

# INTEGRATION OF EQUATIONS OF KAUP SYSTEM KIND WITH SELF-CONSISTENT SOURCE IN CLASS OF PERIODIC FUNCTIONS

A.B. YAKHSHIMURATOV, B.A. BABAJANOV

**Abstract.** In this paper, we consider the equations of Kaup system kind with a self-consistent source in the class of periodic functions. We discuss the complete integrability of the considered nonlinear system of equations, which is based on the transformation to the spectral data of an associated quadratic pencil of Sturm-Liouville equations with periodic coefficients. In particular, Dubrovin-type equations are derived for the time-evolution of the spectral data corresponding to the solutions of equations of Kaup system kind with self-consistent source in the class of periodic functions. Moreover, it is shown that spectrum of the quadratic pencil of Sturm-Liouville equations with periodic coefficients associated with considering nonlinear system does not depend on time. In a one-gap case, we write the explicit formulae for solutions of the problem under consideration expressed in terms of the Jacobi elliptic functions. We show that if  $p_0(x)$  and  $q_0(x)$  are real analytical functions, the lengths of the gaps corresponding to these coefficients decrease exponentially. The gaps corresponding to the coefficients  $p(x, t)$  and  $q(x, t)$  are same. This implies that the solutions of considered problem  $p(x, t)$  and  $q(x, t)$  are real analytical functions in  $x$ .

**Keywords:** equations of Kaup system kind, quadratic pencil of Sturm-Liouville equations, inverse spectral problem, trace formulas, periodical potential.

**Mathematics Subject Classification:** 39A23, 35Q51, 34K13, 34K29

## 1. INTRODUCTION

In [1], D.J. Kaup proved that the nonlinear system of equations

$$\begin{cases} \eta_\tau = \Phi_{xx} + \beta^2 \Phi_{xxxx} - \varepsilon \cdot (\Phi_x \eta)_x \\ \eta = \Phi_\tau + \frac{1}{2} \varepsilon \cdot \Phi_x^2, \end{cases}$$

is completely integrable. The system describes the waves propagation in a shallow water. In [2], complex finite-gap multiphase solutions expressed in terms of the Riemann theta functions were considered. Multi-soliton solutions were found and the asymptotic behavior of these solutions was studied. In [3], [4] and [5], a real finite-gap regular solutions of Kaup system were studied. In [6], the 'Inverse Scattering Transform' was employed to solve a class of nonlinear equations associated with the inverse problem for the one-dimensional Schrodinger equation with the energy-dependent potential.

It is not difficult (see [2]) to confirm that after the transformations

$$\eta = \frac{4\beta^2}{\varepsilon}(q + p^2) + \frac{1}{\varepsilon},$$

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A.B. YAKHSHIMURATOV, B.A. BABAJANOV, INTEGRATION OF EQUATIONS OF KAUP SYSTEM KIND WITH SELF-CONSISTENT SOURCE IN CLASS OF PERIODIC FUNCTIONS.

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Поступила 25 февраля 2019 г.

$$\begin{aligned}\Phi_\tau &= \frac{4\beta^2}{\varepsilon}(q + 3p^2) + \frac{1}{\varepsilon}, \\ \Phi_x &= -\frac{4\beta}{i\varepsilon}p, \quad t = i\beta\tau,\end{aligned}$$

the system of Kaup equations casts into a simpler form

$$\begin{cases} p_t = -6pp_x - q_x \\ q_t = p_{xxx} - 4qp_x - 2pq_x. \end{cases}$$

We shall also call it the Kaup system.

The Kaup system can be considered as a compatibility condition (see [2])

$$y_{xxt} - y_{txx} \equiv [(q_t - p_{xxx} + 4qp_x + 2pq_x) + 2\lambda(p_t + 6pp_x + q_x)]y = 0$$

for the system of the linear equations

$$\begin{cases} -y_{xx} + qy + 2\lambda py - \lambda^2 y = 0 \\ y_t + 2\lambda y_x + 2py_x - p_x y = 0. \end{cases}$$

The first of these equations is called the quadratic pencil of Sturm-Liouville equations.

In [7], the Kaup system with self-consistent sources was studied by means of the inverse problem for the quadratic pencil of Sturm-Liouville equations with periodic potential. The inverse problem for the quadratic pencil of Sturm-Liouville equations with periodical potential on a half-line and the entire line was solved in the works [8]-[13].

Recently, nonlinear evolution equations with self-consistent sources have received much attention in the scientific literature. Physically, the sources appear in solitary waves with non-constant velocity and lead to a variety of dynamics in physical models. They have important applications in plasma physics, hydrodynamics, solid-state physics, etc. [14]-[19]. For example, the KdV equation, which is included an integral type self-consistent source, was considered in [17]. This type equation can be employed to describe the interaction of long and short capillary-gravity waves [18]. Other important soliton equations with self-consistent source are the nonlinear Schrodinger equation, which describes the nonlinear interaction of an ion acoustic wave in the two component homogeneous plasma with the electrostatic high frequency wave [19]. Other aspects on integration of nonlinear systems were presented in [20]-[25].

In this paper, the method of the inverse spectral problem for the quadratic pencil of Sturm-Liouville equations with periodic coefficients is used to integrate the equation of Kaup system kind with a self-consistent source in the class of periodic functions. In the one-gap case, we write the explicit formulae for solutions of the problem under consideration, expressed in terms of the Jacobi elliptic functions.

The paper is organized as follows. In section 2, we present the formulation of the considered problem and we provide some basic information about the spectral theory of the quadratic pencil of Sturm-Liouville equations with periodic coefficients. Section 3 is devoted to describing the evolution of the spectral data corresponding to the problem in question. In section 4, we illustrate the application of the main result for the one-gap case.

## 2. FORMULATION OF PROBLEM

We consider the system of equations with a self-consistent source

$$\begin{cases} p_t = p_{xxx} - 6p_xq - 6pq_x - 30p^2p_x + \sum_{k=-\infty}^{\infty} \alpha_k(t)s(\pi, \lambda_k, t)(\psi_+^2(x, \lambda_k, t))_x, \\ q_t = q_{xxx} + 6pp_{xxx} + 18p_xp_{xx} - 6qq_x - 24pqp_x - 6p^2q_x \\ \quad + 2 \sum_{k=-\infty}^{\infty} \alpha_k(t)s(\pi, \lambda_k, t) (-p_x \cdot \psi_+^2(x, \lambda_k, t) + (\lambda_k - 2p)(\psi_+^2(x, \lambda_k, t))_x), \end{cases} \quad (2.1)$$

in the class of real-valued  $\pi$ -periodic in the spatial variable  $x$  functions  $p = p(x, t)$  and  $q = q(x, t)$  possessing the following regularity

$$p(x, t), q(x, t) \in C_x^3(t \geq 0) \cap C_t^1(t > 0) \cap C(t \geq 0)$$

and satisfying the initial conditions

$$p(x, t)|_{t=0} = p_0(x), \quad q(x, t)|_{t=0} = q_0(x). \quad (2.2)$$

Here  $p_0(x)$  and  $q_0(x)$  are the given real-valued  $\pi$ -periodic functions such that for each nontrivial function  $y(x) \in W_2^2[0, \pi]$  satisfying the identities

$$y'(0)\bar{y}(0) - y'(\pi)\bar{y}(\pi) = 0, \quad (y, y) = 1,$$

the following inequality holds:

$$(p_0y, y)^2 + (q_0y, y) + (y', y') > 0,$$

where  $(\cdot, \cdot)$  is a scalar product of the space  $L_2(0, \pi)$ . We observe that for  $p(x) = 0$  the equation (2.1) reduces to the Korteweg-de Vries equation with a self-consistent source. In system (2.1),  $\alpha_k(t)$ ,  $k \in \mathbb{Z}$  is a given sequences of real-valued continuous functions having a uniform asymptotic behavior

$$\alpha_k = O\left(\frac{1}{k^3}\right), \quad k \rightarrow \pm\infty,$$

and  $\psi_+(x, \lambda_k, t)$  is the Floquet solution normalized by the condition  $\psi_+(0, \lambda_k, t) = 1$  of the quadratic pencil of Sturm-Liouville equations

$$T(\lambda, t)y \equiv -y'' + qy + 2\lambda py - \lambda^2y = 0, \quad x \in \mathbb{R}. \quad (2.3)$$

Here  $\lambda_k$  are the zeroes of the function  $\Delta^2(\lambda) - 4$ , where  $\Delta(\lambda) = c(\pi, \lambda, t) + s'(\pi, \lambda, t)$ . We denote by  $c(x, \lambda, t)$  and  $s(x, \lambda, t)$  the solutions of equation (2.3) satisfying the initial conditions

$$c(0, \lambda, t) = 1, \quad c'(0, \lambda, t) = 0, \quad s(0, \lambda, t) = 0, \quad s'(0, \lambda, t) = 1,$$

respectively.

The spectrum of the quadratic pencil (2.3) is real and coincides with the set [8-9]

$$\sigma(T) = \{\lambda \in \mathbb{R} \mid -2 \leq \Delta(\lambda) \leq 2\} = \mathbb{R} \setminus \bigcup_{n=-\infty}^{\infty} (\lambda_{2n-1}, \lambda_{2n}).$$

The intervals  $(\lambda_{2n-1}, \lambda_{2n})$ ,  $n \in \mathbb{Z}$ , are called the gaps or lacunas. The numbering is introduced in such a way that  $\lambda_{-1} < 0 < \lambda_0$ .

We denote by  $\xi_n(t)$ ,  $n \in \mathbb{Z} \setminus \{0\}$  the eigenvalues of the Dirichlet problem ( $y(0) = y(\pi) = 0$ ) for equation (2.3). The inclusions  $\xi_n(t) \in [\lambda_{2n-1}, \lambda_{2n}]$  and the identity

$$s(\pi, \lambda, t) = \pi \prod_{0 \neq k=-\infty}^{\infty} \frac{\xi_k(t) - \lambda}{k} \quad (2.4)$$

are satisfied.

The numbers  $\xi_n = \xi_n(t)$  with the signs

$$\sigma_n = \sigma_n(t) = \text{sign} \{s'(\pi, \xi_n) - c(\pi, \xi_n)\}, \quad n \in \mathbb{Z} \setminus \{0\},$$

are called the spectral parameters of quadratic pencil (2.3).

The boundaries  $\lambda_n$  of the spectrum and the spectral parameters  $\xi_n, \sigma_n$  are called the spectral data of problem (2.3).

The aim of this work is to develop a procedure for constructing the solution  $(p(x, t), q(x, t), \psi_+(x, \lambda_k, t))$  of problem (2.1)-(2.3) by means of the inverse spectral problem for the quadratic pencil of Sturm-Liouville equations (2.3).

### 3. MAIN RESULT

The main result of the paper is presented in the following theorem.

**Theorem 3.1.** *Let  $p(x, t), q(x, t)$  and  $\psi_+(x, \lambda_k, t)$  be solution of problem (2.1)-(2.3). Then the spectrum of problem (2.3) is independent of  $t$ , and the spectral parameters  $\xi_n(t), n \in \mathbb{Z} \setminus \{0\}$  satisfy the following analogue of the system of Dubrovin equations*

$$\begin{aligned} \dot{\xi}_n(t) = & 2(-1)^n \sigma_n(t) \text{sign}(n) \cdot \sqrt{(\xi_n(t) - \lambda_{2n-1})(\lambda_{2n} - \xi_n(t))} \\ & \cdot \sqrt{(\xi_n(t) - \lambda_{-1})(\xi_n(t) - \lambda_0) \prod_{k \neq n, 0} \frac{(\xi_n(t) - \lambda_{2k-1})(\xi_n(t) - \lambda_{2k})}{(\xi_n(t) - \xi_k(t))^2}} \\ & \cdot \left( \left( 4\xi_n(t) + 2(\lambda_{-1} + \lambda_0) + 4 \sum_{0 \neq k = -\infty}^{\infty} \left( \frac{\lambda_{2k-1} + \lambda_{2k}}{2} - \xi_k(t) \right) \right) \xi_n(t) \right. \\ & + 2 \left( \frac{\lambda_{-1} + \lambda_0}{2} + \sum_{0 \neq k = -\infty}^{\infty} \left( \frac{\lambda_{2k-1} + \lambda_{2k}}{2} - \xi_k(t) \right) \right)^2 + \lambda_{-1}^2 + \lambda_0^2 \\ & \left. + \sum_{0 \neq k = -\infty}^{\infty} (\lambda_{2k-1}^2 + \lambda_{2k}^2 - 2\xi_k^2(t)) + \sum_{k = -\infty}^{\infty} \frac{\alpha_k(t) s(\pi, \lambda_k, t, \tau)}{\xi_n(t) - \lambda_k} \right). \end{aligned} \tag{3.1}$$

The sign  $\sigma_n(t) = \pm 1$  changes at each collision of the point  $\xi_n(t)$  with the boundaries of its gap  $[\lambda_{2n-1}, \lambda_{2n}]$ . Moreover, the following initial conditions are satisfied

$$\xi_n(t)|_{t=0} = \xi_n^0, \quad \sigma_n(t)|_{t=0} = \sigma_n^0, \quad n \in \mathbb{Z} \setminus \{0\}, \tag{3.2}$$

where  $\xi_n^0, \sigma_n^0, n \in \mathbb{Z} \setminus \{0\}$  are the spectral parameters of the quadratic pencil of Sturm-Liouville equations corresponding to the coefficients  $p_0(x)$  and  $q_0(x)$ .

*Proof.* Let  $y_n(x, t)$  be the normalized eigenfunction of the Dirichlet problem for equation (2.3) corresponding to the eigenvalue  $\xi_n = \xi_n(t)$ . It is easy to see that

$$y_n(x, t) = \frac{1}{c_n(t)} s(x, \xi_n(t), t), \tag{3.3}$$

where

$$c_n^2(t) = \int_0^\pi s^2(x, \xi_n(t), t) dx.$$

Differentiating the identity

$$-(y_n'', y_n) + (qy_n, y_n) + 2\xi_n(py_n, y_n) - \xi_n^2 = 0,$$

with respect to  $t$ , we get

$$-(\dot{y}_n'', y_n) - (y_n'', \dot{y}_n) + (q_t y_n + q \dot{y}_n, y_n) + (qy_n, \dot{y}_n) + 2\dot{\xi}_n(py_n, y_n)$$

$$+ 2\xi_n(p_t y_n + p \dot{y}_n, y_n) + 2\xi_n(p y_n, \dot{y}_n) - 2\xi_n \dot{\xi}_n = 0.$$

By the last identity we obtain

$$\begin{aligned} & (-\dot{y}_n'' + q \dot{y}_n + 2\xi_n p \dot{y}_n, y_n) + (-y_n'' + q y_n + 2\xi_n p y_n, \dot{y}_n) \\ & + (q_t y_n + 2\xi_n p_t y_n, y_n) + 2\dot{\xi}_n(p y_n, y_n) - 2\xi_n \dot{\xi}_n = 0, \\ & 2\dot{\xi}_n[\xi_n - (p y_n, y_n)] = (q_t y_n + 2\xi_n p_t y_n, y_n), \end{aligned}$$

and hence,

$$2\dot{\xi}_n \left( \xi_n - \int_0^\pi p y_n^2 dx \right) = \int_0^\pi (q_t + 2\xi_n p_t) y_n^2 dx. \quad (3.4)$$

According to (2.1) we have

$$\begin{aligned} q_t + 2\xi_n p_t = & q_{xxx} + 6pp_{xxx} + 18p_x p_{xx} - 6qq_x - 24pqp_x - 6p^2 q_x \\ & + 2\xi_n(p_{xxx} - 6p_x q - 6pq_x - 30p^2 p_x) + G_2 + 2\xi_n G_1, \end{aligned} \quad (3.5)$$

where

$$G_1(x, t) = \sum_{k=-\infty}^{\infty} \alpha_k(t) s(\pi, \lambda_k, t) (\psi_+^2(x, \lambda_k, t))_x$$

and

$$G_2(x, t) = 2 \sum_{k=-\infty}^{\infty} \alpha_k(t) s(\pi, \lambda_k, t) \left\{ -p_x \cdot \psi_+^2(x, \lambda_k, t) + (\lambda_k - 2p)(\psi_+^2(x, \lambda_k, t))_x \right\}.$$

We introduce the polynomial

$$c(x, t, \xi_n) = c_0(x, t) \xi_n^2 + c_1(x, t) \xi_n + c_2(x, t)$$

and we rewrite identity (3.5) as

$$q_t + 2\xi_n p_t = \frac{1}{2} c''' + 2c' \cdot (\xi_n^2 - 2p\xi_n - q) - c \cdot (2p'\xi_n + q') + G_2 + 2\xi_n G_1, \quad (3.6)$$

where

$$c_0(x, t) = 4, \quad c_1(x, t) = 4p(x, t), \quad c_2(x, t) = 2[q(x, t) + 3p^2(x, t)].$$

Using the identity

$$q y_n = \xi_n^2 y_n + y_n'' - 2\xi_n p y_n$$

it is easy to show that

$$\begin{aligned} & \left( \left( \frac{1}{2} c'' + c \cdot (\xi_n^2 - 2p\xi_n - q) \right) y_n^2 - c' y_n y_n' + c (y_n^2)' \right)' \\ & = \left( \frac{1}{2} c''' + 2c' \cdot (\xi_n^2 - 2p\xi_n - q) - c \cdot (2p'\xi_n + q') \right) y_n^2. \end{aligned} \quad (3.7)$$

Substituting expressions (3.6) into formula (3.4) and taking into consideration (3.7), we arrive at the identity

$$\begin{aligned}
2\dot{\xi}_n \left( \xi_n - \int_0^\pi p y_n^2 dx \right) &= \left( \left( \frac{1}{2} c'' + c \cdot (\xi_n^2 - 2p\xi_n - q) \right) y_n^2 - c' y_n y_n' + c (y_n^2)' \right) \Big|_0^\pi \\
&\quad + \int_0^\pi (G_2 + 2\xi_n G_1) y_n^2 dx \\
&= c(0, t, \xi_n) (y_n'^2(\pi, t) - y_n'^2(0, t)) + \int_0^\pi (G_2 + 2\xi_n G_1) y_n^2 dx.
\end{aligned} \tag{3.8}$$

Now we calculate the last integral in (3.8)

$$\begin{aligned}
I &\equiv \int_0^\pi (G_2 + 2\xi_n G_1) y_n^2 dx \\
&= \sum_{k=-\infty}^{\infty} \alpha_k s(\pi, \lambda_k, t) \\
&\quad \cdot \int_0^\pi \left( -2p_x y_n^2 \cdot \psi_+^2(x, \lambda_k, t) + 2(\xi_n + \lambda_k - 2p) y_n^2 \cdot (\psi_+^2(x, \lambda_k, t))_x \right) dx.
\end{aligned} \tag{3.9}$$

It is easy to confirm that

$$\begin{aligned}
J &\equiv -2 \int_0^\pi p_x y_n^2 \psi_k^2 dx + 2 \int_0^\pi (\xi_n + \lambda_k - 2p) y_n^2 (\psi_k^2)_x dx \\
&= -2 \int_0^\pi p_x y_n^2 \psi_k^2 dx + \int_0^\pi (\xi_n + \lambda_k - 2p) y_n^2 (\psi_k^2)_x dx + \int_0^\pi (\xi_n + \lambda_k - 2p) y_n^2 d(\psi_k^2) \\
&= -2 \int_0^\pi p_x y_n^2 \psi_k^2 dx + \int_0^\pi 2(\xi_n + \lambda_k - 2p) y_n^2 \psi_k \psi_k' dx \\
&\quad - \int_0^\pi \left( -2p_x y_n^2 + 2(\xi_n + \lambda_k - 2p) y_n y_n' \right) \psi_k^2 dx \\
&= \int_0^\pi 2(\xi_n + \lambda_k - 2p) y_n \psi_k (y_n \psi_k' - y_n' \psi_k) dx,
\end{aligned} \tag{3.10}$$

where  $\psi_k = \psi_+(x, \lambda_k, t)$ . Using the identity

$$(\xi_n + \lambda_k - 2p) y_n \psi_k = \frac{(y_n \psi_k' - y_n' \psi_k)'}{\xi_n - \lambda_k},$$

by (3.10) we obtain that

$$J = \frac{1}{\xi_n - \lambda_k} (y_n'^2(\pi, t) - y_n'^2(0, t)). \tag{3.11}$$

Substituting (3.11) into (3.9), we get

$$I = \sum_{k=-\infty}^{\infty} \frac{\alpha_k(t)s(\pi, \lambda_k, t)}{\xi_n - \lambda_k} \cdot (y_n'^2(\pi, t) - y_n'^2(0, t)). \quad (3.12)$$

Hence, by means of expression (3.12), we conclude that

$$\begin{aligned} 2\dot{\xi}_n \left( \xi_n - \int_0^{\pi} p y_n^2 dx \right) &= (y_n'^2(\pi, t) - y_n'^2(0, t)) \sum_{k=0}^2 c_k(0, t) \xi_n^{2-k} \\ &+ (y_n'^2(\pi, t) - y_n'^2(0, t)) \sum_{k=-\infty}^{\infty} \frac{\alpha_k(t)s(\pi, \lambda_k, t)}{\xi_n - \lambda_k}. \end{aligned} \quad (3.13)$$

By virtue of (3.3), identity (3.13) can be written as

$$\begin{aligned} 2\dot{\xi}_n(t) \left( \xi_n(t)c_n^2(t) - \int_0^{\pi} p s^2(x, \xi_n(t), t) dx \right) &= (s'^2(\pi, \xi_n(t), t) - 1) \sum_{k=0}^2 c_k(0, t) \xi_n^{2-k} \\ &+ (s'^2(\pi, \xi_n(t), t) - 1) \sum_{k=-\infty}^{\infty} \frac{\alpha_k(t)s(\pi, \lambda_k, t)}{\xi_n - \lambda_k}. \end{aligned}$$

It follows from the identity

$$2\xi_n(t)c_n^2(t) - 2 \int_0^{\pi} p(x, t)s^2(x, \xi_n(t), t) dx = s'(\pi, \xi_n(t), t) \frac{\partial s(\pi, \xi_n(t), t)}{\partial \lambda}$$

that

$$\begin{aligned} \dot{\xi}_n(t) \frac{\partial s(\pi, \xi_n(t), t)}{\partial \lambda} &= \left( s'(\pi, \xi_n(t), t) - \frac{1}{s'(\pi, \xi_n(t), t)} \right) \sum_{k=0}^2 c_k(0, t) \xi_n^{2-k} \\ &+ \left( s'(\pi, \xi_n(t), t) - \frac{1}{s'(\pi, \xi_n(t), t)} \right) \sum_{k=-\infty}^{\infty} \frac{\alpha_k(t)s(\pi, \lambda_k, t)}{\xi_n - \lambda_k}. \end{aligned} \quad (3.14)$$

Now, substituting the values  $x = \pi$  and  $\lambda = \xi_n(t)$  into the identity

$$c(x, \lambda, t)s'(x, \lambda, t) - c'(x, \lambda, t)s(x, \lambda, t) = 1,$$

we find that

$$c(\pi, \xi_n(t), t) = \frac{1}{s'(\pi, \xi_n(t), t)}. \quad (3.15)$$

By (3.15) and the identity

$$[c(\pi, \lambda, t) - s'(\pi, \lambda, t)]^2 = (\Delta^2(\lambda) - 4) - 4c'(\pi, \lambda, t)s(\pi, \lambda, t),$$

we arrive at the identity

$$s'(\pi, \xi_n(t), t) - \frac{1}{s'(\pi, \xi_n(t), t)} = \sigma_n(t) \sqrt{\Delta^2(\xi_n(t)) - 4}. \quad (3.16)$$

Using (2.4), (3.16) and the expansion

$$\Delta^2(\lambda) - 4 = -4\pi^2(\lambda - \lambda_{-1})(\lambda - \lambda_0) \prod_{0 \neq k=-\infty}^{\infty} \frac{(\lambda - \lambda_{2k-1})(\lambda - \lambda_{2k})}{k^2},$$

we find:

$$\begin{aligned} \frac{s'(\pi, \xi_n(t), t) - \frac{1}{s'(x, \xi_n(t), t)}}{\frac{\partial s(\pi, \xi_n(t), t)}{\partial \lambda}} &= \frac{\sigma_n(t) \sqrt{\Delta^2(\xi_n(t)) - 4}}{\frac{\partial s(\pi, \xi_n(t), t)}{\partial \lambda}} \\ &= 2(-1)^n \sigma_n(t) \text{sign}(n) \cdot \sqrt{(\xi_n(t) - \lambda_{2n-1})(\lambda_{2n} - \xi_n(t))} \\ &\quad \cdot \sqrt{(\xi_n(t) - \lambda_{-1})(\xi_n(t) - \lambda_0) \prod_{k \neq n, 0} \frac{(\xi_n(t) - \lambda_{2k-1})(\xi_n(t) - \lambda_{2k})}{(\xi_n(t) - \xi_k(t))^2}}. \end{aligned} \quad (3.17)$$

Here we have also employed the identity

$$\text{sign} \left( -\frac{\pi}{n} \prod_{k \neq n, 0} \frac{\xi_k(t) - \xi_n(t)}{k} \right) = (-1)^n \text{sign}(n).$$

From (3.14), (3.17) and the trace formulae

$$\begin{aligned} p(t) &= \frac{\lambda_{-1} + \lambda_0}{2} + \sum_{0 \neq k = -\infty}^{\infty} \left( \frac{\lambda_{2k-1} + \lambda_{2k}}{2} - \xi_k(t) \right), \\ q(t) + 2p^2(t) &= \frac{\lambda_{-1}^2 + \lambda_0^2}{2} + \sum_{0 \neq k = -\infty}^{\infty} \left( \frac{\lambda_{2k-1}^2 + \lambda_{2k}^2}{2} - \xi_k^2(t) \right) \end{aligned}$$

we get (3.1).

We note that if instead of the Dirichlet boundary conditions we consider periodic or anti-periodic boundary value conditions, then equation (3.13) becomes  $\dot{\lambda}_n(t) = 0$ ,  $n \in \mathbb{Z}$ . Hence, the spectrum of problem (2.3) is independent of the parameter  $t$ , and this completes the proof.  $\square$

**Remark 1.** *If instead of  $p(x, t)$  and  $q(x, t)$  we consider the functions  $p(x + \tau, t)$  and  $q(x + \tau, t)$ , then, as we have seen in the previous section, the eigenvalues of the periodic and antiperiodic problems are independent of the parameters  $\tau$  and  $t$ . However, the eigenvalues  $\xi_n$  of the Dirichlet problem and the signs  $\sigma_n$  do depend on  $\tau$  and  $t$ :  $\xi_n = \xi_n(\tau, t)$ ,  $\sigma_n = \sigma_n(\tau, t) = \pm 1$ .*

**Remark 2.** *The theorem gives a method for solving problem (2.1)-(2.3). First we find the spectral data  $\lambda_n$ ,  $n \in \mathbb{Z}$ ,  $\xi_n^0(\tau)$ ,  $\sigma_n^0(\tau)$ ,  $n \in \mathbb{Z} \setminus \{0\}$  of the quadratic pencil of Sturm-Liouville equations corresponding to the coefficients  $p_0(x + \tau)$  and  $q_0(x + \tau)$ . Then we solve the Cauchy problem for Dubrovin system (3.1) with the initial conditions*

$$\xi_n(\tau, t) |_{t=0} = \xi_n^0(\tau), \quad \sigma_n(\tau, t) |_{t=0} = \sigma_n^0(\tau), \quad n \in \mathbb{Z} \setminus \{0\}.$$

Finally, by using the trace formulae

$$\begin{aligned} p(\tau, t) &= \frac{\lambda_{-1} + \lambda_0}{2} + \sum_{0 \neq k = -\infty}^{\infty} \left( \frac{\lambda_{2k-1} + \lambda_{2k}}{2} - \xi_k(\tau, t) \right), \\ q(\tau, t) + 2p^2(\tau, t) &= \frac{\lambda_{-1}^2 + \lambda_0^2}{2} + \sum_{0 \neq k = -\infty}^{\infty} \left( \frac{\lambda_{2k-1}^2 + \lambda_{2k}^2}{2} - \xi_k^2(\tau, t) \right) \end{aligned}$$

we get the expressions of  $p(\tau, t)$  and  $q(\tau, t)$ . After that the Floquet solutions  $\psi_+(x, \lambda_k, t)$  of equation (2.3) can be found easily.

**Remark 3.** *If the number of zones is finite, that is, there are two nonnegative integer numbers  $N$  and  $M$  such that  $\lambda_{2k-1} = \lambda_{2k} = \xi_k$  for all  $k > N$  and  $-M < k$ , then system (3.1) reads as*

$$\frac{\partial \xi_n}{\partial t} = 2(-1)^n \sigma_n(\tau, t) \text{sign}(n) \cdot \sqrt{(\xi_n - \lambda_{2n-1})(\lambda_{2n} - \xi_n)}$$



$$\begin{aligned}
 & \cdot \sqrt{(\xi_n - \lambda_{-1})(\xi_n - \lambda_0) \prod_{\substack{k=-M \\ k \neq n, 0}}^N \frac{(\xi_n - \lambda_{2k-1})(\xi_n - \lambda_{2k})}{(\xi_n - \xi_k)^2}} \\
 & \cdot \left( \left( 4\xi_n + 2(\lambda_{-1} + \lambda_0) + 4 \sum_{0 \neq k=-M}^N \left( \frac{\lambda_{2k-1} + \lambda_{2k}}{2} - \xi_k(\tau, t) \right) \right) \xi_n \right. \\
 & + 2 \left( \frac{\lambda_{-1} + \lambda_0}{2} + \sum_{0 \neq k=-M}^N \left( \frac{\lambda_{2k-1} + \lambda_{2k}}{2} - \xi_k(\tau, t) \right) \right)^2 + \lambda_{-1}^2 + \lambda_0^2 \\
 & \left. + \sum_{0 \neq k=-M}^N (\lambda_{2k-1}^2 + \lambda_{2k}^2 - 2\xi_k^2(\tau, t)) + \sum_{k=-M}^N \frac{\alpha_k(t)s(\pi, \lambda_k, t, \tau)}{\xi_n - \lambda_k} \right),
 \end{aligned}$$

where  $n = -M, \dots, -1, 1, \dots, N$ .

**Remark 4.** In [13], there was proved the theorem stating that the lengths of the gaps of the quadratic pencil of Sturm-Liouville equations with  $\pi$ -periodic real-valued coefficients decrease exponentially if and only if the coefficients are analytic. From this theorem we conclude that if  $p_0(x)$  and  $q_0(x)$  are real analytical functions, then the lengths of the gaps corresponding to these coefficients decrease exponentially. The gaps corresponding to the coefficients  $p(x, t)$  and  $q(x, t)$  are same. Hence, the solutions  $p(x, t)$  and  $q(x, t)$  of problem (2.1)-(2.3) are real analytical functions on  $x$ .

**Remark 5.** In [26], an analogue of Borg inverse theorem was proved: the number  $\frac{\pi}{2}$  is a period of the coefficients of the quadratic pencil of Sturm-Liouville equations with  $\pi$ -periodic real-valued coefficients if and only if all eigenvalues of antiperiodic problem are double. By this theorem we conclude that if the functions  $p_0(x)$  and  $q_0(x)$  have the period  $\frac{\pi}{2}$ , then all eigenvalues of antiperiodic problem corresponding to these coefficients are double. The gaps corresponding to the coefficients  $p(x, t)$  and  $q(x, t)$  are same. Hence, the solutions  $p(x, t)$  and  $q(x, t)$  of problem (2.1)-(2.3) are  $\frac{\pi}{2}$ -periodic functions in  $x$ .

#### 4. EXAMPLE

We now illustrate the application of Theorem to solve the problem (2.1)-(2.2).

Let us consider the following initial value conditions

$$\begin{aligned}
 p(x, t)|_{t=0} = p_0(x) &= \frac{3 - 4sn^2(3x, \frac{2}{3})}{1 + 2cn^2(3x, \frac{2}{3})}, \\
 q(x, t)|_{t=0} = q_0(x) &= \frac{81 - 156sn^2(3x, \frac{2}{3}) + 72sn^4(3x, \frac{2}{3})}{(1 + 2cn^2(3x, \frac{2}{3}))^2},
 \end{aligned}$$

for equations of Kaup system kind (2.1) with  $\alpha_k(t) = \frac{1}{\pi k^3}$ . Let us find the spectral data of the problem (2.3) for  $p_0(x)$  and  $q_0(x)$ . It has the form

$$\lambda_{-1} = -1, \quad \lambda_0 = 1, \quad \lambda_1 = 2, \quad \lambda_2 = 4, \quad \xi_1^0(0) = 2, \quad \sigma_1^0(0) = 1.$$

In this case the system (3.1) reads as

$$\frac{\partial \xi_1}{\partial t} = -226\sigma_n(\tau, t) \cdot \sqrt{(4 - \xi_1)(\xi_1 - 2)(\xi_1 - 1)(\xi_1 + 1)}. \tag{4.1}$$

We consider this system subject to the initial condition

$$\xi_1(\tau, t)|_{t=0} = \xi_1^0(\tau), \quad \sigma_1(\tau, t)|_{t=0} = \sigma_1^0(\tau), \tag{4.2}$$

where  $\xi_1^0(\tau)$  solves the differential equation

$$\frac{d\xi_1^0(\tau)}{d\tau} = 2\sigma_1^0(\tau)\sqrt{(4 - \xi_1^0(\tau))(\xi_1^0(\tau) - 2)(\xi_1^0(\tau) - 1)(\xi_1^0(\tau) + 1)}$$

and satisfies the initial condition

$$\xi_1^0(\tau)|_{\tau=0} = 2, \quad \sigma_1^0(\tau)|_{\tau=0} = 1.$$

Solving the Cauchy problem (4.1)-(4.2) we find that (see [27])

$$\xi_1(\tau, t) = \frac{6 - 2\operatorname{sn}^2(-339t + 3\tau, \frac{2}{3})}{1 + 2\operatorname{cn}^2(-339t + 3\tau, \frac{2}{3})}.$$

Substituting this into the first and second trace formulae, we obtain the solution of the given problem:

$$p(\tau, t) = \frac{3 - 4\operatorname{sn}^2(-339t + 3\tau, \frac{2}{3})}{1 + 2\operatorname{cn}^2(-339t + 3\tau, \frac{2}{3})},$$

$$q(\tau, t) = \frac{81 - 156\operatorname{sn}^2(-339t + 3\tau, \frac{2}{3}) + 72\operatorname{sn}^4(-339t + 3\tau, \frac{2}{3})}{(1 + 2\operatorname{cn}^2(-339t + 3\tau, \frac{2}{3}))^2},$$

where  $\operatorname{sn}$  and  $\operatorname{cn}$  are the Jacobi elliptic functions.

#### ACKNOWLEDGEMENTS

The authors express their gratitude to Prof. Aknazar Khasanov (Samarkand State University, Uzbekistan) for discussion and valuable advice, as well as to the International Erasmus+ Program KA106-2, Keele University, UK.

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