ISSN 2074-1871 Уфимский математический журнал. Том 12. № 3 (2020). С. 109-124.

SOLVABILITY OF HIGHER ORDER THREE-POINT ITERATIVE SYSTEMS

K.R. PRASAD, M. RASHMITA, N. SREEDHAR

Abstract. In this paper, we consider an iterative system of nonlinear n^{th} order differential equations:

$$y_i^{(n)}(t) + \lambda_i p_i(t) f_i(y_{i+1}(t)) = 0, \qquad 1 \leqslant i \leqslant m, \qquad y_{m+1}(t) = y_1(t), \qquad t \in [0,1],$$

with three-point non-homogeneous boundary conditions

$$y_i(0) = y_i{}'(0) = \cdots = y_i^{(n-2)}(0) = 0,$$
$$\alpha_i y_i^{(n-2)}(1) - \beta_i y_i^{(n-2)}(\eta) = \mu_i, \qquad 1 \leqslant i \leqslant m,$$

where $n \geqslant 3$, $\eta \in (0,1)$, $\mu_i \in (0,\infty)$ is a parameter, $f_i : \mathbb{R}^+ \to \mathbb{R}^+$ is continuous, $p_i : [0,1] \to \mathbb{R}^+$ is continuous and p_i does not vanish identically on any closed subinterval of $[0,1]$ for $1 \leqslant i \leqslant m$. We express the solution of the boundary value problem as a solution of an equivalent integral equation involving kernels and obtain bounds for these kernels. By an application of Guo–Krasnosel'skii fixed point theorem on a cone in a Banach space, we determine intervals of the eigenvalues $\lambda_1, \lambda_2, \cdots, \lambda_m$ for which the boundary value problem possesses a positive solution. As applications, we provide examples demonstrating our results.

Keywords: boundary value problem, iterative system, kernel, three-point, eigenvalues, cone, positive solution. **Mathematics Subject Classification:** 334B18, 34A40, 34B15

1. Introduction

The existence of positive solutions for multi-point boundary value problems associated with ordinary differential equations are of a high interest and play a vital role in different areas of applied mathematics and physics. Multi-point boundary value problems appear in the mathematical modelling of deflection of a curve beam having a constant or varying cross section, three layer beam, electromagnetic waves and so on. For example, the vibration of a guy wire of a uniform cross-section and composed of different parts with different densities can be formulated as multi-point boundary value problems.

Due to the importance in both theory and applications, much attention is focussed on obtaining optimal eigenvalue intervals for the existence of positive solutions of the iterative systems of nonlinear multi-point boundary value problems by an application of Guo–Krasnosel'skii fixed point theorem. A few papers along these lines are Henderson and Ntouyas [5], Henderson, Ntouyas and Purnaras [6, 7] and Prasad, Sreedhar and Kumar [14]. In the past, the researchers have focussed and established the existence of positive solutions of the boundary value problems associated with homogeneous boundary conditions,

K.R. Prasad, M. Rashmita, N. Sreedhar, Solvability of Higher order three-point iterative systems.

Поступила 12 декабря 2019 г.

M. Rashmita is thankful to DST-INSPIRE, Government of India, New Delhi for awarding JRF.

see [3, 13, 2, 12, 9, 18]. However, some works have been carried out in establishing the existence of positive solutions of the boundary value problems with non-homogeneous boundary conditions, see [15, 17, 11, 16, 10].

Motivated by the papers mentioned above, in this paper, we determine intervals of the eigenvalues $\lambda_1, \lambda_2, \cdots, \lambda_m$, which will give guarantee for the existence of positive solutions of the iterative system of nonlinear n^{th} order differential equations

$$y_i^{(n)}(t) + \lambda_i p_i(t) f_i(y_{i+1}(t)) = 0, \qquad 1 \leqslant i \leqslant m, \qquad y_{m+1}(t) = y_1(t), \qquad t \in [0,1], \tag{1.1}$$

satisfying three-point non-homogeneous boundary conditions

$$y_i(0) = y_i'(0) = \cdots = y_i^{(n-2)}(0) = 0, \qquad \alpha_i y_i^{(n-2)}(1) - \beta_i y_i^{(n-2)}(\eta) = \mu_i, \qquad 1 \leqslant i \leqslant m, \tag{1.2}$$

where $n \geqslant 3$, $\eta \in (0,1)$ and $\mu_i \in (0,\infty)$ is a parameter for $1 \leqslant i \leqslant m$. Our approach is based on application of Guo–Krasnosel'skii fixed point theorem on a cone in a Banach space.

Throughout the paper, we assume that the following conditions hold true:

($B1$) $f_i : R^+ \to R^+$ is continuous for $1 \leqslant i \leqslant m$,

($B2$) $p_i : [0,1] \to R^+$ is continuous and p_i does not vanish identically on any closed subinterval of $[0,1]$ for $1 \leqslant i \leqslant m$,

($B3$) α_i and β_i are constants such that $\alpha_i > 0$ and $\beta_i \in (0, \frac{\alpha_i}{\eta})$ for $1 \leqslant i \leqslant m$,

($B4$) each of

$$f_{i0} = \lim_{x \to 0^+} \frac{f_i(x)}{x} \quad \text{and} \quad f_{i\infty} = \lim_{x \to \infty} \frac{f_i(x)}{x}$$

for $1 \leqslant i \leqslant m$ exists as positive real number.

The rest of the paper is organized as follows. In Section 2, we express the solution of the boundary value problem (1.1)–(1.2) as a solution of an equivalent integral equation involving kernels and find bounds for the these kernels. In Section 3, we establish the criteria determining the eigenvalues, for which the boundary value problems (1.1)-(1.2) has at least one positive solution in a cone; this is done by using the Guo–Krasnosel'skiis fixed point theorem. In Section 4, as an application, we provide some examples to illustrate our results.

2. Kernels and bounds

In this section, we express the solution of the boundary value problem (1.1)-(1.2) into an equivalent integral equation involving kernels by determining integral equation of y_i for $1 \leqslant i \leqslant m$ and find bounds for the kernels, which will be needed to establish the main results.

Lemma 2.1. *If $h(t) \in C([0,1], R^+)$, then the boundary value problem*

$$y_i^{(n)}(t) + h(t) = 0, \quad 1 \leqslant i \leqslant m, \quad t \in [0,1], \tag{2.1}$$

with (1.2) *has a unique solution and is given by*

$$y_i(t) = \frac{\mu_i t^{n-1}}{(n-1)!(\alpha_i - \eta\beta_i)} + \int_0^1 \Big[G(t,s) + \frac{\beta_i t^{n-1}}{(n-1)!(\alpha_i - \eta\beta_i)} G_1(\eta,s)\Big] h(s)\, ds, \tag{2.2}$$

where

$$G(t,s) = \frac{1}{(n-1)!} \begin{cases} \big[t^{n-1}(1-s) - (t-s)^{n-1}\big], & 0 \leqslant s \leqslant t \leqslant 1, \\ t^{n-1}(1-s), & 0 \leqslant t \leqslant s \leqslant 1, \end{cases} \tag{2.3}$$

and

$$G_1(\eta,s) = \begin{cases} s(1-\eta), & 0 \leqslant s \leqslant \eta \leqslant 1, \\ \eta(1-s), & 0 \leqslant \eta \leqslant s \leqslant 1. \end{cases} \tag{2.4}$$

Proof. Let $y_i(t)$, $1 \leqslant i \leqslant m$, be the solution of boundary value problem (2.1), (1.2). Then an equivalent integral equation of (2.1) is given by

$$y_i(t) = C_0 + C_1 t + C_2 t^2 + \cdots + C_{n-1} t^{n-1} - \frac{1}{(n-1)!}\int_0^t (t-s)^{n-1} h(s) ds.$$

Using the boundary conditions (1.2), we can determine C_j as

$$C_j = 0 \quad \text{as} \quad j = 0, 1, 2, \cdots, n-2$$

and

$$C_{n-1} = \frac{\mu_i}{(n-1)!(\alpha_i - \eta\beta_i)} + \frac{\alpha_i}{(n-1)!(\alpha_i - \eta\beta_i)}\int_0^1 (1-s)h(s)\, ds - \frac{\beta_i}{(n-1)!(\alpha_i - \eta\beta_i)}\int_0^\eta (\eta - s)h(s)\, ds.$$

Thus, the unique solution of boundary value problem (2.1), (1.2) is

$$y_i(t) = \frac{\mu_i t^{n-1}}{(n-1)!(\alpha_i - \eta\beta_i)} + \int_0^1 \left(G(t,s) + \frac{\beta_i t^{n-1}}{(n-1)!(\alpha_i - \eta\beta_i)} G_1(\eta, s) \right) h(s)\, ds.$$

□

Lemma 2.2. *Assume that the condition (B3) is satisfied. Then the kernels $G(t,s)$ and $G_1(t,s)$ are satisfies the following inequalities:*

(i) $G(t,s) \geqslant 0$ *and* $G_1(t,s) \geqslant 0$ *for all* $t, s \in [0,1]$,
(ii) $G(t,s) \leqslant G(1,s)$ *for all* $t, s \in [0,1]$,
(iii) $G(t,s) \geqslant \frac{1}{4^{n-1}} G(1,s)$ *for all* $t \in I$ *and* $s \in [0,1]$, *where* $I = \left[\frac{1}{4}, \frac{3}{4}\right]$.

Proof. We prove the inequality (i). For $0 \leqslant s \leqslant t \leqslant 1$, then we have

$$\begin{aligned} G(t,s) &= \frac{1}{(n-1)!}(t^{n-1}(1-s) - (t-s)^{n-1}) \geqslant \frac{1}{(n-1)!}(t^{n-1}(1-s) - (t-st)^{n-1}) \\ &= \frac{t^{n-1}}{(n-1)!}((1-s) - (1-s)^{n-1}) \geqslant 0 \end{aligned}$$

and

$$G_1(t,s) = s(1-t) \geqslant 0.$$

For $0 \leqslant t \leqslant s \leqslant 1$, then we have

$$G(t,s) = \frac{t^{n-1}(1-s)}{(n-1)!} \geqslant 0$$

and

$$G_1(t,s) = t(1-s) \geqslant 0.$$

Now, we prove the inequality (ii). For $0 \leqslant s \leqslant t \leqslant 1$, then we have

$$\frac{\partial}{\partial t} G(t,s) = \frac{1}{(n-2)!}\left(t^{n-2}(1-s) - (t-s)^{n-2}\right) \geqslant \frac{t^{n-2}}{(n-2)!}\left((1-s) - (1-s)^{n-2}\right) \geqslant 0.$$

For $0 \leqslant t \leqslant s \leqslant 1$, then we have

$$\frac{\partial}{\partial t}G(t,s) = \frac{t^{n-2}}{(n-2)!}(1-s) \geqslant 0.$$

Therefore, $G(t,s)$ is increasing in t, which implies that $G(t,s) \leqslant G(1,s)$.

We proceed to proving inequality (iii). Hence, as $0 \leqslant s \leqslant t \leqslant 1$ and $t \in I$, we have

$$\begin{aligned} G(t,s) &= \frac{1}{(n-1)!}\big(t^{n-1}(1-s) - (t-s)^{n-1}\big) \\ &\geqslant \frac{t^{n-1}}{(n-1)!}\big((1-s) - (1-s)^{n-1}\big) \geqslant \frac{1}{4^{n-1}}G(1,s). \end{aligned}$$

As $0 \leqslant t \leqslant s \leqslant 1$ and $t \in I$ we then get

$$G(t,s) = \frac{t^{n-1}}{(n-1)!}(1-s) \geqslant \frac{1}{4^{n-1}}G(1,s).$$

□

We note that an m-tuple $(y_1(t), y_2(t), \cdots, y_m(t))$ is a solution of the boundary value problem (1.1)-(1.2) if and only if $y_i(t)$ satisfies the following equations

$$\begin{aligned} y_i(t) =& \frac{\mu_i t^{n-1}}{(n-1)!(\alpha_i - \eta\beta_i)} \\ &+ \lambda_i \int_0^1 \Big[G(t,s) + \frac{\beta_i t^{n-1}}{(n-1)!(\alpha_i - \eta\beta_i)}G_1(\eta,s)\Big]p_i(s)f_i(y_{i+1}(s))ds, \quad 1 \leqslant i \leqslant m, \quad t \in [0,1], \end{aligned}$$

and

$$y_{m+1}(t) = y_1(t), \quad t \in [0,1],$$

so that, in particular,

$$\begin{aligned} y_1(t) = & \frac{\mu_1 t^{n-1}}{(n-1)!(\alpha_1 - \eta\beta_1)} + \lambda_1 \int_0^1 \Big(G(t,s_1) + \frac{\beta_1 t^{n-1}}{(n-1)!(\alpha_1 - \eta\beta_1)}G_1(\eta,s_1)\Big)p_1(s_1) \\ & f_1\Bigg(\frac{\mu_2 s_1^{n-1}}{(n-1)!(\alpha_2 - \eta\beta_2)} + \lambda_2 \int_0^1 \Big(G(s_1,s_2) + \frac{\beta_2 s_1^{n-1}}{(n-1)!(\alpha_2 - \eta\beta_2)}G_1(\eta,s_2)\Big)p_2(s_2) \\ & \cdots f_{m-1}\bigg(\frac{\mu_m s_{m-1}^{n-1}}{(n-1)!(\alpha_m - \eta\beta_m)} + \lambda_m \int_0^1 \Big(G(s_{m-1},s_m) + \frac{\beta_m s_{m-1}^{n-1}}{(n-1)!(\alpha_m - \eta\beta_m)}G_1(\eta,s_m)\Big) \\ & p_m(s_m)f_m(y_1(s_m))ds_m\bigg)\cdots ds_2\Bigg)ds_1. \end{aligned}$$

The following Guo–Krasnosel'skii fixed point theorem is a fundamental tool to establish our main results.

Theorem 2.1. [1, 4, 8] *Let B be a Banach space, $P \subseteq B$ be a cone and suppose that Ω_1, Ω_2 are open subsets of B with $0 \in \Omega_1$ and $\overline{\Omega}_1 \subset \Omega_2$. Suppose further that $T : P \cap (\overline{\Omega}_2 \setminus \Omega_1) \to P$ is completely continuous operator such that either*

(i) $\|Ty\| \leqslant \|y\|$, $y \in P \cap \partial\Omega_1$, *and* $\|Ty\| \geqslant \|y\|$, $y \in P \cap \partial\Omega_2$,

or

(ii) $\|Ty\| \geqslant \|y\|$, $y \in P \cap \partial\Omega_1$, *and* $\|Ty\| \leqslant \|y\|$, $y \in P \cap \partial\Omega_2$.

Then T has a fixed point in $P \cap (\overline{\Omega}_2 \setminus \Omega_1)$.

3. Positive solutions in a cone

In this section, we establish a criteria to determine the eigenvalues, for which the iterative system of three-point non-homogeneous boundary value problem (1.1)–(1.2) has at least one positive solution in a cone.

Let $B = \{x : x \in C[0,1]\}$ be the Banach space equipped with the norm

$$\|x\| = \max_{t \in [0,1]} |x(t)|.$$

Define a cone $P \subset B$ by

$$P = \left\{ x \in B : x(t) \geqslant 0 \quad \text{on} \quad t \in [0,1] \quad \text{and} \quad \min_{t \in I} \quad x(t) \geqslant \frac{1}{4^{n-1}} \|x(t)\| \right\}.$$

We define an operator $T : P \to B$ for $y_1 \in P$ by

$$\begin{aligned} Ty_1(t) = & \frac{\mu_1 t^{n-1}}{(n-1)!(\alpha_1 - \eta\beta_1)} + \lambda_1 \int_0^1 \Big(G(t, s_1) + \frac{\beta_1 t^{n-1}}{(n-1)!(\alpha_1 - \eta\beta_1)} G_1(\eta, s_1) \Big) p_1(s_1) \\ & f_1 \Bigg(\frac{\mu_2 s_1^{n-1}}{(n-1)!(\alpha_2 - \eta\beta_2)} + \lambda_2 \int_0^1 \Bigg(G(s_1, s_2) + \frac{\beta_2 s_1^{n-1}}{(n-1)!(\alpha_2 - \eta\beta_2)} G_1(\eta, s_2) \Bigg) \\ & p_2(s_2) \cdots f_{m-1} \Bigg(\frac{\mu_m s_{m-1}^{n-1}}{(n-1)!(\alpha_m - \eta\beta_m)} + \lambda_m \int_0^1 \Bigg(G(s_{m-1}, s_m) \\ & + \frac{\beta_m s_{m-1}^{n-1}}{(n-1)!(\alpha_m - \eta\beta_m)} G_1(\eta, s_m) \Bigg) p_m(s_m) f_m(y_1(s_m)) ds_m \Bigg) \cdots ds_2 \Bigg) ds_1. \end{aligned} \tag{3.1}$$

Lemma 3.1. *The operator $T : P \to B$ is a self map on P.*

Proof. From the positivity of the kernels $G(t, s)$ and $G_1(t, s)$ in Lemma 2.2 that for $y_1 \in P$, $Ty_1(t) \geqslant 0$ on $t \in [0,1]$. Also, for $y_1 \in P$, and by Lemma 2.2, we have

$$\begin{aligned} Ty_1(t) = & \frac{\mu_1 t^{n-1}}{(n-1)!(\alpha_1 - \eta\beta_1)} + \lambda_1 \int_0^1 \Bigg(G(t, s_1) + \frac{\beta_1 t^{n-1}}{(n-1)!(\alpha_1 - \eta\beta_1)} G_1(\eta, s_1) \Bigg) \\ & p_1(s_1) f_1 \Bigg(\frac{\mu_2 s_1^{n-1}}{(n-1)!(\alpha_2 - \eta\beta_2)} + \lambda_2 \int_0^1 \Bigg(G(s_1, s_2) + \frac{\beta_2 s_1^{n-1}}{(n-1)!(\alpha_2 - \eta\beta_2)} \\ & G_1(\eta, s_2) \Bigg) p_2(s_2) \cdots f_{m-1} \Bigg(\frac{\mu_m s_{m-1}^{n-1}}{(n-1)!(\alpha_m - \eta\beta_m)} + \lambda_m \int_0^1 \Bigg(G(s_{m-1}, s_m) \\ & + \frac{\beta_m s_{m-1}^{n-1}}{(n-1)!(\alpha_m - \eta\beta_m)} G_1(\eta, s_m) \Bigg) p_m(s_m) f_m(y_1(s_m)) ds_m \Bigg) \cdots ds_2 \Bigg) ds_1 \end{aligned}$$

$$\leqslant \frac{\mu_1}{(n-1)!(\alpha_1-\eta\beta_1)} + \lambda_1 \int_0^1 \Big(G(1,s_1) + \frac{\beta_1}{(n-1)!(\alpha_1-\eta\beta_1)} G_1(\eta,s_1)\Big)$$

$$p_1(s_1) f_1\Big(\frac{\mu_2 s_1^{n-1}}{(n-1)!(\alpha_2-\eta\beta_2)} + \lambda_2 \int_0^1 \Big(G(s_1,s_2) + \frac{\beta_2 s_1^{n-1}}{(n-1)!(\alpha_2-\eta\beta_2)}$$

$$G_1(\eta,s_2)\Big) p_2(s_2) \cdots f_{m-1}\Big(\frac{\mu_m s_{m-1}^{n-1}}{(n-1)!(\alpha_m-\eta\beta_m)} + \lambda_m \int_0^1 \Big(G(s_{m-1},s_m)$$

$$+ \frac{\beta_m s_{m-1}^{n-1}}{(n-1)!(\alpha_m-\eta\beta_m)} G_1(\eta,s_m)\Big) p_m(s_m) f_m(y_1(s_m))\, ds_m\Big) \cdots ds_2\Big)\, ds_1,$$

so that,

$$\|Ty_1(t)\| \leqslant \frac{\mu_1}{(n-1)!(\alpha_1-\eta\beta_1)} + \lambda_1 \int_0^1 \Big(G(1,s_1) + \frac{\beta_1}{(n-1)!(\alpha_1-\eta\beta_1)} G_1(\eta,s_1)\Big)$$

$$p_1(s_1) f_1\Big(\frac{\mu_2 s_1^{n-1}}{(n-1)!(\alpha_2-\eta\beta_2)} + \lambda_2 \int_0^1 \Big(G(s_1,s_2) + \frac{\beta_2 s_1^{n-1}}{(n-1)!(\alpha_2-\eta\beta_2)}$$

$$G_1(\eta,s_2)\Big) p_2(s_2) \cdots f_{m-1}\Big(\frac{\mu_m s_{m-1}^{n-1}}{(n-1)!(\alpha_m-\eta\beta_m)} + \lambda_m \int_0^1 \Big(G(s_{m-1},s_m)$$

$$+ \frac{\beta_m s_{m-1}^{n-1}}{(n-1)!(\alpha_m-\eta\beta_m)} G_1(\eta,s_m)\Big) p_m(s_m) f_m(y_1(s_m))\, ds_m\Big) \cdots ds_2\Big)\, ds_1.$$

Further, if $y_1 \in P$, we have from Lemma 2.2 and the above inequalities that

$$\min_{t\in I}(Ty_1)(t) = \min_{t\in I}\Big\{\frac{\mu_1 t^{n-1}}{(n-1)!(\alpha_1-\eta\beta_1)} + \lambda_1 \int_0^1 \Big(G(t,s_1) + \frac{\beta_1 t^{n-1}}{(n-1)!(\alpha_1-\eta\beta_1)} G_1(\eta,s_1)\Big)$$

$$p_1(s_1) f_1\Big(\frac{\mu_2 s_1^{n-1}}{(n-1)!(\alpha_2-\eta\beta_2)} + \lambda_2 \int_0^1 \Big(G(s_1,s_2) + \frac{\beta_2 s_1^{n-1}}{(n-1)!(\alpha_2-\eta\beta_2)} G_1(\eta,s_2)\Big)$$

$$p_2(s_2) \cdots f_{m-1}\Big(\frac{\mu_m s_{m-1}^{n-1}}{(n-1)!(\alpha_m-\eta\beta_m)} + \lambda_m \int_0^1 \Big(G(s_{m-1},s_m) +$$

$$\frac{\beta_m s_{m-1}^{n-1}}{(n-1)!(\alpha_m-\eta\beta_m)} G_1(\eta,s_m)\Big) p_m(s_m) f_m(y_1(s_m)) ds_m\Big) \cdots ds_2\Big) ds_1\Big\}$$

$$\geqslant \frac{1}{4^{n-1}}\Big(\frac{\mu_1}{(n-1)!(\alpha_1-\eta\beta_1)} + \lambda_1 \int_0^1 \Big(G(1,s_1) + \frac{\beta_1}{(n-1)!(\alpha_1-\eta\beta_1)} G_1(\eta,s_1)\Big)$$

$$p_1(s_1) f_1\Big(\frac{\mu_2 s_1^{n-1}}{(n-1)!(\alpha_2-\eta\beta_2)} + \lambda_2 \int_0^1 \Big(G(s_1,s_2) + \frac{\beta_2 s_1^{n-1}}{(n-1)!(\alpha_2-\eta\beta_2)} G_1(\eta,s_2)\Big)$$

$$p_2(s_2)\cdots f_{m-1}\left(\frac{\mu_m s_{m-1}^{n-1}}{(n-1)!(\alpha_m-\eta\beta_m)}+\lambda_m\int_0^1\left(G(s_{m-1},s_m)+\right.\right.$$

$$\left.\left.\frac{\beta_m s_{m-1}^{n-1}}{(n-1)!(\alpha_m-\eta\beta_m)}G_1(\eta,s_m)\right)p_m(s_m)f_m(y_1(s_m))ds_m\right)\cdots ds_2\Big)ds_1\Big)$$

$$\geqslant\frac{1}{4^{n-1}}\|Ty_1(t)\|.$$

Therefore, $T:P\to P$ and this completes the proof. □

Next, by an applications of the Arzela-Ascoli theorem, one can easily prove the operator T is completely continuous.

Now, we seek suitable fixed point of the operator T belonging to the cone P. To prove our result, we define positive numbers F_1, F_2, F_3 and F_4 by the formulae

$$F_1=\max_{1\leqslant i\leqslant m}\left\{\left(\frac{f_{i\infty}}{4^{2n-2}}\int_{s\in I}\left(G(1,s)+\frac{\beta_i}{(n-1)!(\alpha_i-\eta\beta_i)}G_1(\eta,s)\right)p_i(s)\,ds\right)^{-1}\right\},\tag{3.2}$$

$$F_2=\min_{1\leqslant i\leqslant m}\left\{\frac{1}{2}\left(f_{i0}\int_0^1\left(G(1,s)+\frac{\beta_i}{(n-1)!(\alpha_i-\eta\beta_i)}G_1(\eta,s)\right)p_i(s)\,ds\right)^{-1}\right\},\tag{3.3}$$

$$F_3=\max_{1\leqslant i\leqslant m}\left\{\left(\frac{f_{i0}}{4^{2n-2}}\int_{s\in I}\left(G(1,s)+\frac{\beta_i}{(n-1)!(\alpha_i-\eta\beta_i)}G_1(\eta,s)\right)p_i(s)\,ds\right)^{-1}\right\}\tag{3.4}$$

$$F_4=\min_{1\leqslant i\leqslant m}\left\{\frac{1}{2}\left(f_{i\infty}\int_0^1\left(G(1,s)+\frac{\beta_i}{(n-1)!(\alpha_i-\eta\beta_i)}G_1(\eta,s)\right)p_i(s)\,ds\right)^{-1}\right\}.\tag{3.5}$$

Theorem 3.1. *Assume that the conditions (B1)–(B4) are satisfied. Then, for each* $\lambda_1,\lambda_2,\cdots,\lambda_m$ *satisfying either*

$$F_1<\lambda_1<F_2,\quad F_1<\lambda_2<F_2,\cdots,F_1<\lambda_m<F_2,\tag{3.6}$$

or

$$F_3<\lambda_1<F_4,\quad F_3<\lambda_2<F_4,\cdots,F_3<\lambda_m<F_4,\tag{3.7}$$

there exists an m-tuple $(y_1,y_2,\cdots,y_m)$ *satisfying* (1.1)–(1.2) *such that* $y_i(t)>0$ *on* $(0,1)$ *and* $\mu_i\in(0,\infty)$ *is sufficiently small for* $1\leqslant i\leqslant m$.

Proof. Let $\lambda_i, 1\leqslant i\leqslant m$, be given as in (3.6). Let $\epsilon>0$ be chosen such that

$$\max_{1\leqslant i\leqslant m}\left\{\left(\frac{f_{i\infty}-\epsilon}{4^{2n-2}}\int_{s\in I}\left[G(1,s)+\frac{\beta_i}{(n-1)!(\alpha_i-\eta\beta_i)}G_1(\eta,s)\right]p_i(s)ds\right)^{-1}\right\}\leqslant\min\{\lambda_1,\lambda_2,\cdots,\lambda_m\}$$

and

$$\max\{\lambda_1,\lambda_2,\cdots,\lambda_m\}$$
$$\leqslant\min_{1\leqslant i\leqslant m}\left\{\frac{1}{2}\left((f_{i0}+\epsilon)\int_0^1\left(G(1,s)+\frac{\beta_i}{(n-1)!(\alpha_i-\eta\beta_i)}G_1(\eta,s)\right)p_i(s)\,ds\right)^{-1}\right\}.$$

Now, we seek a fixed point of the completely continuous operator $T : P \to P$ defined by (3.1). By the definition of f_{i0}, $1 \leqslant i \leqslant m$, there exists an $H_1 > 0$ such that, for each $1 \leqslant i \leqslant m$, the inequality

$$f_i(x) \leqslant (f_{i0} + \epsilon)x, \quad 0 < x \leqslant H_1,$$

holds true.

Let μ_i, $1 \leqslant i \leqslant m$, be such that

$$0 < \mu_i \leqslant \frac{(n-1)!(\alpha_i - \eta\beta_i)H_1}{2}.$$

Let $y_1 \in P$ with $\|y_1\| = H_1$. By Lemma 2.2 and the choice of ϵ, for $0 \leqslant s_{m-1} \leqslant 1$ we have

$$\begin{aligned}
&\frac{\mu_m s_{m-1}^{n-1}}{(n-1)!(\alpha_m - \eta\beta_m)} + \lambda_m \int_0^1 \left(G(s_{m-1}, s_m) + \frac{\beta_m s_{m-1}^{n-1}}{(n-1)!(\alpha_m - \eta\beta_m)} G_1(\eta, s_m) \right) \\
&\qquad p_m(s_m) f_m(y_1(s_m))\, ds_m \\
&\leqslant \frac{\mu_m}{(n-1)!(\alpha_m - \eta\beta_m)} + \lambda_m \int_0^1 \left(G(1, s_m) + \frac{\beta_m}{(n-1)!(\alpha_m - \eta\beta_m)} G_1(\eta, s_m) \right) \\
&\qquad p_m(s_m)(f_{m0} + \epsilon) y_1(s_m)\, ds_m \\
&\leqslant \frac{H_1}{2} + \lambda_m \int_0^1 \left(G(1, s_m) + \frac{\beta_m}{(n-1)!(\alpha_m - \eta\beta_m)} G_1(\eta, s_m) \right) p_m(s_m) ds_m (f_{m0} + \epsilon)\|y_1\| \\
&\leqslant \frac{H_1}{2} + \frac{H_1}{2} = H_1.
\end{aligned}$$

In the same way, it follows from Lemma 2.2 and the choice of ϵ that, for $0 \leqslant s_{m-2} \leqslant 1$,

$$\begin{aligned}
&\frac{\mu_{m-1} s_{m-2}^{n-1}}{(n-1)!(\alpha_{m-1} - \eta\beta_{m-1})} + \lambda_{m-1} \int_0^1 \left(G(s_{m-2}, s_{m-1}) + \frac{\beta_{m-1} s_{m-2}^{n-1}}{(n-1)!(\alpha_{m-1} - \eta\beta_{m-1})} \right. \\
&\left. G_1(\eta, s_{m-1}) \right) p_{m-1}(s_{m-1}) f_{m-1} \left(\frac{\mu s_{m-1}^{n-1}}{(n-1)!(\alpha_m - \eta\beta_m)} + \lambda_m \int_0^1 \left(G(s_{m-1}, s_m) \right. \right. \\
&\left. \left. + \frac{\beta_m s_{m-1}^{n-1}}{(n-1)!(\alpha_m - \eta\beta_m)} G_1(\eta, s_m) \right) p_m(s_m) f_m(y_1(s_m)) ds_m \right) ds_{m-1} \\
&\leqslant \frac{\mu_{m-1}}{(n-1)!(\alpha_{m-1} - \eta\beta_{m-1})} + \lambda_{m-1} \int_0^1 \left(G(1, s_{m-1}) + \frac{\beta_{m-1}}{(n-1)!(\alpha_{m-1} - \eta\beta_{m-1})} \right. \\
&\left. G_1(\eta, s_{m-1}) \right) p_{m-1}(s_{m-1}) ds_{m-1} (f_{m-1,0} + \epsilon) H_1 \\
&\leqslant \frac{H_1}{2} + \frac{H_1}{2} = H_1.
\end{aligned}$$

Continuing with this bootstrapping argument, we have, for $0 \leqslant t \leqslant 1$,

$$\frac{\mu_1 t^{n-1}}{(n-1)!(\alpha_1 - \eta\beta_1)} + \lambda_1 \int_0^1 \left(G(t, s_1) + \frac{\beta_1 t^{n-1}}{(n-1)!(\alpha_1 - \eta\beta_1)} G_1(\eta, s_1) \right) p_1(s_1)$$

$$f_1\left(\frac{\mu_2 s_1^{n-1}}{(n-1)!(\alpha_2-\eta\beta_2)}+\lambda_2\int_0^1\left(G(s_1,s_2)+\frac{\beta_2 s_1^{n-1}}{(n-1)!(\alpha_2-\eta\beta_2)}G_1(\eta,s_2)\right)p_2(s_2)\right.$$
$$\cdots f_{m-1}\left(\frac{\mu_m s_{m-1}^{n-1}}{(n-1)!(\alpha_m-\eta\beta_m)}+\lambda_m\int_0^1\left(G(s_{m-1},s_m)+\frac{\beta_m s_{m-1}^{n-1}}{(n-1)!(\alpha_m-\eta\beta_m)}\right.\right.$$
$$\left.\left.\left.G_1(\eta,s_m)\right)p_m(s_m)f_m(y_1(s_m))ds_m\right)\cdots ds_2\right)ds_1\leqslant H_1,$$

so that for $0\leqslant t\leqslant 1$,

$$Ty_1(t)\leqslant H_1.$$

Hence, $\|Ty_1\|\leqslant H_1=\|y_1\|$. If we let

$$\Omega_1=\left\{x\in B:\|x\|<H_1\right\},$$

then

$$\|Ty_1\|\leqslant\|y_1\|\quad\text{for}\quad y_1\in P\cap\partial\Omega_1. \tag{3.8}$$

By the definitions of $f_{i\infty}, 1\leqslant i\leqslant m$, there exists $\bar{H}_2\geqslant 0$, such that, for each $1\leqslant i\leqslant m$,

$$f_i(x)\geqslant(f_{i\infty}-\epsilon)x,\quad x\geqslant\bar{H}_2.$$

Let

$$H_2=\max\left\{2H_1,4^{n-1}\bar{H}_2\right\}.$$

We choose $y_1\in P$ and $\|y_1\|=H_2$. Then

$$\min_{t\in I}y_1(t)\geqslant\frac{1}{4^{n-1}}\|y_1\|\geqslant\bar{H}_2.$$

By Lemma 2.2 and the choice of ϵ, for $\frac{1}{4}\leqslant s_{m-1}\leqslant\frac{3}{4}$, we have:

$$\frac{\mu_m s_{m-1}^{n-1}}{(n-1)!(\alpha_m-\eta\beta_m)}+\lambda_m\int_0^1\left(G(s_{m-1},s_m)+\frac{\beta_m s_{m-1}^{n-1}}{(n-1)!(\alpha_m-\eta\beta_m)}G_1(\eta,s_m)\right)$$
$$mp_m(s_m)f_m(y_1(s_m))\,ds_m$$
$$\geqslant\lambda_m\int_0^1\left(G(s_{m-1},s_m)+\frac{\beta_m s_{m-1}^{n-1}}{(n-1)!(\alpha_m-\eta\beta_m)}G_1(\eta,s_m)\right)p_m(s_m)f_m(y_1(s_m))\,ds_m$$
$$\geqslant\frac{1}{4^{n-1}}\lambda_m\int_{s\in I}\left(G(1,s_m)+\frac{\beta_m}{(n-1)!(\alpha_m-\eta\beta_m)}G_1(\eta,s_m)\right)p_m(s_m)(f_{m\infty}-\epsilon)y_1(s_m)\,ds_m$$
$$\geqslant\frac{1}{4^{2n-2}}\lambda_m\int_{s\in I}\left(G(1,s_m)+\frac{\beta_m}{(n-1)!(\alpha_m-\eta\beta_m)}G_1(\eta,s_m)\right)p_m(s_m)\,ds_m(f_{m\infty}-\epsilon)\|y_1\|$$
$$\geqslant\|y_1\|=H_2.$$

In the same way, it follows from Lemma 2.2 and the choice of ϵ, that for $\frac{1}{4}\leqslant s_{m-1}\leqslant\frac{3}{4}$, we have

$$\frac{\mu_{m-1}s_{m-2}^{n-1}}{(n-1)!(\alpha_{m-1}-\eta\beta_{m-1})}+\lambda_{m-1}\int_0^1\left(G(s_{m-2},s_{m-1})+\frac{\beta_{m-1}s_{m-2}^{n-1}}{(n-1)!(\alpha_{m-1}-\eta\beta_{m-1})}\right.$$

$$G_1(\eta,s_{m-1})\Big)p_{m-1}(s_{m-1})f_{m-1}\Bigg(\frac{\mu_m s_{m-1}^{n-1}}{(n-1)!(\alpha_{m-1}-\eta\beta_{m-1})}+\lambda_m\int_0^1\Bigg(G(s_{m-1},s_m)$$
$$+\frac{\beta_{m-1}s_{m-1}^{n-1}}{(n-1)!(\alpha_{m-1}-\eta\beta_{m-1})}G_1(\eta,s_m)\Bigg)p_m(s_m)f_m(y_1(s_m))ds_m\Bigg)ds_{m-1}$$
$$\geqslant\frac{1}{4^{n-1}}\lambda_{m-1}\int_{s\in I}\Bigg(G(1,s_{m-1})+\frac{\beta_{m-1}}{(n-1)!(\alpha_{m-1}-\eta\beta_{m-1})}G_1(\eta,s_{m-1})\Bigg)$$
$$p_{m-1}(s_{m-1})ds_{m-1}(f_{m-1,\infty}-\epsilon)H_2$$
$$\geqslant\frac{1}{4^{2n-2}}\lambda_{m-1}\int_{s\in I}\Bigg(G(1,s_{m-1})+\frac{\beta_{m-1}}{(n-1)!(\alpha_{m-1}-\eta\beta_{m-1})}G_1(\eta,s_{m-1})\Bigg)$$
$$p_{m-1}(s_{m-1})ds_{m-1}(f_{m-1,\infty}-\epsilon)H_2\geqslant H_2.$$

Proceeding as above, we get:

$$\frac{\mu_1 t^{n-1}}{(n-1)!(\alpha_1-\eta\beta_1)}+\lambda_1\int_0^1\Bigg(G(t,s_1)+\frac{\beta_1 t^{n-1}}{(n-1)!(\alpha_1-\eta\beta_1)}G_1(\eta,s_1)\Bigg)p_1(s_1)$$
$$f_1\Bigg(\frac{\mu_2 s_1^{n-1}}{(n-1)!(\alpha_2-\eta\beta_2)}+\lambda_2\int_0^1\Bigg(G(s_1,s_2)+\frac{\beta_2 s_1^{n-1}}{(n-1)!(\alpha_2-\eta\beta_2)}G_1(\eta,s_2)\Bigg)p_2(s_2)$$
$$\cdots f_{m-1}\Bigg(\frac{\mu_m s_{m-1}^{n-1}}{(n-1)!(\alpha_{m-1}-\eta\beta_{m-1})}+\lambda_m\int_0^1\Bigg(G(s_{m-1},s_m)+\frac{\beta_m s_{m-1}^{n-1}}{(n-1)!(\alpha_m-\eta\beta_m)}$$
$$G_1(\eta,s_m)\Bigg)p_m(s_m)f_m(y_1(s_m))ds_m\Bigg)\cdots ds_2\Bigg)ds_1\geqslant H_2,$$

so that, for $0\leqslant t\leqslant 1$,

$$Ty_1(t)\geqslant H_2=\|y_1\|.$$

Hence, $\|Ty_1\|\geqslant\|y_1\|$. If we let

$$\Omega_2=\big\{x\in B:\|x\|<H_2\big\},$$

then

$$\|Ty_1\|\geqslant\|y_1\|,\quad\text{for}\quad y_1\in P\cap\partial\Omega_2. \tag{3.9}$$

Applying Theorem 2.1 to (3.8) and (3.9), we obtain that T has a fixed point $y_1\in P\cap(\bar\Omega_2\setminus\Omega_1)$. Since $y_{m+1}=y_1$, we obtain a positive solution $(y_1,y_2,\cdots,y_m)$ of (1.1)–(1.2) given iteratively by

$$y_i(t)=\frac{\mu_i t^{n-1}}{(n-1)!(\alpha_i-\eta\beta_i)}+\lambda_i\int_0^1\Bigg(G(t,s)+\frac{\beta_i t^{n-1}}{(n-1)!(\alpha_i-\eta\beta_i)}G_1(\eta,s)\Bigg)p_i(s)f_i(y_{i+1}(s))ds,$$

as $i=m,m-1,\cdots,1$.

Let λ_i, $1\leqslant i\leqslant m$, be given as in (3.7) and let $\epsilon>0$ be chosen such that

$$\max_{1\leqslant i\leqslant m}\Bigg\{\Bigg(\frac{f_{i0}-\epsilon}{4^{2n-2}}\int_{s\in I}\Bigg(G(1,s)+\frac{\beta_i}{(n-1)!(\alpha_i-\eta\beta_i)}G_1(\eta,s)\Bigg)p_i(s)\,ds\Bigg)^{-1}\Bigg\}\leqslant\min\{\lambda_1,\lambda_2,\cdots,\lambda_m\}$$

and

$$\max\{\lambda_1, \lambda_2, \cdots, \lambda_m\}$$
$$\leqslant \min_{1\leqslant i\leqslant m}\left\{\frac{1}{2}\left(\int\limits_0^1\left(G(1,s)+\frac{\beta_i}{(n-1)!(\alpha_i-\eta\beta_i)}G_1(\eta,s)\right)p_i(s)\,ds(f_{i\infty}+\epsilon)\right)^{-1}\right\}.$$

We seek fixed point of the completely continuous operator $T : P \to P$ defined in (3.1). By the definition of $f_{i0}, 1 \leqslant i \leqslant m$, there exists an $\bar{H}_3 > 0$ such that, for each $1 \leqslant i \leqslant m$,

$$f_i(x) \geqslant (f_{i0}-\epsilon)x, \quad 0 < x \leqslant \bar{H}_3.$$

Also, it follows from the definition of f_{i0} that $f_{i0} = 0, 1 \leqslant i \leqslant m$, and so there exists $0 < l_m < l_{m-1} < \cdots < l_2 < \bar{H}_3$ such that

$$\lambda_i f_i(x) \leqslant \frac{l_{i-1}}{2\int\limits_0^1\left(G(1,s_m)+\frac{\beta_i}{(n-1)!(\alpha_i-\eta\beta_i)}G_1(\eta,s_m)\right)p_i(s)\,ds}, \quad x\in[0,l_i],$$
$$0<\mu_i<\frac{(n-1)!(\alpha_i-\eta\beta_i)l_{i-1}}{2} \quad \text{for} \quad 3\leqslant i\leqslant m,$$

and

$$\lambda_2 f_2(x) \leqslant \frac{\bar{H}_3}{2\int\limits_0^1\left(G(1,s_m)+\frac{\beta_2}{(n-1)!(\alpha_2-\eta\beta_2)}G_1(\eta,s_m)\right)p_2(s)\,ds}, \quad x\in[0,l_2],$$
$$0<\mu_2<\frac{(n-1)!(\alpha_2-\eta\beta_2)\bar{H}_3}{2}.$$

Choose $y_1 \in P$ with $\|y_1\| = l_m$. Then we have

$$\frac{\mu_m s_{m-1}^{n-1}}{(n-1)!(\alpha_m-\eta\beta_m)}+\lambda_m\int\limits_0^1\left(G(s_{m-1},s_m)+\frac{\beta_m s_{m-1}^{n-1}}{(n-1)!(\alpha_m-\eta\beta_m)}\right.$$
$$\left.G_1(\eta,s_m)\right)p_m(s_m)f_m(y_1(s_m))ds_m$$
$$\leqslant\frac{\mu_m}{(n-1)!(\alpha_m-\eta\beta_m)}+\lambda_m\int\limits_0^1\left(G(1,s_m)+\frac{\beta_m}{(n-1)!(\alpha_m-\eta\beta_m)}G_1(\eta,s_m)\right)$$
$$p_m(s_m)f_m(y_1(s_m))\,ds_m$$
$$\leqslant\frac{l_{m-1}}{2}+\frac{\int\limits_0^1\left(G(1,s_m)+\frac{\beta_m}{(n-1)!(\alpha_m-\eta\beta_m)}G_1(\eta,s_m)\right)l_{m-1}p_m(s_m)ds_m}{2\int\limits_0^1\left[G(1,s_m)+\frac{\beta_m}{(n-1)!(\alpha_m-\eta\beta_m)}G_1(\eta,s_m)\right]p_m(s_m)ds_m}\leqslant l_{m-1}.$$

Continuing as above, we get:

$$\frac{\mu_2 s_1^{n-1}}{(n-1)!(\alpha_2-\eta\beta_2)}+\lambda_2\int\limits_0^1\left(G(s_1,s_2)+\frac{\beta_2 s_1^{n-1}}{(n-1)!(\alpha_2-\eta\beta_2)}G_1(\eta,s_2)\right)p_2(s_2)$$
$$f_2\left(\frac{\mu_3 s_2^{n-1}}{(n-1)!(\alpha_3-\eta\beta_3)}+\lambda_3\int\limits_0^1\left(G(s_2,s_3)+\frac{\beta_3 s_2^{n-1}}{(n-1)!(\alpha_3-\eta\beta_3)}G_1(\eta,s_3)\right)\right.$$

$$p_3(s_3)\cdots f_m(y_1(s_m))ds_m\cdots ds_3\Bigg) ds_2 \leqslant \bar{H}_3.$$

Then,

$$Ty_1(t) = \frac{\mu_1 t^{n-1}}{(n-1)!(\alpha_1-\eta\beta_1)} + \lambda_1\int_0^1 \Bigg(G(t,s_1) + \frac{\beta_1 t^{n-1}}{(n-1)!(\alpha_1-\eta\beta_1)}G_1(\eta,s_1)\Bigg)$$

$$p_1(s_1)f_1\Bigg(\frac{\mu_2 s_1^{n-1}}{(n-1)!(\alpha_2-\eta\beta_2)} + \lambda_2\int_0^1 \Bigg(G(s_1,s_2) + \frac{\beta_2 s_1^{n-1}}{(n-1)!(\alpha_2-\eta\beta_2)}$$

$$G_1(\eta,s_2)\Bigg)p_2(s_2)\cdots f_m(y_1(s_m))ds_m\cdots ds_2\Bigg) ds_1$$

$$\geqslant \frac{(f_{10}-\epsilon)\|y_1\|}{4^{2n-2}}\lambda_1\int_{s\in I}\Bigg(G(1,s_1) + \frac{\beta_1}{(n-1)!(\alpha_1-\eta\beta_1)}G_1(\eta,s_1)\Bigg)p_1(s_1)\,ds_1$$

$$\geqslant \|y_1\|.$$

So, $\|Ty_1\| \geqslant \|y_1\|$. We let

$$\Omega_1 = \Big\{x\in B : \|x\| < l_m\Big\},$$

then

$$\|Ty_1\| \geqslant \|y_1\| \quad \text{for} \quad y_1 \in P\cap\partial\Omega_1. \tag{3.10}$$

Since each $f_{i\infty}$ is assumed to be a positive real number, it follows that f_i, $1\leqslant i\leqslant m$, is unbounded at ∞. For each $1\leqslant i\leqslant m$, let

$$f_i^*(x) = \sup_{0\leqslant s\leqslant x} f_i(s).$$

Then, it is straightforward that, for each $1\leqslant i\leqslant m$, $f_i^*(x)$ is a non-decreasing real-valued function, $f_i\leqslant f_i^*$ and

$$\lim_{x\to\infty}\frac{f_i^*(x)}{x} = f_{i\infty}.$$

By the definition of $f_{i\infty}$, $1\leqslant i\leqslant m$, there exists $\bar{H}_4$ such that, for each $1\leqslant i\leqslant m$,

$$f_i^*(x) \leqslant (f_{i\infty}+\epsilon)x, \quad x\geqslant \bar{H}_4.$$

This implies that there exists $H_4 > \max\Big\{2\bar{H}_3, \bar{H}_4\Big\}$ such that, for each $1\leqslant i\leqslant m$,

$$f_i^*(x) \leqslant f_i^*(H_4), \quad 0 < x \leqslant H_4.$$

Let μ_i, $1\leqslant i\leqslant m$, satisfy

$$0 < \mu_i < \frac{(n-1)!(\alpha_i-\eta\beta_i)H_4}{2}.$$

We choose $y_1\in P$ with $\|y_1\| = H_4$. Then, using the above bootstrapping argument, we obtain

$$Ty_1(t) = \frac{\mu_1 t^{n-1}}{(n-1)!(\alpha_1-\eta\beta_1)} + \lambda_1\int_0^1 \Bigg(G(t,s_1) + \frac{\beta_1 t^{n-1}}{(n-1)!(\alpha_1-\eta\beta_1)}G_1(\eta,s_1)\Bigg)$$

$$p_1(s_1)f_1\Bigg(\frac{\mu_2 s_1^{n-1}}{(n-1)!(\alpha_2-\eta\beta_2)} + \lambda_2\int_0^1 \Bigg(G(s_1,s_2) + \frac{\beta_2 s_1^{n-1}}{(n-1)!(\alpha_2-\eta\beta_2)}$$

$$G_1(\eta, s_2)\Big)p_2(s_2)\cdots f_m(y_1(s_m))ds_m\cdots ds_2\Bigg)\,ds_1$$

$$\leqslant\frac{\mu_1}{(n-1)!(\alpha_1-\eta\beta_1)}+\lambda_1\int_0^1\left(G(t,s_1)+\frac{\beta_1}{(n-1)!(\alpha_1-\eta\beta_1)}G_1(\eta,s_1)\right)$$

$$p_1(s_1)f_1^*\Bigg(\frac{\mu_2 s_1^{n-1}}{(n-1)!(\alpha_2-\eta\beta_2)}+\lambda_2\int_0^1\Bigg(G(s_1,s_2)+\frac{\beta_2 s_1^{n-1}}{(n-1)!(\alpha_2-\eta\beta_2)}$$

$$G_1(\eta, s_2)\Big)p_2(s_2)\cdots f_m(y_1(s_m))ds_m\cdots ds_2\Bigg)\,ds_1$$

$$\leqslant\frac{H_4}{2}+\lambda_1\int_0^1\left(G(1,s_1)+\frac{\beta_1}{(n-1)!(\alpha_1-\eta\beta_1)}G_1(\eta,s_1)\right)p_1(s_1)f_1^*(H_4)\,ds_1$$

$$\leqslant\frac{H_4}{2}+\lambda_1(f_{1\infty}+\epsilon)H_4\int_0^1\left(G(1,s_1)+\frac{\beta_1}{(n-1)!(\alpha_1-\eta\beta_1)}G_1(\eta,s_1)\right)p_1(s_1)\,ds_1$$

$$\leqslant\frac{H_4}{2}+\frac{H_4}{2}=H_4.$$

Hence, $\|Ty_1\|\leqslant\|y_1\|$. So, if we set

$$\Omega_2=\big\{x\in B:\|x\|<H_4\big\},$$

then

$$\|Ty_1\|\leqslant\|y_1\|,\quad \text{for}\quad y_1\in P\cap\partial\Omega_2. \tag{3.11}$$

Applying Theorem 2.1 to (3.10) and (3.11), we obtain that T has a fixed point $y_1\in P\cap(\bar\Omega_2\backslash\Omega_1)$. In view of the identity $y_{m+1}=y_1$ this yields that the m-tuple $(y_1,y_2,\cdots,y_m)$ satisfies boundary value problem (1.1)-(1.2) for the values λ_i, $1\leqslant i\leqslant m$. The proof is complete. □

4. Examples

Here we consider two examples demonstrating our results.

Example 1. Consider the iterative system of third order three-point non-homogeneous boundary value problem

$$\begin{aligned}
&y_1'''+\lambda_1 f_1(y_2(t))=0,\quad t\in[0,1],\\
&y_2'''+\lambda_2 f_2(y_3(t))=0,\quad t\in[0,1],\\
&y_3'''+\lambda_3 f_3(y_1(t))=0,\quad t\in[0,1],
\end{aligned}\tag{4.1}$$

$$\begin{aligned}
&y_1(0)=0,\quad y_1'(0)=0,\quad 2y_1'(1)-3y_1'\left(\frac12\right)=\mu_1,\\
&y_2(0)=0,\quad y_2'(0)=0,\quad 3y_2'(1)-4y_2'\left(\frac12\right)=\mu_2,\\
&y_3(0)=0,\quad y_3'(0)=0,\quad 3y_3'(1)-2y_3'\left(\frac12\right)=\mu_3,
\end{aligned}\tag{4.2}$$

where

$$f_1(y_2)=y_2(476.5-468.7e^{-y_2})(210-202.7e^{-3y_2}),$$

$$f_2(y_3) = y_3(872.5 - 867.2e^{-5y_3})(162 - 149.5e^{-2y_3}),$$
$$f_3(y_1) = y_1(374.6 - 366.5e^{-3y_1})(250 - 238.5e^{-2y_1}).$$

and

$$p_1(t) = p_2(t) = p_3(t) = 1.$$

The kernels $G(t,s)$ and $G_1(t,s)$ are

$$G(t,s) = \frac{1}{2}\begin{cases} t^2(1-s) - (t-s)^2, & 0 \leqslant s \leqslant t \leqslant 1, \\ t^2(1-s), & 0 \leqslant t \leqslant s \leqslant 1, \end{cases}$$

and

$$G_1\big(\frac{1}{2}, s\big) = \begin{cases} \frac{s}{2}, & 0 \leqslant s \leqslant \frac{1}{2} \leqslant 1, \\ \frac{(1-s)}{2}, & 0 \leqslant \frac{1}{2} \leqslant s \leqslant 1. \end{cases}$$

By direct calculation, we find that

$$\begin{aligned} f_{10} =& 56.94, \qquad f_{20} = 66.25, \qquad f_{30} = 93.15, \\ f_{1\infty} =& 100065, \qquad f_{2\infty} = 141345, \qquad f_{3\infty} = 93650, \\ F_1 =& \max\Bigg\{\Bigg(\frac{1}{256}\int_{0.25}^{0.75}\Big(G(1,s) + 3G_1\big(\frac{1}{2}, s\big)\Big)\, ds(100065)\Bigg)^{-1}, \\ & \Bigg(\frac{1}{256}\int_{0.25}^{0.75}\Big(G(1,s) + 2G_1\big(\frac{1}{2}, s\big)\Big)\, ds(141345)\Bigg)^{-1}, \\ & \Bigg(\frac{1}{256}\int_{0.25}^{0.75}\Big(G(1,s) + (0.5)G_1\big(\frac{1}{2}, s\big)\Big)\, ds(93650)\Bigg)^{-1}\Bigg\} \\ =& \max\Big\{0.0031692, 0.0032499, 0.0149956\Big\} \\ =& 0.0149956 \\ F_2 =& \min\Big\{0.019158895, 0.022641543, 0.0368069256\Big\} = 0.019158895. \end{aligned}$$

Applying Theorem 3.1, we get an optimal eigenvalue interval $0.0149956 < \lambda_i < 0.019158895$, $i = 1,2,3$ for which the boundary value problem (4.1)–(4.2) has at least one positive solution by choosing μ_1, μ_2 and μ_3 are sufficiently small.

Example 2. Here we consider the iterative system of third order three-point non-homogeneous boundary value problem

$$\begin{aligned} y_1''' + \lambda_1 f_1(y_2(t)) = 0, \quad t \in [0,1], \\ y_2''' + \lambda_2 f_2(y_3(t)) = 0, \quad t \in [0,1], \\ y_3''' + \lambda_3 f_3(y_1(t)) = 0, \quad t \in [0,1], \end{aligned} \tag{4.3}$$

$$\begin{aligned} &y_1(0)=0, \quad y_1'(0)=0, \quad 2y_1'(1)-3y_1'\left(\frac{1}{2}\right)=\mu_1,\\ &y_2(0)=0, \quad y_2'(0)=0, \quad 3y_2'(1)-4y_2'\left(\frac{1}{2}\right)=\mu_2,\\ &y_3(0)=0, \quad y_3'(0)=0, \quad 3y_3'(1)-2y_3'\left(\frac{1}{2}\right)=\mu_3, \end{aligned} \tag{4.4}$$

where

$$\begin{aligned} f_1(y_2)&=y_2(11+1001e^{-2y_2})(11+1101e^{-4y_2}),\\ f_2(y_3)&=y_3(21+1011e^{-3y_1})(21+1111e^{-5y_3}),\\ f_3(y_1)&=y_1(31+1021e^{-4y_1})(31+1121e^{-6y_1}), \end{aligned}$$

and

$$p_1(t)=p_2(t)=p_3(t)=1.$$

By direct calculation, we find that

$$\begin{aligned} &f_{10}=1125344, \quad f_{20}=1168224, \quad f_{30}=1211904,\\ &f_{1\infty}=121, \quad f_{2\infty}=441, \quad f_{3\infty}=961,\\ &F_3=\max\Big\{0.00028178, 0.0003932, 0.00115891\Big\}=0.00115891,\\ &F_4=\min\Big\{0.009015, 0.0034013, 0.0035676\Big\}=0.0034013. \end{aligned}$$

Applying Theorem 3.1, we get an optimal eigenvalue interval $0.00115891 < \lambda_i < 0.0034013$, $i = 1, 2, 3$ for which boundary value problem (4.3)-(4.4) has at least one positive solution once μ_1, μ_2 and μ_3 are sufficiently small.

Acknowledgements

The authors thank the referees for their valuable suggestions and comments.

REFERENCES

1. K. Deimling. *Nonlinear functional analysis.* Springer-Verlag, New York (1985).
2. J.R. Graef, J. Henderson, B. Yang. *Positive solutions to a fourth order three-point boundary value problem* // Discr. Contin. Dyn. Syst. **2009**(Special), 269–275 (2009).
3. J.R. Graef, B. Yang. *Multiple positive solutions to a three-point third order boundary value problem* // Discr. Contin. Dyn. Syst. **2005**(Special), 337–344 (2005).
4. D. Guo, V. Lakshmikantham. *Nonlinear problems in abstract cones.* Academic Press, Orlando (1988).
5. J. Henderson, S.K. Ntouyas. *Positive solutions for systems of n^{th} order three-point nonlocal boundary value problems* // Electron. J. Qual. Theory Differ. Equ. **2007**, 1–12 (2007).
6. J. Henderson, S.K. Ntouyas, I.K. Purnaras. *Positive solutions for systems of generalized three-point nonlinear boundary value problems* // Comment. Math. Univ. Carolin. **49**:1, 79–91 (2008).
7. J. Henderson, S.K. Ntouyas, I.K. Purnaras. *Positive solutions for systems of second order four-point nonlinear boundary value problems* // Comm. Appl. Anal. **12**:1, 29–40 (2008).
8. M.A. Krasnosel'skii. *Positive solutions of operator equations.* P. Noordhoff Ltd, Groningen (1964).
9. A.G. Lakoud, L. Zenkoufi. *Existence of positive solutions for a fourth order three-point boundary value problem* // J. Appl. Math. Comput. **50**, 139–155 (2016).
10. X. Lin, Z. Zhao. *Iterative technique for a third order differential equation with three-point nonlinear boundary value conditions* // Electron. J. Qual. Theory Differ. Equ. **2016**, 1–10 (2016).

11. Z. Liu, H. Chen, C. Liu. *Positive solutions for singular third order non-homogeneous boundary value problems* // J. Appl. Math. Comput. **38**, 161–172 (2012).
12. D. Liu, Z. Ouyang. *Solvability of third order three-point boundary value problems* // Abstr. Appl. Anal. **2014**, id 793639 (2014).
13. A.P. Palamides, G. Smyrlis. *Positive solutions to a singular third order three-point boundary value problem with an indefinitely signed Green's function* // Nonlinear Anal. TMA. **68**:7, 2104–2118 (2008).
14. K.R. Prasad, N. Sreedhar, K.R. Kumar. *Solvability of iterative systems of three-point boundary value problems* // TWMS J. APP. Eng. Math. **3**:2, 147–159 (2013).
15. Y. Sun. *Positive solutions for third order three-point non-homogeneous boundary value problems* // Appl. Math. Lett. **22**, 45–51 (2009).
16. Y. Sun, C. Zhu. *Existence of positive solutions for singular fourth order three-point boundary value problems* // Adv. Difference Equ. **2013**:51, 1–13 (2013).
17. C.X. Wang, H.R. Sun. *Positive solutions for a class of singular third order three-point non-homogeneous boundary value problem* // Dynam. Syst. Appl. **19**, 225–234 (2010).
18. L. Zhao, W. Wang, C. Zhai. *Existence and uniqueness of monotone positive solutions for a third order three-point boundary value problem* // Differ. Equ. Appl. **10**:3, 251–260 (2018).

Kapula Rajendra Prasad
Department of Applied Mathematics,
College of Science and Technology,
Andhra University,
Visakhapatnam, 530 003, India
E-mail: rajendra92@rediffmail.com

Mahanty Rashmita
Department of Applied Mathematics,
College of Science and Technology,
Andhra University,
Visakhapatnam, 530 003, India
E-mail: rashmita.mahanty@gmail.com

Sreedhar Namburi,
Department of Mathematics,
Institute of Science,
GITAM Deemed to be University),
Visakhapatnam, 530 045, India
E-mail: sreedharnamburi13@gmail.com