ABSTRACTS

A.B. Borisov

On integrability of $O(3)$–model

Abstract. A three-dimensional $O(3)$ model for a unit vector $\mathbf{n}(\mathbf{r})$ has numerous application in the field theory and in the physics of condensed matter. We prove that this model is integrable under some differential constraint, that is, under certain restrictions for the gradients of fields $\Theta(\mathbf{r})$, $\Phi(\mathbf{r})$ parametrizing the vector $\mathbf{n}(\mathbf{r})$). Under the presence of the differential constraint, the equations of the models are reduced to a one-dimensional sine-Gordon equation determining the dependence of the field $\Theta(\mathbf{r})$ on an auxiliary field $a(\mathbf{r})$ and to a system of two equations $(\nabla S)(\nabla S) = 0$, $\Delta S = 0$ for a complex-valued function $S(\mathbf{r}) = a(\mathbf{r}) + \mathrm{i}\Phi(\mathbf{r})$. We show the solution of this system provide all known before exact solutions of models, namely, two-dimensional magnetic instantons and three-dimensional structures of hedgehog type. We find an exact solution for the field $S(\mathbf{r})$ as an arbitrary implicity function of two variables, which immediately represent the solution for the fields $\Theta(\mathbf{r})$, $\Phi(\mathbf{r})$ in an implicit form. We show that the found in this way exact solution of the system for the field $S(\mathbf{r})$ leads one to exact solution of equations of $O(3)$–model in the form of an arbitrary implicit function of two variables.

Keywords: integrable system, $O(3)$–model, differential substitution, quasilinear equation, general solution

B.S. Bychkov, G.B. Shabat

On generalizations of Chebyshev polynomials and Catalan numbers

Abstract. We provide possible directions of generalizations of earlier found relations between the Chebyshev polynomials and the Catalan numbers arising in studying commuting difference operators. These generalizations are mostly related with ideas proposed by N.H. Abel in his publication in 1826, which then were reproduced by many authors in a modern language. As generalization of Chebyshev polynomials, we propose to consider polynomials with exactly two critical values well-studied in a so-called theory of dessins d'enfants. The Catalan numbers are located in the first column of the table of Harer–Zagier numbers related with the distribution by genius of orientable sewing of polygons with even number of sides. The commuting difference operators are implicitly contained in the Abel theory, who studied quasi-elliptic integrals, namely, the elliptic integrals of 3rd kind integrated in terms of logarithms. In the present work we formulate conjectures on relation between the main Abel theorem and commuting self-infinite matrices. In the work we provide calculations supporting the conjectured relations.

Keywords: Chebyshev polynomials, Catalan numbers, Harer-Zagier numbers, polynomial Pell equation, dessins d'enfants.

R.N. Garifullin

On integrability of semi-discrete Tzitzeica equation

Abstract. In the paper we consider a semi-discrete version of Tzitzeica equation

$$\frac{du_{n+1}}{dx} = \frac{du_n}{dx} + (e^{-2u_n} + e^{-2u_{n+1}}) + \sqrt{e^{2u_n} + e^{2u_{n+1}}},$$

which was found in a recent paper [R.N. Garifullin and I.T. Habibullin 2021 J. Phys. A: Math. Theor. 54 205201]. It was shown that this equation possess higher symmetries along the discrete and continuous directions. These higher symmetries are equations of Sawada-Kotera equation type and of discere Sawada-Kotera equation type. In the work we construct the Lax pair for this equation and for its higher symmetries. The found Lax pair is written out in terms of 3×3 matrices and this indicates the integrability of the found equations. To solve this problem, we employ the known relation between one of the higher symmetries with a well-studied КаupвЪ“Kupershmidt equation. The found Lax pairs can be employed in further studies of this equation, namely, for finding its conservations laws, the recursion operators and wide classes of solutions. Moreover, we write out two Lax representations in the form of scalar operators. The first representation is written in terms of the powers of the differentiation operators with respect to the continuous variable x, while the other is written via the powers of the operator of the shift along the discrete variable n.

Keywords: integrability, Lax pairs, higher symmetries, Tzitzeica equation.

A.V. Zhiber, M.N. Kuznetsova

Integrals and characteristic Lie rings of semi-discrete systems of equations

Abstract. The paper is devoted to the study of systems of semi-discrete equations $\bar{r}_{n+1,x} = \bar{h}(x, n, \bar{r}_n, \bar{r}_{n+1}, \bar{r}_{n,x})$ within the framework of an approach based on the concept of a characteristic Lie ring. Here $\bar{r}_n = (r_n^1, r_n^2, \ldots, r_n^N)$, $\bar{h} = (h^1, h^2, \ldots, h^N)$, $n \in \mathbb{Z}$. Among integrable nonlinear partial differential equations and systems, we find Darboux integrable nonlinear hyperbolic equations and systems. A feature of such equations is the existence of integrals along each characteristic direction, the so-called x - and y -integrals. This allows us to reduce the integration of a partial differential equation to integrating a system of ordinary differential equations. Darboux integrable equations and systems can be efficiently investigated and classified by means of characteristic Lie rings. Papers by Leznov, Smirnov, Shabat, Yamilov underlie an algebraic approach for studying nonlinear hyperbolic systems. Currently, the algebraic approach is extended to semi-discrete and discrete equations. In this paper, we prove that the system has N essentially independent x-integrals if and only if the characteristic Lie ring corresponding to a continuous characteristic direction is finite-dimensional.

Keywords:semi-discrete system of equations, characteristic ring, x-integral, Darboux integrable system.

O.V. Kaptsov, M.M. Mirzaokhmedov

General solutions of some linear equations with variable coefficients

Abstract. In the work we find general solutions to some classes of linear wave equations with variable coefficients. Such equations describe the oscillations of rods, acoustic waves, and also some models of gas dynamics are reduced to these equations.

To construct general solutions, we employ special types of Euler-Darboux transformations, namely, Levi type transformations. These transformations are first order differential substitutions. For constructing each transformation, we need to solve two linear second order ordinary differential equations. The solutions of one of these equations are determined by the solutions of the other equations by means of a differential substitution and Liouville formula. In the general case, it is not easy to solve these ordinary differential equations. However, it is possible to provide some formula for the superposition of the transformation of Levi type.

Starting with a classical wave equation with constant coefficients and employing the found transformations, we can construct infinite series of equations possessing explicit general solutions. By means of Matveev method we obtain limiting forms of iterated transformations. We provide a series of particular examples of the equations possessing general solutions.

Keywords: linear equations with variable coefficients, general solutions, limiting Levi transformations.

D.V. Millionschikov, S.V. Smirnov

Characteristic algebras and integrable exponential systems

Abstract. In the present work we study characteristic algebras for exponential systems corresponding to degenerate Cartan matrices. These systems generalize hyperbolic sine-Gordon and Tzitzeica equations well-known in the theory of integrable systems. For such systems, corresponding to Cartan matrices of rank 2, we describe explicitly characteristic algebras in terms of generators and relations and we prove that they have linear growth. We study the relations between the higher symmetries of these systems and the structure of their characteristic algebras. We describe completely the higher symmetries of exponential systems corresponding to the Cartan matrix of affine Lie algebra $A_2^{(1)}$. We also obtain partial results on symmetries of such systems corresponding to other degenerate Cartan matrices of rank 2. We propose a conjecture on the structure of higher symmetries of arbitrary exponential system corresponding to a degenerate Cartan matrix. We study an interesting combinatorics related with an operator generating a characteristic algebra in the simplest case for a Darboux integrable Liouville equation. The found combinatorial properties can be very useful for proving the aforementioned conjecture on the structure of higher symmetries. Moreover, in the present paper we give a rigorous meaning to notion of a characteristic algebra of a hyperbolic system used for a long time in the literature. We do this by means of the notion of Lie-Rinehart algebra and at the examples we demonstrate that such formalization is indeed needed.

Keywords: characteristic algebra, higher symmetry, Liouville equation, exponential system.

V.Yu. Novokshenov

Discrete Riemann-Hilbert problem and interpolation of entire functions

Abstract. We consider two problems in complex analysis which were developed in Ufa in 1970s years. These are a Riemann-Hilbert problem about jump of a piecewise-analytic function on a contour and a problem of interpolation of entire functions on a countable set in the complex plane. A progress in recent years led to comprehension that they have much common in subject. The first problem arrives as an equivalent of the inverse scattering problem applied for integrating nonlinear differential equations of mathematical physics. The second problem is a natural generalization of Lagrange formula for polynomial with given values on a finite set of points. It is shown that both problems can be united by generalization of the Riemann-Hilbert problem on a case of "discrete contour", where a "jump" of analytic function takes place. This formulation of discrete matrix Riemann-Hilbert (dmRH) problem is applied now for various problems of exactly solvable difference equations as well as estimates of spectrum of random matrices. It is shown in the paper how dmRH provides a way to integrate nonlinear difference equations such as a discrete Painlevé equation. On the other hand, it is shown how assignment of residues to meromorphic matrix functions is effectively reduced to an interpolation problem of entire functions on a countable set in $\mathbb{C}$ with the only accumulation point at infinity. Other application of dmRH includes calculation of Fredholm determinants emerging in combinatorics and representation of groups theory.

Keywords: Riemann-Hilbert problem, inverse scattering problem, entire functions, interpolation, canonical product, discrete Painlevé equations, Fredholm determinant, asymptotic expansions.

A.O. Smirnov, V.B. Matveev

Finite-gap solutions of nonlocal equations in Ablowitz-Kaup-Newell-Segur hierarchy

Abstract. Nonlinear nonlocal models exist in many fields of physics. The most known of them are models possessing $\mathcal{PT}$-symmetries. Apart of $\mathcal{PT}$-symmetric models, nonlocal models with inverse time and/or coordinates are actively studied. Other types of nonlocalities arise much rare. As a rule, in works devoted to nonlinear nonlocal equations, soliton or quasi-rational solutions to such equations are studied.

In the present work we consider nonlocal symmetries, to which all equations in the Ablowitz-Kaup-Newell-Segur hierarchy. On the base of the properties of solutions satisfying nonlocal reductions of the equations in the Ablowitz-Kaup-Newell-Segur hierarchy, we propose a modification of theta-functional formula for Baker-Akhiezer functions. We find the conditions for the parameters of spectral curves associated with multi-phase solutions possessing no exponential growth at infinity. We show that under these conditions, the variables separate. The most part of statement of our work remain true for soliton and quasi-rational solutions since they are limiting cases for the multi-phase solutions.

Keywords: Nonlinear Schrödinger equation, Ablowitz-Kaup-Newell-Segur hierarchy, nonlocal equation, PT-symmetry, finite-gap solution, spectral curve, theta function.

B.I. Suleimanov, A.M. Shavlukov

Integrable Abel equation and asymptotics of symmetry solutions of Korteweg-de Vries equation

Abstract. We provide a general solution for a first order ordinary differential equation with a rational right-hand side, which arises in constructing asymptotics for large time of simultaneous solutions of the Korteweg - de Vries equation and the stationary part of its higher non-autonomous symmetry. This symmetry is determined by a linear combination of the first higher autonomous symmetry of the Korteweg-de Vries equation and of its classical Galileo symmetry. This general solution depends on an arbitrary parameter. By the implicit function theorem, locally it is determined by the first integral explicitly written in terms of hypergeometric functions. A particular case of the general solution defines self-similar solutions of the Whitham equations, found earlier by G.V. Potemin in 1988. In the well-known works by A.V. Gurevich and L.P. Pitaevsky in early 1970s, it was established that these solutions of the Whitham equations describe the origination in the leading term of non-damping oscillating waves in a wide range of problems with a small dispersion. The result of this article supports once again an empirical rule saying under various passages to the limits, integrable equations can produce only integrable, in certain sense, equations. We propose a general conjecture: integrable ordinary differential equations similar to that considered in the present paper should also arise in describing the asymptotics at large times for other symmetry solutions to evolution equations admitting the application of the method of inverse scattering problem.

Keywords: integrability, Abel equation, Korteweg-de Vries equation, asymptotics.

V.E. Adler

Differential substitutions for non-Abelian equations of KdV type

Abstract. The work is devoted to constructing differential substitutions connecting the non-Abelian KdV equation with other third-order evolution equations. One of the main results is the construction of a non-Abelian analog of the exponential Calogero–Degasperis equation in a rational form. Some generalizations of the Schwarzian KdV equation are also obtained. Equations and differential substitutions under study contain arbitrary non-Abelian parameters. The construction method is based on the auxiliary linear problem for KdV, in which the usual spectral parameter is replaced by a non-Abelian one. The wave function, corresponding to a fixed value of this parameter, also satisfies a certain evolution equation. Passing to the left and right logarithmic derivatives of the wave function leads one to two versions of the modified KdV equation. In addition, a gauge transformation of the original linear problem leads to a linear problem for one of these versions, mKdV-2. After that, the described procedure is repeated, and the resulting evolution equation for the wave function contains already two arbitrary non-Abelian parameters. For the logarithmic derivative, we obtain an analog of the Calogero–Degasperis equation, which is thus a second modification of the KdV equation. Combining the found Miura-type transformations with discrete symmetries makes it possible to obtain chains of Bäcklund transformations for the modified equations.

Keywords: non-Abelian equation, Lax pair, Miura transformation.

V.S. Gerdjikov

On mKdV equations related to Kac-Moody algebras $A_5^{(1)}$ and $A_5^{(2)}$

Abstract. We outline the derivation of the mKdV equations related to the Kac–Moody algebras $A_5^{(1)}$ and $A_5^{(2)}$. First we formulate their Lax representations and provide details how they can be obtained from generic Lax operators related to the algebra $sl(6)$ by applying proper Mikhailov type reduction groups $\mathbb{Z}_h$. Here h is the Coxeter number of the relevant Kac–Moody algebra. Next we adapt Shabat's method for constructing the fundamental analytic solutions of the Lax operators L. Thus we are able to reduce the direct and inverse spectral problems for L to Riemann–Hilbert problems (RHP) on the union of $2h$ rays l_ν. They leave the origin of the complex λ-plane partitioning it into equal angles π/h. To each l_ν we associate a subalgebra $\mathfrak{g}_\nu$ which is a direct sum of $sl(2)$–subalgebras. In this way, to each regular solution of the RHP we can associate scattering data of L consisting of scattering matrices $T_\nu \in \mathcal{G}_\nu$ and their Gauss decompositions. The main result of the paper states how to find the minimal sets of scattering data $\mathcal{T}_k$, $k = 1, 2$, from T_0 and T_1 related to the rays l_0 and l_1. We prove that each of the minimal sets $\mathcal{T}_1$ and $\mathcal{T}_2$ allows one to reconstruct both the scattering matrices T_ν, $\nu = 0, 1, \dots 2h$ and the corresponding potentials of the Lax operators L.

Keywords: mKdV equations, Kac–Moody algebras, Lax operators, minimal sets of scattering data.

I.T. Habibullin, A.R. Khakimova, A.O. Smirnov

Generalized invariant manifolds for integrable equations and their applications

Abstract. In the article we discuss the notion of the generalized invariant manifold introduced in our previous study. In the literature, the method of the differential constraints is well known as a tool for constructing particular solutions for the nonlinear partial differential equations. Its essence is in adding to a given nonlinear PDE, another much simpler, as a rule ordinary, differential equation, consistent with the given one. Then any solution of the ODE is a particular solution of the PDE as well. However the main problem is to find this consistent ODE. Our generalization is that we look for an ordinary differential equation that is consistent not with the nonlinear partial differential equation itself, but with its linearization. Such generalized invariant manifold is effectively sought. Moreover, it allows one to construct such important attributes of integrability theory as Lax pairs and recursion operators for integrable nonlinear equations. In this paper, we show that they provide a way to construct particular solutions to the equation as well.

Keywords: invariant manifold, integrable system, recursion operator, Lax pair, algebro-geometric solutions, Dubrovin equations, spectral curves.

D. Levi, M.A. Rodríguez

Yamilov's theorem for differential and difference equations

Abstract. S-integrable scalar evolutionary differential difference equations in $1+1$ dimensions have a very particular form described by Yamilov's theorem. We look for similar results in the case of S-integrable 2-dimensional partial difference equations and 2-dimensional partial differential equations. To do so, on one side we discuss the semi-continuous limit of S-integrable quad equations and on the other, we semi-discretize partial differential equations. For partial differential equations, we show that any equation can be semi-discretized in such a way to satisfy Yamilov's theorem. In the case of partial difference equations, we are not able to find a form of the equation such that its semi-continuous limit always satisfies Yamilov's theorem. So we just present a few examples, in which to get evolutionary equations, we need to carry out a skew limit. We also consider an S-integrable quad equation with non-constant coefficients which in the skew limit satisfies an extended Yamilov's theorem as it has non-constant coefficients. This equation turns out to be a subcase of the Yamilov discretization of the Krichever-Novikov equation with non-constant coefficient, an equation suggested to be integrable by Levi and Yamilov in 1997 and whose integrability has been proved only recently by algebraic entropy. If we do a strait limit, we get non-local evolutionary equations, which show that an extension of Yamilov's theorem may exist in this case.

Keywords: differential difference equations, continuous and discrete integrable systems, Yamilov's theorem.

S.Ya. Startsev

On Darboux non-integrability of Hietarinta equation

Abstract. The autonomous Hietarinta equation is a well-known example of the quad-graph discrete equation which is consistent around the cube. In a recent work, it was conjectured that this equation is Darboux integrable, that is, for each of two independent discrete variables there exist non-trivial functions that remain unchanged on solutions of the equation after the shift in this discrete variable. We demonstrate that this conjecture is not true for generic values of the equation coefficients.

To do this, we employ two-point invertible transformations introduced by R.I. Yamilov. We prove that an autonomous difference equation on the quad-graph cannot be Darboux integrable if a transformation of the above type maps solutions of this equation into its solutions. This implies that the generic Hietarinta equation is not Darboux integrable since the Hietarinta equation in the general case possesses the two-point invertible auto-transformations. Along the way, all Darboux integrable subcases of the Hietarinta equation are found. All of them are reduced by point transformations to already known integrable equations.

At the end of the article, we also briefly describe another way to prove the Darboux non-integrability of the Hietarinta equation. This alternative way is based on the known fact that a difference substitution relates this equation to a linear one. Thus, the Hietarinta equation gives us an example of a quad-graph equation that is linearizable but not Darboux integrable.

Keywords: Hietarinta equation, quad-graph equation, Bäcklund auto-transformation, Darboux integrability, C-integrability.

K. Zheltukhin, N. Zheltukhina

On discretization of Darboux integrable systems admitting second-order integrals

Abstract. We consider a discretization problem for hyperbolic Darboux integrable systems. In particular, we discretize continuous systems admitting x- and y-integrals of the first and second order. Such continuous systems were classified by Zhyber and Kostrigina. In the present paper, continuous systems are discretized with respect to one of continuous variables and the resulting semi-discrete system is required to be also Darboux integrable.

To obtain such discretization, we take x- or y-integrals of a given continuous system and look for a semi-discrete systems admitting the chosen integrals as n-integrals. This method was proposed by Habibullin. For all considered systems and corresponding sets of integrals we were able to find such semi-discrete systems. In general, the obtained semi-discrete systems are given in terms of solutions of some first order quasilinear differential systems. For all such first order quasilinear differential systems we find implicit solutions. New examples of semi-discrete Darboux integrable systems are obtained. Also for each of considered continuous systems we determine a corresponding semi-discrete system that gives the original system in the continuum limit.

Keywords: Darboux integrability, discretization.