\left(i\dot{\gamma}_j\right) =& \frac{\nu(\gamma_j)}{\prod_{k\neq j}(\gamma_j-\gamma_k)}\left[-\sum_{s\neq j}\left(\sum_{k\neq s}\gamma_k\right)\frac{\nu(\gamma_s)}{(\gamma_j-\gamma_s)\prod_{k\neq s}(\gamma_s-\gamma_k)}\right.\\
&\left.+\sum_{s\neq j}\frac{\nu(\gamma_s)}{\prod_{k\neq s}(\gamma_s-\gamma_k)}+\sum_{k\neq j}\gamma_k\sum_{s\neq j}\frac{\nu(\gamma_s)}{(\gamma_j-\gamma_s)\prod_{k\neq s}(\gamma_s-\gamma_k)}\right].
\end{aligned}
\tag{3.16}
$$

We observe that the first two terms in the brackets can be simplified as

$$
\begin{aligned}
-\sum_{s\neq j}\left(\sum_{k\neq s}\gamma_k\right)&\frac{\nu(\gamma_s)}{(\gamma_j-\gamma_s)\prod_{k\neq s}(\gamma_s-\gamma_k)}+\sum_{s\neq j}\frac{\nu(\gamma_s)}{\prod_{k\neq s}(\gamma_s-\gamma_k)}\\
&=\sum_{s\neq j}\left(\gamma_j-\gamma_s-\sum_{k\neq s}\gamma_k\right)\frac{\nu(\gamma_s)}{(\gamma_j-\gamma_s)\prod_{k\neq s}(\gamma_s-\gamma_k)}\\
&=\sum_{s\neq j}\left(-\sum_{k\neq j}\gamma_k\right)\frac{\nu(\gamma_s)}{(\gamma_j-\gamma_s)\prod_{k\neq s}(\gamma_s-\gamma_k)}.
\end{aligned}
$$

Then, taking into consideration the obtained relation, we see that identity (3.16) becomes

$$
i\frac{d}{dt}\left(\gamma_j'\right)-\frac{d}{dx}\left(i\dot{\gamma}_j\right)=0.
$$

This completes the proof. □

3.2. Consistent pair of linear invariant manifolds. Here we present an example supporting Conjecture 2.1. From nonlinear invariant manifold (3.4) we derive a consistent pair of linear invariant manifolds for the system (3.1). We differentiate both equations in (3.4) with respect to x

$$U_{xx} = \lambda U_x - 2u_x\sqrt{C-UV} + u\frac{U_xV+V_xU}{\sqrt{C-UV}},$$
$$V_{xx} = -\lambda V_x - 2v_x\sqrt{C-UV} + v\frac{U_xV+V_xU}{\sqrt{C-UV}}$$

and then exclude irrationalities in the obtained equations due to relations:

$$\frac{U_xV+V_xU}{\sqrt{C-UV}} = -2(uV+vU),$$
$$\sqrt{C-UV} = \frac{\lambda U - U_x}{2u} = \frac{-\lambda V - V_x}{2v}.$$

As a result, we obtain a linear relation

$$L_2W = \lambda L_1 W, \tag{3.17}$$

where $W = (U,V)^T$ and the operators are as follows

$$L_1 = \begin{pmatrix} D_x - \frac{u_x}{u} & 0 \\ 0 & -D_x + \frac{v_x}{v} \end{pmatrix},$$

and

$$L_2 = \begin{pmatrix} D_x^2 - \frac{u_x}{u}D_x + 2uv & 2u^2 \\ 2v^2 & D_x^2 - \frac{v_x}{v}D_x + 2uv \end{pmatrix}.$$

We confirm straightforwardly that the linear constraint defined by equation (3.17) is consistent with linearized equation (3.2), i.e., it defines a linear generalized invariant manifold with a parameter λ for the NLS equation. It is easily verified that a pseudo differential operator $L_1^{-1}L_2$ coincides with the recursion operator for NLS system (3.1)

$$R = \begin{pmatrix} D_x + 2uD_x^{-1}v & 2uD_x^{-1}u \\ -2vD_x^{-1}v & -D_x - 2vD_x^{-1}u \end{pmatrix}.$$

3.3. Integrals of systems. The overdetermined system of equations (3.13), (3.14) admits integrals of the form

$$\sum_{j=1}^{N}\int_{\gamma_j(0,0)}^{\gamma_j(x,t)} \gamma^k \frac{d\gamma}{\nu(\gamma)} = 0, \quad k = 0,1,\dots,N-3, \tag{3.18}$$

$$\sum_{j=1}^{N}\int_{\gamma_j(0,0)}^{\gamma_j(x,t)} \gamma^{N-2} \frac{d\gamma}{\nu(\gamma)} = t, \tag{3.19}$$

$$\sum_{j=1}^{N}\int_{\gamma_j(0,0)}^{\gamma_j(x,t)} \gamma^{N-1} \frac{d\gamma}{\nu(\gamma)} = x + \frac{1}{2}t\sum_{k=1}^{2N+2}\lambda_k \tag{3.20}$$

that are derived directly from the systems by using some elementary manipulations and subsequent integration.

3.4. Novikov equation. We express coefficients of the polynomial $P = (\Phi,\Psi)^T$ in terms of the solution (u,v) of the NLS equation (3.1) obtained due to (3.10). Then we rewrite equation (3.17) in a convenient form

$$\widetilde{R}P = \lambda P, \tag{3.21}$$

where

$$\widetilde{R} = \begin{pmatrix} \frac{1}{u} & 0 \\ 0 & \frac{1}{v} \end{pmatrix} R \begin{pmatrix} u & 0 \\ 0 & v \end{pmatrix}.$$

We introduce notations for the coefficients of the polynomial P

$$P = \begin{pmatrix} 1 \\ -1 \end{pmatrix} \lambda^N + \begin{pmatrix} r_1 \\ s_1 \end{pmatrix} \lambda^{N-1} + \ldots + \begin{pmatrix} r_N \\ s_N \end{pmatrix}. \tag{3.22}$$

By virtue of the above expansion, equation (3.21) gives rise to

$$\begin{aligned} &\widetilde{R}\left(\begin{pmatrix} 1 \\ -1 \end{pmatrix} \lambda^N + \begin{pmatrix} r_1 \\ s_1 \end{pmatrix} \lambda^{N-1} + \ldots + \begin{pmatrix} r_N \\ s_N \end{pmatrix}\right) \\ &= \begin{pmatrix} 1 \\ -1 \end{pmatrix} \lambda^{N+1} + \begin{pmatrix} r_1 \\ s_1 \end{pmatrix} \lambda^{N} + \ldots + \begin{pmatrix} r_N \\ s_N \end{pmatrix}\lambda. \end{aligned}$$

Comparing the coefficients at λ^{N+1} in (3.1), we find:

$$\widetilde{R}\begin{pmatrix} 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 1 \\ -1 \end{pmatrix}. \tag{3.23}$$

We recall that the operator $\widetilde{R}$ involves an integration. It is easy to confirm that equation (3.23) is satisfied for an appropriate choice of the constants in the integration.

We proceed to the coefficients at λ^N and we get

$$\widetilde{R}\begin{pmatrix} 1 \\ -1 \end{pmatrix} = \begin{pmatrix} r_1 \\ s_1 \end{pmatrix}$$

that implies

$$\begin{pmatrix} r_1 \\ s_1 \end{pmatrix} = \begin{pmatrix} \frac{1}{u}(u_x + c_1 u) \\ \frac{1}{v}(v_x - c_1 v) \end{pmatrix}.$$

By continuing this process we find for $k \geq 1$:

$$\begin{pmatrix} r_k \\ s_k \end{pmatrix} = \begin{pmatrix} \frac{1}{u}(g_k + c_1 g_{k-1} + \ldots + c_k u) \\ \frac{1}{v}(h_k + c_1 h_{k-1} + \ldots + c_k(-v)) \end{pmatrix},$$

where c_i, $i = \overline{1,k}$ are arbitrary constants, the vector (g_j, h_j) coincides with the generator of the homogeneous symmetry of the order k

$$u_{\tau_j} = g_j, \quad v_{\tau_j} = h_j$$

of NLS system (3.1). Finally, comparing the coefficients at λ^0, we find

$$r_{N+1} = 0, \quad s_{N+1} = 0$$

that actually coincides with the Novikov equation

$$\begin{aligned} &g_N + c_1 g_{N-1} + \ldots + c_N u = 0, \\ &h_N + c_1 h_{N-1} + \ldots + c_N(-v) = 0. \end{aligned}$$

3.5. Examples. Below we present two examples illustrating the use of the Dubrovin equations (3.13), (3.14) by taking $N = 1$ and $N = 2$.

Example 1. In the particular case when $N = 1$ and

$$\nu(\gamma) = (\gamma - \lambda_1)(\gamma - \lambda_2) \quad \text{with} \quad \lambda_1 = \eta + i\xi, \quad \lambda_2 = -\eta + i\xi$$

we get a system of consistent equations for determining the unknown $\gamma = \gamma(x,t)$:

$$\begin{aligned} &\gamma' = (\gamma - \lambda_1)(\gamma - \lambda_2), \\ &i\dot{\gamma} = 2i\xi(\gamma - \lambda_1)(\gamma - \lambda_2). \end{aligned}$$

It can be solved easily:

$$\gamma = \eta \tanh(2\xi\eta t + \eta x + s_0) + i\xi.$$

In order to find $u = u(x,t)$, we first solve the equation

$$\frac{u_x}{u} = -\gamma + 2i\xi.$$

Integration of this equation yields

$$u = \frac{e^{i\xi x A(t)}}{\cosh(2\xi\eta t + \eta x + s_0)}.$$

We substitute the obtained ansatz into the NLS equation $iu_t = u_{xx} + 2\,|u|^2\,u$ and find

$$A(t) = \eta e^{i(\eta^2-\xi^2)t+i\varphi_0}.$$

Then finally we get the well-known soliton solution

$$u(x,t) = \frac{\eta e^{i\left(\xi x+(\eta^2-\xi^2)t+\varphi_0\right)}}{\cosh(2\xi\eta t + \eta x + s_0)}.$$

Example 2. Let us take $N = 2$ and assume that the hyperelliptic curve is as follows

$$\nu^2(\lambda) = (\lambda^2 - 4)^3.$$

In order to find the functions $\gamma_1(x,t)$ and $\gamma_2(x,t)$, we use the integrals (3.18)-(3.20), which in this case take the form:

$$\sum_{j=1}^{2} \int_{\gamma_j(0,0)}^{\gamma_j(x,t)} \frac{d\gamma}{\nu(\gamma)} = t,$$

$$\sum_{j=1}^{2} \int_{\gamma_j(0,0)}^{\gamma_j(x,t)} \gamma\frac{d\gamma}{\nu(\gamma)} = x + \frac{1}{2}t\sum_{k=1}^{4}\lambda_k.$$

These integrals are evaluated in a closed form and generate a system of algebraic equations for $\gamma_1(x,t)$, $\gamma_2(x,t)$, which is easily solved and gives rise to

$$\gamma_j(x,t) = -2\frac{X(1-iT) + (-1)^j(iT - X - 1)(iT + X - 1)R}{(X^2+T^2+1)(X^2+T^2+4iT-3)}, \qquad j = 1,2,$$

where $T = 4t$, $X = 2x$ and

$$R = \sqrt{(T^2 + 2iT + X^2 + 3)(X - iT + 1)(X + iT - 1)}.$$

Then we find $u(x,t)$ according to (3.10):

$$u(x,t) = \left(1 - \frac{4(1-iT)}{X^2+T^2+1}\right)e^{-2it}.$$

The obtained solution obviously coincides with the well known two-phase Peregrine soliton [14].

4. Invariant manifolds for mKdV equation: complex-valued case

A complex-valued version of the modified KdV equations

$$u_\tau + u_{xxx} + 6|u|^2 u_x = 0 \tag{4.1}$$

has important physical applications, see [15].

The equation is obtained by imposing an involution of the form $v = \bar{u}$, where the bar over the letter means complex conjugation, in the system of equations

$$\begin{aligned} &u_\tau + u_{xxx} + 6uvu_x = 0,\\ &v_\tau + v_{xxx} + 6uvv_x = 0. \end{aligned} \tag{4.2}$$

Linearization of (4.2) leads us to the system

$$\begin{aligned} &U_\tau + U_{xxx} + 6uvU_x + 6vu_xU + 6uu_xV = 0, \\ &V_\tau + V_{xxx} + 6uvV_x + 6vu_xU + 6uu_xV = 0. \end{aligned} \tag{4.3}$$

It should be stressed that generalized invariant manifold (3.4) found in the previous section for the NLS equation, is also a generalized invariant manifold for mKdV equation (4.2). This is not surprising since the systems (3.1) and (4.2) mutually commute. It can be easily confirmed that system

$$\begin{aligned} &U_x = \lambda U - 2u\sqrt{C - UV}, \\ &V_x = -\lambda V - 2v\sqrt{C - UV} \end{aligned} \tag{4.4}$$

is consistent with (4.3) for each solution (u, v) of (4.2). Due to (4.4), linearized equation (4.3) is reduced to the form

$$\begin{aligned} &U_\tau = 2\left(u_{xx} + 2u^2v + \lambda u_x + \lambda^2 u\right)\sqrt{-UV + C} - \left(2uv_x - 2vu_x - 2\lambda uv - \lambda^3\right)U, \\ &V_\tau = 2\left(v_{xx} + 2uv^2 - \lambda v_x + \lambda^2 v\right)\sqrt{-UV + C} - \left(2uv_x - 2vu_x - 2\lambda uv - \lambda^3\right)V. \end{aligned} \tag{4.5}$$

It is easy to confirm that (4.4), (4.5) define a nonlinear Lax pair for system (4.2).

Since systems (4.4) and (4.5) are very similar to those studied in the previous section (see (3.4), (3.5)), we investigate them in the same way. First we change the variables as $U = u\Phi$, $V = v\Psi$ and get

$$\begin{aligned} &\frac{u_x}{u}\Phi + \Phi_x - \lambda\Phi = -2\sqrt{C - \Phi\Psi uv}, \\ &\frac{v_x}{v}\Psi + \Psi_x + \lambda\Psi = -2\sqrt{C - \Phi\Psi uv} \end{aligned} \tag{4.6}$$

and

$$\begin{aligned} &\frac{u_\tau}{u}\Phi + \Phi_\tau = 2\left(\frac{u_{xx}}{u} + 2uv + \lambda\frac{u_x}{u} + \lambda^2\right)\sqrt{C - \Phi\Psi uv} - \left(2uv_x - 2vu_x - 2\lambda uv - \lambda^3\right)\Phi, \\ &\frac{v_\tau}{v}\Psi + \Psi_\tau = 2\left(\frac{v_{xx}}{v} + 2uv - \lambda\frac{v_x}{v} + \lambda^2\right)\sqrt{C - \Phi\Psi uv} - \left(2uv_x - 2vu_x - 2\lambda uv - \lambda^3\right)\Psi. \end{aligned} \tag{4.7}$$

We use the same spectral curve:

$$C = \frac{1}{4}\prod_{k=1}^{2N+2}(\lambda - \lambda_k) = \frac{1}{4}\nu^2(\lambda) \tag{4.8}$$

and seek solutions to the nonlinear Lax equations in the same form:

$$\Phi = \prod_{k=1}^{N}(\lambda - \gamma_k), \qquad \Psi = -\prod_{k=1}^{N}(\lambda - \beta_k). \tag{4.9}$$

We substitute representations (4.8) and (4.9) into system (4.6) and by comparing the coefficients at λ^N, we derive trace formulae (3.10). The next step is to substitute polynomials (4.9) into system (4.6). Then we set $\lambda = \gamma_j$ in the first equation and $\lambda = \beta_j$ in the second and get the system of ordinary differential equations defining the dynamics of the roots on x: [8]

$$\gamma_j' = \frac{\nu(\gamma_j)}{\prod_{k\neq j}(\gamma_j - \gamma_k)}, \qquad \beta_j' = -\frac{\nu(\beta_j)}{\prod_{k\neq j}(\beta_j - \beta_k)}, \tag{4.10}$$

where $\gamma_j' = \frac{d\gamma_j}{dx}$, $\beta_j' = \frac{d\beta_j}{dx}$.

Let us derive equations describing the time evolution of the functions $\gamma(x,t)$, $\beta(x,t)$. We substitute explicit representations of the functions Φ, Ψ and C into (4.7) and set $\lambda = \gamma_j$ in the first equation and $\lambda = \beta_j$ in the second:

$$\begin{aligned} \dot{\gamma}_j &= \left(\frac{u_{xx}}{u} + 2uv + \lambda\frac{u_x}{u} + \lambda^2\right)\frac{\nu(\gamma_j)}{\prod_{k\neq j}(\gamma_j-\gamma_k)}, \\ \dot{\beta}_j &= -\left(\frac{v_{xx}}{v} + 2uv - \lambda\frac{v_x}{v} + \lambda^2\right)\frac{\nu(\beta_j)}{\prod_{k\neq j}(\beta_j-\beta_k)}, \end{aligned} \tag{4.11}$$

where we used notations $\dot{\gamma}_j = \frac{d\gamma_j}{d\tau}$, $\dot{\beta}_j = \frac{d\beta_j}{d\tau}$. To get a closed system for γ_j, β_j, we exclude $\frac{u_x}{u}$, $\frac{v_x}{v}$, $\frac{v_{xx}}{v}$ and $\frac{u_{xx}}{u}$ due to (3.10). For the term uv we deduce two equations:

$$4uv = 2\sum_{k\neq j}\gamma_k' + \sum_{k=1}^{2N+2}\lambda_k\sum_{k\neq j}\gamma_k - 2\sum_{k\neq s}\gamma_k\gamma_s - 2\sum_{k=1}^{N}(\gamma_k)^2 + \frac{1}{4}\left(\sum_{k=1}^{2N+2}\lambda_k\right)^2 - \sum_{k\neq s}\lambda_k\lambda_s,$$

$$4uv = -2\sum_{k\neq j}\beta_k' + \sum_{k=1}^{2N+2}\lambda_k\sum_{k\neq j}\beta_k - 2\sum_{k\neq s}\beta_k\beta_s - 2\sum_{k=1}^{N}(\beta_k)^2 + \frac{1}{4}\left(\sum_{k=1}^{2N+2}\lambda_k\right)^2 - \sum_{k\neq s}\lambda_k\lambda_s$$

by comparing coefficients at the power λ^{N+1} in (4.7). As a result, system (4.11) converts into a system of equations describing the time evolution of the roots

$$\begin{aligned} \dot{\gamma}_j &= \left(-\frac{1}{2}\sum_{k=1}^{2N+2}\lambda_k\sum_{k\neq j}\gamma_k + \sum_{k\neq s\neq j}\gamma_k\gamma_s + \frac{3}{8}\left(\sum_{k=1}^{2N+2}\lambda_k\right)^2 - \frac{1}{2}\sum_{k\neq s}\lambda_k\lambda_s\right)\frac{\nu(\gamma_j)}{\prod_{k\neq j}(\gamma_j-\gamma_k)}, \\ \dot{\beta}_j &= -\left(\frac{1}{2}\sum_{k=1}^{2N+2}\lambda_k\sum_{k\neq j}\beta_k - \sum_{k\neq s\neq j}\beta_k\beta_s - \frac{3}{8}\left(\sum_{k=1}^{2N+2}\lambda_k\right)^2 + \frac{1}{2}\sum_{k\neq s}\lambda_k\lambda_s\right)\frac{\nu(\beta_j)}{\prod_{k\neq j}(\beta_j-\beta_k)}. \end{aligned} \tag{4.12}$$

It can be proved by a direct computation that systems (4.10) and (4.12) mutually commute.

In what follows, we impose reductions of two types on system (4.1). Complex reduction $v = \bar{u}$ in (4.1) is related to the involution

$$\bar{\Phi}\left(-\bar{\lambda}\right) = (-1)^{N+1}\Psi(\lambda)$$

of the eigenfunctions, that generates conditions for the zeros $\bar{\gamma}_k = -\beta_j$ and $\bar{\lambda}_k = -\lambda_j$ of the polynomials Φ, Ψ, C such that

$$C(\lambda) = \prod_{j=1}^{N+1}\left(\lambda^2 - 2\lambda\, Re\lambda_j + |\lambda_j|^2\right).$$

In the case of the reduction $v = u$ we obtain

$$\Phi\left(-\lambda\right) = (-1)^{N+1}\Psi(\lambda)$$

and $\gamma_k = -\beta_j$, $\lambda_k = -\lambda_j$, such that

$$C(\lambda) = \prod_{j=1}^{N+1}\left(\lambda^2 - \lambda_k^2\right).$$

It is obvious that both reductions are consistent with dynamics (4.10), (4.12). By using the found solution $\{\gamma_j\}_{j=1}^{N}$ of the Dubrovin equations, one can find solution u of equation (4.1) due to formula (3.10). Solutions of such kind were earlier studied in [16].

The overdetermined system of equations (4.10), (4.12) admits integrals of the form

$$\sum_{j=1}^{N}\int_{\gamma_j(0,0)}^{\gamma_j(x,\tau)}\gamma^k\frac{d\gamma}{\nu(\gamma)}=0,\qquad k=0,1,\ldots,N-4, \tag{4.13}$$

$$\sum_{j=1}^{N}\int_{\gamma_j(0,0)}^{\gamma_j(x,\tau)}\gamma^{N-3}\frac{d\gamma}{\nu(\gamma)}=\tau, \tag{4.14}$$

$$\sum_{j=1}^{N}\int_{\gamma_j(0,0)}^{\gamma_j(x,\tau)}\gamma^{N-2}\frac{d\gamma}{\nu(\gamma)}=\left(\frac{1}{2}\sum_{k=1}^{2N+2}\lambda_k\right)\tau, \tag{4.15}$$

$$\sum_{j=1}^{N}\int_{\gamma_j(0,0)}^{\gamma_j(x,\tau)}\gamma^{N-1}\frac{d\gamma}{\nu(\gamma)}=x+\left(\frac{3}{8}\left(\sum_{k=1}^{2N+2}\lambda_k\right)^2-\frac{1}{2}\sum_{k\neq s}\lambda_k\lambda_s\right)\tau. \tag{4.16}$$

Taking into account the dependence on t and τ in equations (3.18)-(3.20) and (4.13)-(4.16), we find that

$$\sum_{j=1}^{N}\int_{\gamma_j(0,0,0)}^{\gamma_j(x,t,\tau)}\gamma^k\frac{d\gamma}{\nu(\gamma)}=0,\quad k=0,1,\ldots,N-4,$$

$$\sum_{j=1}^{N}\int_{\gamma_j(0,0,0)}^{\gamma_j(x,t,\tau)}\gamma^{N-3}\frac{d\gamma}{\nu(\gamma)}=\tau,$$

$$\sum_{j=1}^{N}\int_{\gamma_j(0,0,0)}^{\gamma_j(x,t,\tau)}\gamma^{N-2}\frac{d\gamma}{\nu(\gamma)}=t+\left(\frac{1}{2}\sum_{k=1}^{2N+2}\lambda_k\right)\tau,$$

$$\sum_{j=1}^{N}\int_{\gamma_j(0,0,0)}^{\gamma_j(x,t,\tau)}\gamma^{N-1}\frac{d\gamma}{\nu(\gamma)}=x+\left(\frac{1}{2}\sum_{k=1}^{2N+2}\lambda_k\right)t+\left(\frac{3}{8}\left(\sum_{k=1}^{2N+2}\lambda_k\right)^2-\frac{1}{2}\sum_{k\neq s}\lambda_k\lambda_s\right)\tau.$$

We note that formulae (3.18)-(3.20) and (4.13)-(4.16) can be easily generalized to solutions which simultaneously satisfy several equations from the AKNS hierarchy. In this case we obtain the following representation for the integrals

$$\sum_{j=1}^{N}\int_{\gamma_j(0,0,\ldots)}^{\gamma_j(t_0,t_1,t_2,\ldots)}\gamma^{N-k}\frac{d\gamma}{\nu(\gamma)}=t_k+\sum_{j>k}c_{k,j}t_j,\qquad k\geq 1,$$

where $t_1=x$, $t_2=t$, $t_3=\tau$ and t_j, for $j>3$ correspond to higher symmetries. Here $c_{k,j}$ are constant coefficients.

5. Invariant manifolds for mKdV equation: real case

In this section we seek for the generalized invariant manifold for the mKdV equation which can be obtained from (4.2) by imposing the real involution $u=v$:

$$u_t=u_{xxx}+6u^2u_x \tag{5.1}$$

and then show how to derive from it the Lax pair, recursion operator and a consistent pair of the linear invariant manifolds.

First, we linearize equation (5.1):

$$U_t=U_{xxx}+6u^2U_x+12uu_xU. \tag{5.2}$$

Since equation (5.1) is reduced to the equation with cubic nonlinearity

$$w_t=w_{xxx}+2w^3$$

by substitution $u = w_x$ the linearizations of these two equations are related by a similar replacement. Indeed, if we put $U = W_x$, then we arrive at the equation

$$W_t = W_{xxx} + 6u^2 W_x \tag{5.3}$$

that is much simpler than (5.2). Hence, it is more convenient to work with this one. Below we will seek an ODE of the form

$$W_{xx} = F(W_x, W, u),$$

consistent with linear equation (5.3) for each solution $u(x,t)$ of the equation (5.1). Omitting the computations, we present only the answer

$$W_{xx} = 2u\sqrt{-W_x^2 + \lambda W^2 + C} + \lambda W, \tag{5.4}$$

where λ and C are arbitrary constants. By virtue of the found equation, linearization (5.3) turns into the form

$$W_t = \left(\lambda + 2u^2\right) W_x + 2u_x\sqrt{-W_x^2 + \lambda W^2 + C}. \tag{5.5}$$

It worth mentioning that the found nonlinear equations provide a Lax pair for (5.1), namely, the following theorem holds.

Theorem 5.1. *A pair of equations* (5.4) *and* (5.5) *are consistent if and only if the function u solves equation* (5.1).

Remark 5.1. *We note that generalized invariant manifolds* (4.4) *and* (5.4) *are related by the following change of the variables. We let $\lambda = \xi^2$ in* (5.4) *and introduce U, V in such a way*

$$U = W_x - \xi W, \quad V = W_x + \xi W.$$

Then we get

$$U_x = \xi U - 2u\sqrt{C - UV},$$
$$V_x = -\xi V - 2v\sqrt{C - UV}.$$

Let us reduce pair of nonlinear equations (5.4), (5.5) for the case when $C = 0$ to the usual Lax pair of equation (5.1). We change the variables in the following way

$$W = 2\varphi\psi, \quad W_x = \sqrt{\lambda}\left(\varphi^2 + \psi^2\right)$$

then in the new variables equation (5.4) becomes a system of linear equations

$$\varphi_x = -iu\varphi + \frac{1}{2}\sqrt{\lambda}\psi,$$
$$\psi_x = \frac{1}{2}\sqrt{\lambda}\varphi + iu\psi$$

and similarly, (5.5) turns into

$$\varphi_t = -i\left(u_{xx} + 2u^3 + u\lambda\right)\varphi + \frac{1}{2}\sqrt{\lambda}\left(2iu_x + 2u^2 + \lambda\right)\psi,$$
$$\psi_t = -\frac{1}{2}\sqrt{\lambda}\left(2iu_x - 2u^2 - \lambda\right)\varphi + i\left(u_{xx} + 2u^3 + u\lambda\right)\psi.$$

In order to bring it to a standard form, we make a replacement

$$\Phi = \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} \tilde{\Phi},$$

where $\Phi = (\varphi, \psi)^T$ and $\tilde{\Phi} = (\tilde{\varphi}, \tilde{\psi})^T$. Then we get

$$\tilde{\varphi}_x = \xi\tilde{\varphi} - iu\tilde{\psi},$$
$$\tilde{\psi}_x = -iu\tilde{\varphi} - \xi\tilde{\psi}$$

and

$$\tilde{\varphi}_t = \left(4\xi^3 + 2u^2\xi\right)\tilde{\varphi} - i\left(4u\xi^2 + 2u_x\xi + u_{xx} + 2u^2\right)\tilde{\psi},$$
$$\tilde{\psi}_t = -i\left(4u\xi^2 - 2u_x\xi + u_{xx} + 2u^2\right)\tilde{\varphi} - \left(4\xi^3 + 2u^2\xi\right)\tilde{\psi},$$

where $\lambda = 4\xi^2$.

We get rid of irrationality in identity (5.4) by squaring and then rewrite the result as

$$\frac{W_{xx}^2}{u^2} - 2\lambda\frac{WW_{xx}}{u^2} + \lambda^2\frac{W^2}{u^2} + 4W_x^2 - 4\lambda W^2 - 4C = 0.$$

Then we differentiate the obtained equation with respect to x. The found equation turns out to be linear

$$W_{xxx} - \frac{u_x}{u}W_{xx} + 4u^2W_x = \lambda\left(W_x - \frac{u_x}{u}W\right).$$

Actually, it defines a linear invariant manifold, consistent with the equation (5.3). Let us rewrite it in the form

$$L_2W = \lambda L_1W, \tag{5.6}$$

where

$$L_1 = D_x - \frac{u_x}{u}, \quad L_2 = D^3 - \frac{u_x}{u}D_{xx} + 4u^2D_x.$$

Since $U = W_x$ from (5.6) we get a relation

$$L_2D_x^{-1}U = \lambda L_1D_x^{-1}U$$

for the solution U of linearization (5.2) of mKdV equation (5.1). The latter allows to get the recursion operator for (5.1):

$$R = D_xL_1^{-1}L_2D_x^{-1} = D_x^2 + 4u^2 + 4u_xD_x^{-1}u. \tag{5.7}$$

6. Appendix

Let us show how equations (3.4) are constructed. The aforementioned consistency condition for equations (3.2) and (3.3) is written as

$$\begin{aligned} &\frac{\partial}{\partial x}\left(U_{xx} + 4uvU + 2u^2V\right) - i\frac{\partial}{\partial t}\left(f(U,V,u,v)\right)\Big|_{(3.1),(3.2),(3.3)} = 0,\\ &\frac{\partial}{\partial x}\left(-V_{xx} - 2v^2U - 4uvV\right) - i\frac{\partial}{\partial t}\left(g(U,V,u,v)\right)\Big|_{(3.1),(3.2),(3.3)} = 0. \end{aligned} \tag{6.1}$$

We regard variables u, v and their derivatives with respect to x and U, V as independent. The variables u_t, v_t, U_t, V_t, D_x^kU, D_x^kV, where $k \geq 1$, in identities (6.1), will be excluded by virtue of equations (3.1),(3.2) and (3.3). Then we get an overdetermined system of equations:

$$F(U, V, u, v, u_x, v_x, v_{xx}) = 0,$$
$$G(U, V, u, v, u_x, v_x, u_{xx}) = 0.$$

In the obtained relations, we equate the coefficients at independent variables u_x, v_x, u_{xx}, v_{xx} and step by step we determine the form of the sought functions $f(U, V, u, v)$ and $g(U, V, u, v)$.

We compare the coefficients at the highest derivatives of functions u and v, i.e., at u_{xx} and v_{xx}, as well as at u_x^2 and v_x^2, then we get:

$$f(U,V,u,v) = f_1(U,V)u + f_2(U,V),$$
$$g(U,V,u,v) = g_1(U,V)v + g_2(U,V).$$

Note that the dependence of functions $f(U,V,u,v)$ and $g(U,V,u,v)$ on the variables u and v is defined, so we can equate the coefficients at these variables.

Analyzing the equations obtained by comparing the coefficients at u_x, v_x, u and v, we determine the form of functions $f_1(U,V)$, $f_2(U,V)$ and $g_1(U,V)$, $g_2(U,V)$:

$$f_1(U,V) = \sqrt{-4UV + c_1}, \quad f_2(U,V) = c_2U + c_3,$$
$$g_1(U,V) = \sqrt{-4UV + c_4}, \quad g_2(U,V) = c_5V + c_6.$$

By considering the remaining equations, we obtain a relationship between the constant parameters c_i, $i = \overline{1,6}$:

$$c_4 = c_1, \qquad c_5 = -c_2, \qquad c_3 = c_6 = 0.$$

Thus, the functions $f(U,V,u,v)$ and $g(U,V,u,v)$ read as

$$f(U,V,u,v) = \lambda U - 2u\sqrt{C - UV},$$
$$g(U,V,u,v) = -\lambda V - 2v\sqrt{C - UV},$$

where $\lambda = c_2$, $C = \frac{1}{4}c_1$.

Conclusion

In the article we have discussed the notion of the generalized invariant manifold for the nonlinear partial differential equations. We have found such manifolds for the nonlinear Scrödinger equation and the modified Korteweg-de Vries equation. We illustrated that this object provides an effective tool for constructing the recursion operator and the Lax pair. We have shown that the well-known Dubrovin equation, which is an important ingredient in the finite-gap integration method, can be easily derived from a suitably selected generalized invariant manifold.

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Ismagil Talgatovich Habibullin,
Institute of Mathematics,
Ufa Federal Research Center, RAS,
Chernyshevsky str., 112,
450008, Ufa, Russia
E-mail: habibullinismagil@gmail.com

Aigul Rinatovna Khakimova,
Institute of Mathematics,
Ufa Federal Research Center, RAS,
Chernyshevsky str., 112,
450008, Ufa, Russia
E-mail: aigul.khakimova@mail.ru

Aleksandr Olegovich Smirnov,
Saint-Petersburg State University of Aerospace Instrumentation,
Bol'shaya Morskaya str., 67,
190000, St. Petersburg, Russia
E-mail: alsmir@guap.ru