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REALIZATION OF HOMOGENEOUS TRIEBEL-LIZORKIN SPACES WITH $p = \infty$ AND CHARACTERIZATIONS VIA DIFFERENCES

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Abstract. In this paper, via the decomposition of Littlewood-Paley, the homogeneous Triebel-Lizorkin space $\dot{F}^s_{\infty,q}$ is defined on $\mathbb{R}^n$ by distributions modulo polynomials in the sense that $\|f\| = 0$ ($\|\cdot\|$ the quasi-seminorm in $\dot{F}^s_{\infty,q}$) if and only if f is a polynomial on $\mathbb{R}^n$. We consider this space as a set of "true" distributions and we are lead to examine the convergence of the Littlewood-Paley sequence of each element in $\dot{F}^s_{\infty,q}$. First we use the realizations and then we obtain the realized space $\dot{\tilde{F}}^s_{\infty,q}$ of $\dot{F}^s_{\infty,q}$.

Our approach is as follows. We first study the commuting translations and dilations of realizations in $\dot{F}^s_{\infty,q}$, and employing distributions vanishing at infinity in the weak sense, we construct $\dot{\tilde{F}}^s_{\infty,q}$. Then, as another possible definition of $\dot{F}^s_{\infty,q}$, in the case $s > 0$, we make use of the differences and describe $\dot{\tilde{F}}^s_{\infty,q}$ as $s > \max(n/q - n, 0)$.

Keywords: Triebel-Lizorkin spaces, Littlewood-Paley decomposition, realizations.

Mathematics Subject Classification: 46E35.

1. Introduction

In this paper we study a realization of homogeneous Triebel-Lizorkin spaces $\dot{F}^s_{\infty,q}$ on $\mathbb{R}^n$. The spaces $\dot{F}^s_{\infty,q}$ are defined by distributions modulo polynomials in the sense that $\|f\|_{\dot{F}^s_{\infty,q}} = 0$ if and only if f is a polynomial on $\mathbb{R}^n$. Some of their properties can be found in [12], [22].

The basic definition of $\dot{F}^s_{\infty,q}$ is given via the Littlewood-Paley decomposition (abbreviated as LP decomposition). To recall this, we introduce some notations.

By ρ we denote an infinitely differentiable radial function obeying the estimates $0 \leqslant \rho \leqslant 1$ such that

$$\rho(\xi) = 1 \quad \text{as} \quad |\xi| \leqslant 1, \qquad \rho(\xi) = 0 \quad \text{as} \quad |\xi| \geqslant \frac{3}{2}.$$

We denote $\gamma(\xi) := \rho(\xi) - \rho(2\xi)$. This function is supported in the annulus $\frac{1}{2} \leqslant |\xi| \leqslant \frac{3}{2}$, and

$$\gamma(\xi) = 1 \quad \text{as} \quad \frac{3}{4} \leqslant |\xi| \leqslant 1, \qquad \sum_{j\in\mathbb{Z}} \gamma(2^j\xi) = 1 \quad \text{as} \quad \xi \neq 0.$$

For $m \in \mathbb{N}$, the symbol $\mathcal{P}_m$ stands for the set of all polynomials on $\mathbb{R}^n$ of degree less than m obeying $\mathcal{P}_0 = \{0\}$. By $\mathcal{P}_\infty$ we denote the set of all polynomials. For $m \in \mathbb{N}_0 \cup \{\infty\}$, the set $\mathcal{S}'_m$ of the tempered distributions modulo polynomials is the dual space of $\mathcal{S}_m$, which is the orthogonal space of $\mathcal{P}_m$ in $\mathcal{S}$, that is, $\mathcal{S}_m$ is the set of all $f \in \mathcal{S}$ such that $\langle u, f\rangle = 0$ for all $u \in \mathcal{P}_m$. For a tempered distributions $f \in \mathcal{S}'$, the symbol $[f]_m$ denotes the equivalence class of f modulo $\mathcal{P}_m$.

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We define the operators Q_j by the formula

$$\widehat{Q_j f} := \gamma(2^{-j}(\cdot))\widehat{f}, \qquad j \in \mathbb{Z}.$$

These operators are defined on $\mathcal{S}'$ as well as on $\mathcal{S}'_m$ since $Q_j f = 0$ if and only if $f \in \mathcal{P}_m$. For instance, we have $Q_j(\mathcal{S}) \subset \mathcal{S}_\infty$. All these operators take values in the space of analytical functions of exponential type, see the Paley-Wiener theorem. Finally, we adopt the following convention: for $f \in \mathcal{S}'_m$, we define $Q_j f := Q_j f_1$ for all $f_1 \in \mathcal{S}'$ such that $[f_1]_m = f$.

We turn to the LP decomposition; for all $f \in \mathcal{S}_\infty$ (or $\mathcal{S}'_\infty$) the identity

$$f = \sum_{j \in \mathbb{Z}} Q_j f \quad \text{in} \quad \mathcal{S}_\infty \quad (\text{or } \mathcal{S}'_\infty) \tag{1}$$

holds; this is an easy application of Lemma 7 below. However, once we work in $\dot{F}^s_{\infty,q}$, it is possible to obtain the convergence of the series of the LP decomposition in $\mathcal{S}'_\mu$ for some integer μ, see (7) below. This leads us to the need to realize $\dot{F}^s_{\infty,q}$ and to obtain the realized spaces by using the notion of realization. For a quasi-Banach distribution space $E \hookrightarrow \mathcal{S}'_\infty$, we need to find a continuous linear mapping $\sigma : E \to \mathcal{S}'_m$ such that $[\sigma(f)]_m$ coincides with f modulo polynomials in $\mathcal{P}_m$ for all $f \in E$, cf. Definition 4 below. If in addition, E is a translation or a dilation invariant, that is,

$$\|\tau_a f\|_E = \|f\|_E \qquad \text{or} \qquad \|h_\lambda f\|_E = \lambda^r \|f\|_E$$

with $r \in \mathbb{R}$, where $\tau_a f(x) := f(x-a)$ and $h_\lambda f(x) := f(x/\lambda)$ for all $x, a \in \mathbb{R}^n$ and all $\lambda > 0$, the existence of a such σ commuting with translation or dilation operators, that is, obeying

$$\tau_a \circ \sigma = \sigma \circ \tau_a \qquad \text{or} \qquad h_\lambda \circ \sigma = \sigma \circ h_\lambda,$$

is nontrivial.

We note that the realizations have been introduced by G. Bourdaud [3] for the homogeneous Besov spaces $\dot{B}^s_{p,q}$; the corresponding integer μ was defined in [7]. In the same way, we know the realizations of both the homogeneous Triebel-Lizorkin spaces $\dot{F}^s_{p,q}$ with $p < \infty$ and the homogeneous Sobolev spaces $\dot{W}^m_p$, and some of their properties, see, for instance, [2], [5], [6], [7], [16], [21]. Also, nowadays there are various papers presenting applications of the realizations to Navier-Stokes equations, pseudodifferential operators, wavelet, etc., see, for instance, [9], [15], [20] and in particular, a comment in [1].

On the other hand, the distributions vanishing at infinity play an important role to characterize such realization. We recall this notion.

Definition 1. *We say that a distribution $f \in \mathcal{S}'$ vanishes at infinity if*

$$\lim_{\lambda \to 0} h_\lambda f = 0 \quad \textit{in} \quad \mathcal{S}'.$$

The set of all such distributions is denoted by $\widetilde{C}_0$.

For instance, we have $f \in \widetilde{C}_0$ if $f \in L_p$ $(1 \leqslant p < \infty)$. If either $f \in L_\infty$ or $f \in \widetilde{C}_0$ then $\partial_j f \in \widetilde{C}_0$ $(j = 1, \ldots, n)$. An easy statement is given by identity $\widetilde{C}_0 \cap \mathcal{P}_\infty = \{0\}$ (see, for instance, [3]).

As usually, $\mathbb{N}$ stands for the natural numbers $\{1, 2, \ldots\}$ and $\mathbb{N}_0 := \mathbb{N} \cup \{0\}$. All function spaces occurring in the paper are defined in the Euclidean space $\mathbb{R}^n$. By $\|\cdot\|_p$ we denote the L_p quasi-norm for $0 < p \leqslant \infty$. For $s \in \mathbb{R}$, the symbol $[s]$ denotes the integer part of s. For all $m \in \mathbb{N}_0$, the standard norms in $\mathcal{S}$ are given by

$$\zeta_m(f) := \sup_{x \in \mathbb{R}^n} \sup_{|\alpha| \leqslant m} (1 + |x|)^m \, |f^{(\alpha)}(x)|.$$

The Fourier transform for a function $f \in L_1$ is defined as

$$\mathcal{F}f(\xi) = \widehat{f}(\xi) := \int_{\mathbb{R}^n} \mathrm{e}^{-\mathrm{i}x\cdot\xi} f(x)\, dx, \qquad \xi \in \mathbb{R}^n.$$

The operator $\mathcal{F}$ can be extended to the whole $\mathcal{S}'$ in the usual way. In the same way we define the inverse Fourier transform $\mathcal{F}^{-1}$,

$$\mathcal{F}^{-1}f(x) := (2\pi)^{-n}\widehat{f}(-x).$$

For an arbitrary function f, we define the difference operators as

$$\Delta_h f = \Delta_h^1 f := \tau_{-h}f - f, \qquad \Delta_h^m f := \Delta_h(\Delta_h^{m-1}f), \quad h \in \mathbb{R}^n, \quad m = 2, 3, \ldots$$

The constants c, c_1, ... are strictly positive and depend only on the fixed parameters as n, s, q and probably on auxiliary functions, their values may vary from line to line. The notation $A \lesssim B$ means that $A \leqslant cB$. The symbol $E \hookrightarrow F$ denotes that we have the embedding $E \subseteq F$ and the natural mapping $E \to F$ is continuous. Throughout the paper, the real numbers s, q satisfy as $s \in \mathbb{R}$ and $0 < q \leqslant \infty$ unless otherwise is stated.

The paper is organized as follows. In Section 2 we recall the definitions and some properties of homogeneous Triebel-Lizorkin spaces $\dot{F}^s_{\infty,q}$ and of inhomogeneous ones $F^s_{\infty,q}$. Section 3 is devoted to the realizations of $\dot{F}^s_{\infty,q}$. In Section 4, by means of the differences, we characterize the realized spaces of $\dot{F}^s_{\infty,q}$ in the case $s > \max(n/q - n,\, 0)$.

2. Preliminaries

2.1. Homogeneous spaces $\dot{F}^s_{\infty,q}$. By $P_{k,\nu}$ ($k \in \mathbb{Z}$, $\nu \in \mathbb{Z}^n$) we denote the dyadic cube with side length 2^{-k}, left lower corner in the point $2^{-k}\nu$ and sides parallel to the coordinate axes, that is,

$$P_{k,\nu} := \big\{x \in \mathbb{R}^n : 2^{-k}\nu_j \leqslant x_j < 2^{-k}(\nu_j + 1), \quad j = 1, 2, \ldots, n\big\}.$$

The definition of $\dot{F}^s_{\infty,q}$ was given by Frazier and Jawerth [12] as follows.

Definition 2. *Let $q \in]0, \infty[$. The space $\dot{F}^s_{\infty,q}$ is the set of $f \in \mathcal{S}'_\infty$ such that*

$$\|f\|_{\dot{F}^s_{\infty,q}} := \sup_{k\in\mathbb{Z}} \sup_{\nu\in\mathbb{Z}^n} \Big(2^{kn} \int_{P_{k,\nu}} \sum_{j\geqslant k} 2^{jsq} |Q_j f(x)|^q dx\Big)^{1/q} < \infty.$$

Remark 1. *For $q = \infty$, the set $\dot{F}^s_{\infty,\infty}$ coincides with the Hölder space $\dot{B}^s_{\infty,\infty}$, see* [14, Eq. (1.3)] *and Lemma 3 below. We let*

$$\|f\|_{\dot{F}^s_{\infty,\infty}} := \sup_{j\in\mathbb{Z}} 2^{js}\|Q_j f\|_\infty < \infty.$$

The space $\dot{F}^s_{\infty,q}$ becomes a quasi-Banach with the above defined quasi-seminorm. On the one hand, its definition is independent of the choice of γ, see [12, Cor. 5.3]. On the other hand, by (1) and Lemma 7 below, we have $\mathcal{S}_\infty \hookrightarrow \dot{F}^s_{\infty,q} \hookrightarrow \mathcal{S}'_\infty$. We also have the following statements.

Lemma 1. *There exist two constants c_1, $c_2 > 0$ such that the inequalities*

$$c_1\|f\|_{\dot{F}^s_{\infty,q}} \leqslant \lambda^s \|h_\lambda f\|_{\dot{F}^s_{\infty,q}} \leqslant c_2\|f\|_{\dot{F}^s_{\infty,q}} \tag{2}$$

holds for all $f \in \dot{F}^s_{\infty,q}$ and all $\lambda > 0$.

Proof. At the first step, we prove (2) with $\lambda := 2^N$, $N \in \mathbb{Z}$. Here by using the identity

$$Q_j(h_{2^N} f) = Q_{j+N} f(2^{-N}(\cdot)),$$

we obtain easily that

$$\|h_{2^N} f\|_{\dot{F}^s_{\infty,q}} = 2^{-Ns}\|f\|_{\dot{F}^s_{\infty,q}}.$$

In the case of arbitrary $\lambda > 0$, we introduce an integer $N \in \mathbb{Z}$ such that $2^N \leqslant \lambda < 2^{N+1}$. Then we use the equivalent quasi-seminorm in $\dot{F}^s_{\infty,q}$ defined by the function $\gamma_1 := \gamma\big(2^N \lambda^{-1} \cdot\big)$ and we get

$$\|f(\lambda\,\cdot)\|_{\dot{F}^s_{\infty,q}} = 2^{Ns}\|f\big(2^{-N}\lambda\,\cdot\big)\|_{\dot{F}^s_{\infty,q}}.$$

Then it is not difficult to prove that

$$c_1\|f\|_{\dot{F}^s_{\infty,q}} \leqslant \|f\big(2^{-N}\lambda\,\cdot\big)\|_{\dot{F}^s_{\infty,q}} \leqslant c_2\|f\|_{\dot{F}^s_{\infty,q}}$$

for some positive constants c_1 and c_2 independent of N, λ and f. This completes the proof. $\square$

The next lemma was proved in [11].

Lemma 2. *There exists a constant $c > 0$ such that*

$$\sup_{x\in P_{j,\nu}} |\varphi(x)| \leqslant c2^{jn/q} \sup_{\eta\in\mathbb{Z}^n} \|\varphi\|_{L_q(P_{j,\eta})} \tag{3}$$

holds for all $j \in \mathbb{Z}$, $\nu \in \mathbb{Z}^n$, and $\varphi \in \mathcal{S}'$ with $\operatorname{supp}\widehat{\varphi} \subset \{\xi \in \mathbb{R}^n : |\xi| \leqslant 2^{j+1}\}$.

Lemma 3. *For all $q > 0$ we have $\dot{F}^s_{\infty,q} \hookrightarrow \dot{F}^s_{\infty,\infty} = \dot{B}^s_{\infty,\infty}$.*

Proof. The identity is known, see, for instance, [12] and here we provide a proof of the embedding for more clarity.

Let $f \in \dot{F}^s_{\infty,q}$. By Lemma 2 we have

$$|Q_j f(x)|^q \leqslant c_1 2^{jn} \sup_{\eta\in\mathbb{Z}^n} \int\limits_{P_{j,\eta}} |Q_j f(y)|^q\,dy \qquad \text{for all} \quad x \in P_{j,\nu},$$

which is bounded by

$$c_1 2^{-jsq}2^{jn} \sup_{\eta\in\mathbb{Z}^n} \int\limits_{P_{j,\eta}} \sum_{l\geqslant j} 2^{lsq}|Q_l f(y)|^q dy,$$

where the constant c_1 is independent of f, j and ν. This inequality implies that

$$|Q_j f(x)| \lesssim 2^{-js}\|f\|_{\dot{F}^s_{\infty,q}} \qquad (\forall x \in P_{j,\nu}).$$

Then we get

$$\|f\|_{\dot{F}^s_{\infty,\infty}} = \sup_{\eta\in\mathbb{Z}^n} \sup_{k\geqslant j} \sup_{z\in P_{j,\eta}} 2^{ks}|Q_k f(z)| \lesssim \|f\|_{\dot{F}^s_{\infty,q}}.$$

The proof is complete. $\square$

Remark 2. *An inequality opposite to* (3) *can be easily proved, and for this, the assumption* $\operatorname{supp}\widehat{\varphi} \subset \{\xi \in \mathbb{R}^n : |\xi| \leqslant 2^{j+1}\}$ *is not needed.*

Remark 3. *In case $1 < q < \infty$, the space $\dot{F}^s_{\infty,q}$ has another definition introduced by Triebel* [19]*, which is compatible with the one of Frazier and Jawerth, see a comment in* [12].

2.2. Inhomogeneous spaces $F^s_{\infty,q}$.

For each $f \in \mathcal{S}$ (or $f \in \mathcal{S}'$), we use the inhomogeneous LP decomposition $f = \mathcal{F}^{-1}\rho * f + \sum_{j>0} Q_j f$ in $\mathcal{S}$ (or $\mathcal{S}'$) and we obtain the inhomogeneous Triebel-Lizorkin spaces $F^s_{\infty,q}$ as introduced in [12].

Definition 3. *The space $F^s_{\infty,q}$ is the set of $f \in \mathcal{S}'$ such that*

$$\|f\|_{F^s_{\infty,q}} := \|\mathcal{F}^{-1}\rho * f\|_\infty + \sup_{k\in\mathbb{N}} \sup_{\nu\in\mathbb{Z}^n} \Big(2^{kn} \int\limits_{P_{k,\nu}} \sum_{j\geqslant k} 2^{jsq}|Q_j f(x)|^q dx\Big)^{1/q} < \infty.$$

Also as above,

$$\|f\|_{F^s_{\infty,\infty}} = \|f\|_{B^s_{\infty,\infty}} := \|\mathcal{F}^{-1}\rho * f\|_\infty + \sup_{j>0} 2^{js}\|Q_j f\|_\infty < \infty,$$

cf. Lemma 3 and see also [19, Sect. 2.3.4, Rem. 3].

For some properties of $F^s_{\infty,q}$, we refer to [12]. The case $s > 0$ is related with the case of the homogeneous space.

Lemma 4. *Let $s > 0$. Then*

(i) $F^s_{\infty,q} \hookrightarrow L_\infty$,

(ii) *$F^s_{\infty,q}$ is the set of $f \in L_\infty$ such that $[f]_\infty \in \dot{F}^s_{\infty,q}$. The expression $\|f\|_\infty + \|[f]_\infty\|_{\dot{F}^s_{\infty,q}}$ is an equivalent quasi-norm in $F^s_{\infty,q}$.*

Proof. Proof of (i). This embedding can be found in [22], see in particular, Statement (iii) in Propositions 2.4 and Proposition 2.6 in the cited work as well as Remark 8 below.

Proof of (ii). Let $f \in L_\infty$ be such that $[f]_\infty \in \dot{F}^s_{\infty,q}$. Thanks to the convolution inequality

$$\|\mathcal{F}^{-1}\rho * f\|_\infty \leqslant \|\mathcal{F}^{-1}\rho\|_1 \|f\|_\infty,$$

we have

$$\|f\|_{F^s_{\infty,q}} \lesssim \|f\|_\infty + \|[f]_\infty\|_{\dot{F}^s_{\infty,q}}.$$

For the opposite inequality, let $f \in F^s_{\infty,q}$. By (i), we first have $\|f\|_\infty \lesssim \|f\|_{F^s_{\infty,q}}$. Then for all $k \leqslant 0$ and all $\nu \in \mathbb{Z}^n$, we obtain

$$\begin{aligned} 2^{kn} \int\limits_{P_{k,\nu}} \sum_{j \geqslant k} 2^{jsq} |Q_j f|^q dx =& 2^{kn} \int\limits_{P_{k,\nu}} \Big(\sum_{k \leqslant j \leqslant 0} + \sum_{j \geqslant 1} \Big) 2^{jsq} |Q_j f|^q dx \\ \lesssim& \|f\|_\infty^q \sum_{j \leqslant 0} 2^{jsq} + 2^{kn} \int\limits_{P_{k,\nu}} \sum_{j \geqslant 1} 2^{jsq} |Q_j f|^q dx. \end{aligned} \tag{4}$$

On the one hand, denoting by $E(x)$ the vector $([x_1], \ldots, [x_n]) \in \mathbb{Z}^n$ for $x \in \mathbb{R}^n$, we get an elementary inequality

$$[2^{1-k}\nu_j] \leqslant 2x_j < [2^{1-k}\nu_j] + 1 + 2^{1-k}, \qquad x \in P_{k,\nu}, \qquad k \leqslant 0,\ j = 1, \ldots, n,$$

and this yields

$$x \in P_{k,\nu} \;\Rightarrow\; x \in \bigcup_{r=0}^{1+2^{1-k}} P_{1,E(2^{1-k}\nu)+rw_0}\,,$$

where $w_0 := (1, 1, \ldots, 1) \in \mathbb{Z}^n$. We then obtain

$$\begin{aligned} \int\limits_{P_{k,\nu}} \sum_{j \geqslant 1} 2^{jsq} |Q_j f|^q dx \leqslant& \sum_{r=0}^{1+2^{1-k}} \int\limits_{P_{1,E(2^{1-k}\nu)+rw_0}} \sum_{j \geqslant 1} 2^{jsq} |Q_j f|^q dx \\ \leqslant& (2 + 2^{1-k}) \sup_{\eta \in \mathbb{Z}^n} \int\limits_{P_{1,\eta}} \sum_{j \geqslant 1} 2^{jsq} |Q_j f|^q dx \\ \leqslant& (2 + 2^{1-k}) \sup_{r \in \mathbb{N}} \sup_{\eta \in \mathbb{Z}^n} 2^{rn} \int\limits_{P_{r,\eta}} \sum_{j \geqslant r} 2^{jsq} |Q_j f|^q dx \\ \leqslant& (2 + 2^{1-k}) \|f\|^q_{F^s_{\infty,q}}. \end{aligned}$$

Finally, by inserting this inequality into (4), and taking into account that $2^{kn}(2+2^{1-k})\leqslant 4$ for $k\leqslant 0$, we get

$$2^{kn}\int\limits_{P_{k,\nu}}\sum_{j\geqslant k}2^{jsq}|Q_jf|^q dx \lesssim \|f\|_\infty^q+\|f\|^q_{F^s_{\infty,q}}\lesssim \|f\|^q_{F^s_{\infty,q}}, \qquad k\leqslant 0. \tag{5}$$

On the other hand, clearly for all $k\in\mathbb{N}$,

$$2^{kn}\int\limits_{P_{k,\nu}}\sum_{j\geqslant k}2^{jsq}|Q_jf|^q dx\leqslant \sup_{r\in\mathbb{N}}2^{rn}\int\limits_{P_{r,\nu}}\sum_{j\geqslant r}2^{jsq}|Q_jf|^q dx\leqslant \|f\|^q_{F^s_{\infty,q}}.$$

Then this estimate and (5) yield the desired result. The proof is complete. □

The space $F^s_{\infty,q}$ can be described via differences. We recall the following statement.

Lemma 5. *Let $m\in\mathbb{N}$ be such that*

$$\max\,(n/q-n,0)<s<m\,. \tag{6}$$

Then

(i) *A function f belongs to $F^s_{\infty,q}$ if and only if $f\in L_\infty$ and*

$$\mathcal{N}^{s,m,1}_{\infty,q}(f):=\sup_{k\in\mathbb{N}_0,\nu\in\mathbb{Z}^n}\Big(2^{kn}\int\limits_0^{2^{1-k}}t^{-sq}\sup_{t/2\leqslant|h|<t}\int\limits_{P_{k,\nu}}|\Delta_h^m f(x)|^q\,dx\frac{dt}{t}\Big)^{\frac1q}<\infty\,.$$

Moreover, the expression $\|f\|_\infty+\mathcal{N}^{s,m,1}_{\infty,q}(f)$ is an equivalent quasi-seminorm in $F^s_{\infty,q}$.

(ii) *The same conclusion holds by replacing in* (i) *the term $\mathcal{N}^{s,m,1}_{\infty,q}(f)$ by*

$$\mathcal{N}^{s,m,2}_{\infty,q}(f):=\sup_{k\in\mathbb{N}_0,\nu\in\mathbb{Z}^n}\Big(2^{kn}\int\limits_0^{2^{1-k}}t^{-sq}\int\limits_{P_{k,\nu}}\Big(t^{-n}\int\limits_{t/2\leqslant|h|<t}|\Delta_h^m f(x)|dh\Big)^q dx\frac{dt}{t}\Big)^{\frac1q},$$

or

$$\mathcal{N}^{s,m,3}_{\infty,q}(f):=\sup_{k\in\mathbb{N}_0,\nu\in\mathbb{Z}^n}\Big(2^{kn}\int\limits_0^{2^{1-k}}t^{-sq}\int\limits_{P_{k,\nu}}t^{-n}\int\limits_{t/2\leqslant|h|<t}|\Delta_h^m f(x)|^q\,dhdx\frac{dt}{t}\Big)^{\frac1q}.$$

Proof. We refer to [22, Rem. 4.8] if $0<q<\infty$, and to [22, Cor. 4.3] as $q=\infty$, in which the statement was proved for the Besov-type spaces $B^{s,\tau}_{\infty,\infty}$, but $B^{s,0}_{\infty,\infty}=B^s_{\infty,\infty}$. □

2.3. Definition of realizations.

Definition 4. *Let $m\in\mathbb{N}_0\cup\{\infty\}$ and $k\in\{0,\dots,m\}$. Let E be a vector subspace of $\mathcal{S}'_m$ endowed with a quasi-norm such that the continuous embedding $E\hookrightarrow\mathcal{S}'_m$ holds. A realization of E into $\mathcal{S}'_k$ is a continuous linear mapping $\sigma:E\to\mathcal{S}'_k$ such that $[\sigma(f)]_m=f$ for all $f\in E$. The image set $\sigma(E)$ is called the realized space of E with respect to σ.*

Remark 4. *In case $k=m$ the identity is the unique realization.*

If a realization is known, then it generates other realizations. We recall the following statement, see [6, Prop. 1].

Lemma 6. *Let $\sigma_0:E\to\mathcal{S}'_k$ be a realization. For all finite families $(\mathcal{L}_\alpha)_{k\leqslant|\alpha|\leqslant N}$ of continuous linear functionals on E, the following formula defines a realization of E in $\mathcal{S}'_k$:*

$$\sigma(f)(x):=\sigma_0(f)(x)+\sum_{k\leqslant|\alpha|\leqslant N}\mathcal{L}_\alpha(f)\,x^\alpha \qquad (modulo\ \mathcal{P}_k)\,.$$

And vice versa, each realization of E modulo $\mathcal{P}_k$ is given in such a way.

3. Realizations of $\dot{F}^s_{\infty,q}$

In what follows, to any space $\dot{F}^s_{\infty,q}$, we associate a number $\mu \in \mathbb{N}_0$ defined by:

$$\mu := \max(0,\, [s]+1). \tag{7}$$

We shall employ the following lemma, a classical consequence of Taylor formula, see, for instance, [16, Prop. 2.5].

Lemma 7. *Let $0 < p \leqslant \infty$ and $N \in \mathbb{N}_0$. There exist $c_1, c_2 > 0$ and $m_1, m_2 \in \mathbb{N}_0$ such that*
(i) $\|Q_j\varphi\|_p \leqslant c_1 2^{-jN}\zeta_{m_1}(\mathcal{F}^{-1}\gamma)\zeta_{m_1}(\varphi)$ for all $\varphi \in \mathcal{S}$ and all $j \in \mathbb{N}_0$.
(ii) $\|Q_j\varphi\|_p \leqslant c_2 2^{jN}\zeta_{m_2}(\mathcal{F}^{-1}\gamma)\zeta_{m_2}(\varphi)$ for all $\varphi \in \mathcal{S}_N$ and all $j \in \mathbb{Z}\setminus\mathbb{N}$.

Our main aim is to prove the following result.

Theorem 1. *Let $f \in \dot{F}^s_{\infty,q}$. Then the series $\sum_{j\in\mathbb{Z}} Q_j f$ converges in $\mathcal{S}'_\mu$. Let us define $\sigma(f)$ as the its sum belonging to $\mathcal{S}'_\mu$. Then the mapping $\sigma : \dot{F}^s_{\infty,q} \to \mathcal{S}'_\mu$ is a translation and a dilation commuting realization of $\dot{F}^s_{\infty,q}$ into $\mathcal{S}'_\mu$. The element $\sigma(f)$ is the unique representative of f in $\mathcal{S}'_\mu$ satisfying $[\sigma(f)]_\infty = f$ in $\mathcal{S}'_\infty$ and $\partial^\alpha\sigma(f) \in \widetilde{C}_0$ for all $|\alpha| = \mu$. Moreover,*

$$\|[\sigma(f)]_\infty\|_{\dot{F}^s_{\infty,q}} = \|f\|_{\dot{F}^s_{\infty,q}}.$$

Proof. Step 1. Let $f \in \dot{F}^s_{\infty,q}$. We introduce a radial and positive function $\widetilde{\gamma} \in \mathcal{D}(\mathbb{R}^n\backslash\{0\})$ such that $\gamma\widetilde{\gamma} = \gamma$. Then we define a sequence of operators $(\widetilde{Q}_j)$ as (Q_j) by taking $\widetilde{\gamma}$ instead of γ.

Let $g \in \mathcal{S}_\mu$. We begin with the inequality

$$|\langle Q_j f, \widetilde{Q}_j g\rangle| \leqslant 2^{js}\|Q_j f\|_\infty (2^{-js}\|\widetilde{Q}_j g\|_1).$$

Then by Lemma 7 with $p = 1$, $\varphi := g$ and an arbitrary N and $\dot{F}^s_{\infty,q} \hookrightarrow \dot{B}^s_{\infty,\infty}$ we get:

$$|\langle Q_j f, \widetilde{Q}_j g\rangle| \lesssim 2^{-js}\min(2^{-jN}, 2^{j\mu})\zeta_m(g)\|f\|_{\dot{F}^s_{\infty,q}}, \qquad j \in \mathbb{Z}, \tag{8}$$

where an integer m depends only on N and μ. We choose N such that $N + s > 0$, and by the definition of μ we have $\mu - s > 0$. Then by the identity $\langle Q_j f, g\rangle = \langle Q_j f, \widetilde{Q}_j g\rangle$ we get

$$\sum_{j\in\mathbb{Z}} |\langle Q_j f, g\rangle| \lesssim \zeta_m(g)\|f\|_{\dot{F}^s_{\infty,q}}. \tag{9}$$

Step 2. Inequality (9) yields

$$\sup_{g\in\mathcal{S}_\mu,\, \zeta_m(g)\leqslant 1} |\langle \sigma(f), g\rangle| \lesssim \|f\|_{\dot{F}^s_{\infty,q}}$$

for all $f \in \dot{F}^s_{\infty,q}$. Then σ is a realization of $\dot{F}^s_{\infty,q}$ into $\mathcal{S}'_\mu$.

Step 3. The identity $[\sigma(f)]_\infty = f$ in $\mathcal{S}'_\infty$ is implied by (1).

Step 4. Let $|\alpha| = \mu$, $\lambda > 0$ and $g \in \mathcal{S}$. We introduce an integer r such that $2^{-r-1} < \lambda \leqslant 2^{-r}$. Then $\operatorname{supp}\mathcal{F}\big(h_\lambda(Q_{j-r}f^{(\alpha)})\big)$ is contained in the annulus $2^{j-1} \leqslant |\xi| \leqslant 3\cdot 2^j$, and

$$\mathcal{F}\big(Q_k h_\lambda(Q_{j-r}f^{(\alpha)})\big) = 0 \quad \text{as} \quad k - j \geqslant 3 \quad \text{or} \quad k - j \leqslant -2.$$

Hence,

$$\langle h_\lambda(Q_{j-r}f^{(\alpha)}), g\rangle = \sum_{k=-2}^{3} \langle h_\lambda(Q_{j-r}f^{(\alpha)}), Q_{j+k}g\rangle.$$

By Bernstein inequality we have

$$\|h_\lambda(Q_{j-r}f^{(\alpha)})\|_\infty \lesssim 2^{(j-r)|\alpha|}\|Q_{j-r}f\|_\infty \lesssim 2^{j(\mu-s)}\lambda^{\mu-s}\|f\|_{\dot{B}^s_{\infty,\infty}},$$

on the one hand. On the other hand, by Lemma 7(i) and the fact that $\|Q_{j+k}g\|_1 \lesssim \|g\|_1$, for some $N \in \mathbb{N}_0$ and $m := m(N) \in \mathbb{N}_0$ we have

$$|\langle h_\lambda \left(\partial^\alpha \sigma(f)\right), g\rangle| \lesssim \lambda^{\mu-s}\|f\|_{\dot{F}^s_{\infty,q}}\Big(\zeta_m(g)\sum_{j\geqslant 0} 2^{j(\mu-s-N)} + \|g\|_1 \sum_{j<0} 2^{j(\mu-s)}\Big).$$

Choosing N such that $N+s-\mu>0$, and taking into account that $\mu - s > 0$ for all $s \in \mathbb{R}$, we pass to limit as λ tends to 0 and arrive at $\partial^\alpha \sigma(f) \in \widetilde{C}_0$.

Step 5. Let $f_i \in \mathcal{S}'_\mu$, $i = 1, 2$, satisfy the identity $[f_1]_\infty = [f_2]_\infty = f$ and $\partial^\alpha f_i \in \widetilde{C}_0$ for all $|\alpha| = \mu$. Then

$$f_1 - f_2 \in \mathcal{P}_\infty \qquad \text{and} \qquad \partial^\alpha(f_1 - f_2) \in \widetilde{C}_0 \cap \mathcal{P}_\infty = \{0\} \qquad \text{for all} \qquad |\alpha| \geqslant \mu.$$

Hence, $f_1 - f_2 \in \mathcal{P}_\mu$.

Step 6. Since each operator Q_j commutes with the mapping τ_a for all $a \in \mathbb{R}^n$, the realization σ commutes also with τ_a.

Let $\lambda > 0$. Since $\dot{F}^s_{\infty,q}$ is dilation invariant, that is, $h_\lambda f \in \dot{F}^s_{\infty,q}$, see Lemma 1, it follows that $\sigma(h_\lambda f) = \sum_{j\in\mathbb{Z}} Q_j(h_\lambda f) \in \mathcal{S}'_\mu$. We define the operators $Q_{j,\lambda}$ as Q_j replacing γ by $h_\lambda \gamma$. It is easy to see that $Q_j(h_\lambda f) = h_\lambda Q_{j,\lambda} f$ in $\mathcal{S}'$ since $Q_j\varphi(\lambda(\cdot)) = Q_{j,\lambda}(h_{\lambda^{-1}}\varphi)$ for all $\varphi \in \mathcal{S}$; recall that $Q_j(\mathcal{S}) \subset \mathcal{S}_\infty$. We now define the realization $\sigma_\lambda(f) := \sum_{j\in\mathbb{Z}} Q_{j,\lambda} f$ of $\dot{F}^s_{\infty,q}$ into $\mathcal{S}'_\mu$. Then

$$\langle \sigma(h_\lambda f), \varphi\rangle = \sum_{j\in\mathbb{Z}} \langle h_\lambda Q_{j,\lambda} f, \varphi\rangle = \lambda^n \sum_{j\in\mathbb{Z}} \big\langle Q_{j,\lambda} f, \varphi(\lambda(\cdot))\big\rangle = \lambda^n \big\langle \sigma_\lambda(f), \varphi(\lambda(\cdot))\big\rangle$$

for all $\varphi \in \mathcal{S}_\mu$. Hence,

$$\sigma(h_\lambda f) = h_\lambda \sigma_\lambda(f) \quad \text{in} \quad \mathcal{S}'_\mu. \tag{10}$$

As above, we also obtain that for σ_λ, the arguing in Steps 1–5 hold true. Then

$$[\sigma(f)]_\infty = [\sigma_\lambda(f)]_\infty = f,$$

and $\sigma(f) - \sigma_\lambda(f) \in \mathcal{P}_\infty$. But $\partial^\alpha(\sigma(f) - \sigma_\lambda(f)) \in \widetilde{C}_0 \cap \mathcal{P}_\infty = \{0\}$ if $|\alpha| \geqslant \mu$, and hence, $\sigma(f) - \sigma_\lambda(f) \in \mathcal{P}_\mu$. This implies $h_\lambda(\sigma(f) - \sigma_\lambda(f)) \in \mathcal{P}_\mu$. Therefore,

$$h_\lambda \sigma(f) = h_\lambda \sigma_\lambda(f) \quad \text{in} \quad \mathcal{S}'_\mu. \tag{11}$$

Now, by (10) and (11) we obtain that $\sigma(h_\lambda f) = h_\lambda \sigma(f)$ in $\mathcal{S}'_\mu$.

Step 7. It is clear that $Q_r Q_j f = 0$ as $|j - r| \geqslant 2$. Then

$$\begin{aligned}\|[\sigma(f)]_\infty\|_{\dot{F}^s_{\infty,q}} &= \sup_{l\in\mathbb{Z}} \sup_{\nu\in\mathbb{Z}^n} \Big(2^{ln} \int_{P_{l,\nu}} \sum_{j\geqslant l} 2^{jsq} \Big| \sum_{j-1\leqslant r\leqslant j+1} Q_r Q_j f\Big|^q dx\Big)^{1/q} \\ &= \sup_{l\in\mathbb{Z}} \sup_{\nu\in\mathbb{Z}^n} \Big(2^{ln} \int_{P_{l,\nu}} \sum_{j\geqslant l} 2^{jsq} \Big| \sum_{m=-1}^{1} Q_{m+j} Q_j f\Big|^q dx\Big)^{1/q}.\end{aligned} \tag{12}$$

We let

$$\widetilde{\gamma}_1 := \sum_{m=-1}^{1} \gamma(2^{-m}\,\cdot)\gamma,$$

and define the operators $\widetilde{Q}_{j,1}$ as

$$\widehat{\widetilde{Q}_{j,1} f} := \widetilde{\gamma}_1(2^{-j}(\cdot))\widehat{f}.$$

Then we get

$$\sum_{m=-1}^{1} Q_{m+j} Q_j = \widetilde{Q}_{j,1} \qquad \text{for all} \qquad j \in \mathbb{Z}. \tag{13}$$

We have

$$\operatorname{supp}\widetilde{\gamma}_1 \subset \left\{\xi \in \mathbb{R}^n : \frac{1}{2} \leqslant |\xi| \leqslant \frac{3}{2}\right\} \qquad \text{and} \qquad \widetilde{\gamma}_1(\xi) \geqslant 1 \quad \text{as} \quad \frac{3}{4} \leqslant |\xi| \leqslant 1$$

since $\widetilde{\gamma}_1(\xi) \geqslant \gamma^2(\xi)$, see the definition of γ in Section 1. Then $\widetilde{\gamma}_1$ satisfies equations (2.1)–(2.3) in [12] and owing to equation (5.1) and Corollary 5.3 in [12], we can replace the operators Q_j by $\widetilde{Q}_{j,1}$ in Definition 2 to obtain

$$\|f\|_{\dot{F}^s_{\infty,q}} \lesssim \sup_{l\in\mathbb{Z}} \sup_{\nu\in\mathbb{Z}^n} \Big(2^{ln} \int_{P_{l,\nu}} \sum_{j\geqslant l} 2^{jsq} \Big| \sum_{m=-1}^{1} Q_{m+j} Q_j f\Big|^q dx\Big)^{1/q} \lesssim \|f\|_{\dot{F}^s_{\infty,q}}.$$

Hence, it follows from (12) that $\|[\sigma(f)]_\infty\|_{\dot{F}^s_{\infty,q}} = \|f\|_{\dot{F}^s_{\infty,q}}$.

Finally, for this identity for quasi-seminorms, we can add the following observation. Let $f_1 \in \mathcal{S}'$ be such that $[f_1]_\infty = [\sigma(f)]_\infty$. We have

$$\|[\sigma(f)]_\infty\|_{\dot{F}^s_{\infty,q}} = \|[f_1]_\infty\|_{\dot{F}^s_{\infty,q}}.$$

Let $f_2 \in \mathcal{S}'$ be such that $[f_2]_\infty = f$. By Step 5, $f_1 - f_2$ is a polynomial; we denote $f_1 - f_2 =: \widetilde{f}$. But $Q_j([\sigma(f)]_\infty) = Q_j f_1 = Q_j f_2$ since $Q_j \widetilde{f} = 0$; we also have $Q_j f_1 = Q_j f_2$ in the sense of functions, since both $Q_j f_1$ and $Q_j f_2$ are smooth functions of exponential type, see Paley-Wiener theorem [13, Thm. 1.7.7]). We again arrive at the desired identity. The proof is complete. □

Remark 5. *For all $s \in \mathbb{R}$, if $f \in \dot{F}^s_{\infty,q}$, the series $\sum_{j\geqslant 0} Q_j f$ converges in $\mathcal{S}'$. Indeed, the inequality* (8) *becomes*

$$|\langle Q_j f, \widetilde{Q}_j g\rangle| \lesssim 2^{-j(N+s)} \zeta_m(g) \|f\|_{\dot{F}^s_{\infty,q}}$$

for all $g \in \mathcal{S}$ and all $j \in \mathbb{N}_0$; here $\widetilde{Q}_j$ is the same as in Step 1 in the proof of Theorem 1.

The next lemma *characterizes* the number μ; the proof of this lemma is similar to that given by G. Bourdaud for Besov spaces [4, Prop. 2.2.1].

Lemma 8. *Let $s \geqslant 0$. Then there exists a function $f \in \dot{F}^s_{\infty,q}$ such that the series $\sum_{j\leqslant 0} Q_j f$ diverges in $\mathcal{S}'_{\mu-1}$.*

Proof. We briefly outline the proof, since in case $q < \infty$ we do not have the same spaces as in [4]. We denote $m := \mu - 1 = [s]$. Let $\varphi \in \mathcal{D}$ be such that $\int_{\mathbb{R}^n} \varphi(x)dx = 1$. As $\partial_1^m \varphi \in \mathcal{S}_m$, we split the sum $\sum_{j\leqslant 0} \langle Q_j f, \partial_1^m \varphi\rangle$ into $I_1 + I_2$, where

$$I_1 := (-1)^m \sum_{j\leqslant 0} \int_{\mathbb{R}^n} \big(\partial_1^m Q_j f(x) - \partial_1^m Q_j f(0)\big)\overline{\varphi}(x)dx, \qquad I_2 := (-1)^m \sum_{j\leqslant 0} \partial_1^m Q_j f(0).$$

It is sufficient to construct a function $f \in \dot{F}^s_{\infty,q}$ such that $|I_1| < \infty$ and $|I_2| = \infty$. For this purpose, let $g \in \mathcal{S}$ be such that

$$\widehat{g} \in \mathcal{D}, \qquad \widehat{g} \geqslant 0, \qquad \operatorname{supp}\widehat{g} \subset \left\{\xi : \frac{3}{4} \leqslant |\xi| \leqslant 1,\, \xi_1 \geqslant 0\right\}.$$

We let

$$f(x) := \sum_{k\geqslant 0} 2^{k(s+m)/2} g(2^{-k}x).$$

Clearly, we have

$$Q_j f(x) = 2^{-j(s+m)/2} g(2^j x) \quad \text{if} \quad j \leqslant 0, \qquad Q_j f(x) = 0 \quad \text{if} \quad j \geqslant 1,$$

since $\gamma(2^{-j}\xi)\widehat{g}(2^k\xi) = 0$ if $k \neq -j$ and $\gamma\widehat{g} = \widehat{g}$; we recall that $\gamma(\xi) = 1$ as $\frac{3}{4} \leqslant |\xi| \leqslant 1$. It is also clear that for all $j \leqslant 0$ the identities hold:

$$\partial_1^m Q_j f(0) = (2\pi)^{-n} i^m 2^{j(m-s)/2} \int\limits_{\mathbb{R}^n} \xi_1^m \widehat{g}(\xi)\, d\xi,$$

$$|\partial_1^m Q_j f(x) - \partial_1^m Q_j f(0)| \leqslant (2\pi)^{-n} 2^{j(m-s+2)/2} \sum_{k=1}^{n} |x_k| \int\limits_{\mathbb{R}^n} |\xi_k|\, \xi_1^m \widehat{g}(\xi)\, d\xi.$$

Then

$$|\sum_{j\leqslant 0} \partial_1^m Q_j f(0)| = \infty, \qquad \sum_{j\leqslant 0} \|\nabla \partial_1^m Q_j f\|_\infty < \infty.$$

It remains to prove that $[f]_\infty \in \dot{F}^s_{\infty,q}$. Since

$$\int\limits_{P_{k,\nu}} |g(2^j x)|^q dx \leqslant 2^{-jn} \|g\|_1^q$$

and $s - m \geqslant 0$, that is, $2^{jq(s-m)/2} \leqslant 1$ for all $j \leqslant 0$, we first have

$$2^{kn} \int\limits_{P_{k,\nu}} \sum_{0\geqslant j\geqslant k} 2^{jq(s-m)/2} |g(2^j x)|^q dx \leqslant \|g\|_1^q \sum_{0\geqslant j\geqslant k} 2^{(k-j)n} \lesssim \|g\|_1^q \tag{14}$$

for all $k \in \mathbb{Z} \setminus \mathbb{N}$. Therefore, by taking the supremum over $k \in \mathbb{Z} \setminus \mathbb{N}$ and $\nu \in \mathbb{Z}^n$ in (14), we get

$$\|[f]_\infty\|_{\dot{F}^s_{\infty,q}} \lesssim 1.$$

The proof is complete. $\square$

Without use the LP decomposition, we define the realized space of $\dot{F}^s_{\infty,q}$.

Definition 5. *The realized space of $\dot{F}^s_{\infty,q}$ denoted by $\dot{\widetilde{F}}^s_{\infty,q}$ is the set of all $f \in \mathcal{S}'_\mu$ such that $[f]_\infty \in \dot{F}^s_{\infty,q}$ and $f^{(\alpha)} \in \widetilde{C}_0$ for all $|\alpha| = \mu$.*

We should be sure of the identity $\sigma(\dot{F}^s_{\infty,q}) = \dot{\widetilde{F}}^s_{\infty,q}$, where the mapping σ was defined in Theorem 1. The direct embedding is by the definition; let us prove the opposite one.

Let $f \in \dot{\widetilde{F}}^s_{\infty,q}$, then $f - \sigma([f]_\infty)$ is a polynomial. Since $\widetilde{C}_0 \cap \mathcal{P}_\infty = \{0\}$ and $f^{(\alpha)} - \partial^\alpha \sigma([f]_\infty) \in \widetilde{C}_0$ for all $|\alpha| \geqslant \mu$, we conclude $f - \sigma([f]_\infty) \in \mathcal{P}_\mu$, that is, $f = \sigma([f]_\infty)$ in $\mathcal{S}'_\mu$.

The space $\dot{\widetilde{F}}^s_{\infty,q}$ is equipped with a quasi-seminorm defined as

$$\|f\|_{\dot{\widetilde{F}}^s_{\infty,q}} := \|[f]_\infty\|_{\dot{F}^s_{\infty,q}}.$$

Of course, one has to justify this definition. If $[f]_\mu = [f_1]_\mu$ and $[f]_\infty = [f_2]_\infty$, then $f_1 - f_2 \in \mathcal{P}_\infty$, but $Q_j(f_1 - f_2) = 0$, which is a sufficient argument. In the case $s \geqslant 0$, $\dot{\widetilde{F}}^s_{\infty,q}$ can be characterized in $\mathcal{S}'$. This is done in the next lemma; for the case $s = 0$ see Remark 6 below.

Lemma 9. *Let $s > 0$. Then $\dot{\widetilde{F}}^s_{\infty,q}$ is the set of $f \in \mathcal{S}'$ such that $[f]_\infty \in \dot{F}^s_{\infty,q}$, and $f^{(\alpha)} \in \widetilde{C}_0$ for all $|\alpha| = \mu$, and moreover:*

(i) If $s \notin \mathbb{N}$, then $f \in C^{\mu-1}$ and $f^{(\alpha)}(0) = 0$ for all $|\alpha| \leqslant \mu - 1$,

(ii) If $s \in \mathbb{N}$, then $f \in C^{\mu-2}$ and $f^{(\alpha)}(0) = 0$ for all $|\alpha| \leqslant \mu - 2$ with $\mu = s + 1 \geqslant 2$.

Proof. The proof is similar to the proofs of Proposition 4.8 in [7] and of Theorem 4.5 in [16] thanks to the embedding $\dot{F}^s_{\infty,q} \hookrightarrow \dot{B}^s_{\infty,\infty}$; let us briefly outline this.

Proof of (i). We first define $\dot{\widetilde{F}}^s_{\infty,q}$ in $\mathcal{S}'$ by replacing each $Q_j f$ by a polynomial of degree less than μ in $\sigma(f)$, see Theorem 1. Then we get a realization denoted σ_1. Since any realization on $\dot{F}^s_{\infty,q}$ is a surjective mapping, then if $f \in \dot{\widetilde{F}}^s_{\infty,q}$, there exists $g \in \dot{F}^s_{\infty,q}$ such that $[f]_\mu = g$, and it is sufficient to take $f := \sigma_1(g)$.

Construction of σ_1. Let $g \in \dot{F}^s_{\infty,q}$. Then the series

$$\sigma_1(g) := \sum_{j\in\mathbb{Z}} \Big(Q_j g - \sum_{|\alpha|<\mu} (Q_j g)^{(\alpha)}(0)\, \frac{x^\alpha}{\alpha!} \Big)$$

converges in $\mathcal{S}'$. The mapping $\sigma_1 : \dot{F}^s_{\infty,q} \to \mathcal{S}'$ is a realization of $\dot{F}^s_{\infty,q}$ into $\mathcal{S}'$, where $\sigma_1(f)$ is the unique representative of g in $\mathcal{S}'$, of class $C^{\mu-1}$, $\partial^\alpha \sigma_1(g)(0) = 0$ for all $|\alpha| \leqslant \mu - 1$, $\partial^\alpha \sigma_1(g) \in \widetilde{C}_0$ for all $|\alpha| = \mu$ and $\|[\sigma_1(g)]_\infty\|_{\dot{F}^s_{\infty,q}} = \|g\|_{\dot{F}^s_{\infty,q}}$.

We now present the role of the assumption $s \notin \mathbb{N}$: by the Bernstein inequality

$$\|(Q_j g)^{(\alpha)}\|_\infty \lesssim 2^{j|\alpha|} \|Q_j g\|_\infty \lesssim 2^{j(|\alpha|-s)} \|g\|_{\dot{B}^s_{\infty,\infty}},$$

we get

$$\begin{aligned} \Big| Q_j g(x) - \sum_{|\alpha|<\mu} (Q_j g)^{(\alpha)}(0)\, \frac{x^\alpha}{\alpha!} \Big| &\leqslant \|Q_j g\|_\infty + \sum_{|\alpha|\leqslant\mu-1} \frac{|x|^{|\alpha|}}{\alpha!} \|(Q_j g)^{(\alpha)}\|_\infty \\ &\lesssim \big(2^{-js} + 2^{j(\mu-1-s)}(1+|x|)^{\mu-1}\big) \|g\|_{\dot{B}^s_{\infty,\infty}} \qquad , x \in \mathbb{R}^n, \quad j \in \mathbb{N}_0. \end{aligned}$$

On the other hand, by the Taylor formula we have

$$\begin{aligned} \Big| Q_j g(x) - \sum_{|\alpha|<\mu} (Q_j g)^{(\alpha)}(0)\, \frac{x^\alpha}{\alpha!} \Big| &\leqslant \mu \sum_{|\alpha|=\mu} \frac{|x|^{|\alpha|}}{\alpha!} \int_0^1 (1-t)^{\mu-1} |(Q_j g)^{(\alpha)}(tx)|\, dt \\ &\lesssim 2^{j(\mu-s)}\, |x|^\mu\, \|g\|_{\dot{B}^s_{\infty,\infty}}. \end{aligned}$$

Therefore,

$$|\sigma_1(g)(x)| \lesssim \Big\{ \sum_{j\geqslant 0} \Big(2^{-js} + 2^{j(\mu-1-s)}(1+|x|)^{\mu-1} \Big) + \sum_{j<0} 2^{j(\mu-s)}\, |x|^\mu \Big\} \|g\|_{\dot{F}^s_{\infty,q}}.$$

Thus, thanks to assumption $s \in \mathbb{R}^+ \backslash \mathbb{N}_0$, we get the convergence of above series with $\mu - 1 - s = [s] - s < 0$ and $\mu - s > 0$.

Proof of (ii). As in the previous step, we consider the mapping:

$$\sigma_2(g) := \sum_{j\geqslant 0} Q_j g + \sum_{j<0} \Big(Q_j g - \sum_{|\alpha|<\mu} (Q_j g)^{(\alpha)}(0)\, \frac{x^\alpha}{\alpha!} \Big) \quad \text{for all} \quad g \in \dot{F}^s_{\infty,q}, \tag{15}$$

where $\sigma_2(g)$ is the unique representative of g in $\mathcal{S}'$, and σ_2 is also a realization of $\dot{F}^s_{\infty,q}$ into $\mathcal{S}'$ satisfying $\partial^\alpha \sigma_2(g) \in \widetilde{C}_0$ for all $|\alpha| = \mu$ and $\|[\sigma_2(g)]_\infty\|_{\dot{F}^s_{\infty,q}} = \|g\|_{\dot{F}^s_{\infty,q}}$. If in addition $s > 0$, then $\sigma_2(g)$ is of class $C^{\mu-2}$.

Owing to Lemma 6, if $f \in \dot{\widetilde{F}}^s_{\infty,q}$, there exists $g \in \dot{F}^s_{\infty,q}$ such that $[f]_\mu = g$ and it is sufficient to take

$$f := \sigma_2(g) - \sum_{|\beta|\leqslant\mu-2} \Big(\sum_{j\geqslant 0} (Q_j g)^{(\beta)}(0) \Big) \frac{x^\beta}{\beta!}.$$

For the realization σ_2 we refer to [7, Rem. 4.9]. In case $s > 0$, for $|\beta| \leqslant \mu - 2$, we have $|\beta| - s \leqslant \mu - 2 - s = -1$, and then

$$\sum_{j\geqslant 0} \|(Q_j g)^{(\beta)}\|_\infty \lesssim \|g\|_{\dot{F}^s_{\infty,q}} \sum_{j\geqslant 0} 2^{(|\beta|-s)j} \lesssim \|g\|_{\dot{F}^s_{\infty,q}};$$

the estimate for the sum

$$\sum_{j<0} |\partial^\beta \{Q_j g - \sum_{|\alpha|<\mu} (Q_j g)^{(\alpha)}(0)\, \frac{x^\alpha}{\alpha!}\}|$$

can be obtained as in [16]. The proof is complete. □

Remark 6. *If $f \in \dot{\widetilde{F}}^0_{\infty,q}$ then $f = \sigma_2(g)$, where $\sigma_2(g)$ is defined in the above proof, see* (15).

Remark 7. *Clearly, we can not identify $\dot{F}^0_{\infty,2}$ with BMO, where the space BMO is as defined in* [10]*, since $\|[f]_\infty\|_{\dot{F}^0_{\infty,2}} = 0$ for all polynomials, while one can easily find a polynomial $f \notin \mathcal{P}_1$ such that $\int_{\mathbb{R}^n} (1+|x|^{n+1})^{-1}|f(x)|dx = \infty$, see* [10].

4. Characterizations by differences

We now present a characterization of realized spaces $\dot{\widetilde{F}}^s_{\infty,q}$ by means of differences. In view of Lemmata 4 and 5, one could think that the scales $\mathcal{N}^{s,m,i}_{\infty,q}(f)$, $i = 1,2,3$, are other equivalent quasi-seminorms in $\dot{F}^s_{\infty,q}$. But this is not the case since for any polynomial f of degree m we can have $\mathcal{N}^{s,m,i}_{\infty,q}(f) \neq 0$, while $\|[f]_\infty\|_{\dot{F}^s_{\infty,q}} = 0$; for instance $f(x) := x_1^m$, then $\Delta_h^m f(x) = m! h_1^m$ and $\mathcal{N}^{s,m,1}_{\infty,q}(f) = m! 2^{m-s}(q(m-s))^{-1/q}$, which tends to infinity as $s \uparrow m$; the kernel of Δ_h^m is $\mathcal{P}_m$.

Lemma 10. *Let* (6) *be satisfied. Then there exists a constant $c > 0$ such that the inequality $\mathcal{N}(f) \leqslant c\|[f]_\infty\|_{\dot{F}^s_{\infty,q}}$ holds for all $f \in F^s_{\infty,q}$, where $\mathcal{N} := \mathcal{N}^{s,m,1}_{\infty,q}$. The same holds if we replace $\mathcal{N}^{s,m,1}_{\infty,q}$ by $\mathcal{N}^{s,m,i}_{\infty,q}$ with $i = 2,3$.*

Proof. Lemmata 4 and 5 we have

$$\mathcal{N}(f) \lesssim \|f\|_\infty + \|[f]_\infty\|_{\dot{F}^s_{\infty,q}}$$

for all $f \in F^s_{\infty,q}$. Replacing f by $f_\lambda := f(\lambda(\cdot))$ arbitrary $\lambda > 0$ in this inequality and using Lemma 1, we obtain:

$$\lim_{\lambda\to\infty} \lambda^{-s}\mathcal{N}(f_\lambda) \leqslant c\,\|[f]_\infty\|_{\dot{F}^s_{\infty,q}} \qquad \text{for all} \qquad f \in F^s_{\infty,q}. \tag{16}$$

Let now $\lambda > 1$ and $N \in \mathbb{N}$ be such that $2^N \leqslant \lambda < 2^{N+1}$. By the elementary inequality

$$\forall x \in P_{k,\nu} :\ [2^N\lambda^{-1}\nu_j] \leqslant 2^{k+N}\lambda^{-1}x_j < [2^N\lambda^{-1}\nu_j] + 2, \qquad j = 1,\ldots,n$$

recall that $2^{-1} < 2^N\lambda^{-1} \leqslant 1$, we obtain

$$x \in P_{k,\nu} \ \Rightarrow\ \lambda^{-1}x \in P_{k+N,E(2^N\lambda^{-1}\nu)} \cup P_{k+N,E(2^N\lambda^{-1}\nu)+w_0},$$

where $w_0 := (1,1,\ldots,1) \in \mathbb{Z}^n$ and we have employed the notation $E(x) = ([x_1],\ldots,[x_n]) \in \mathbb{Z}^n$, $x \in \mathbb{R}^n$. As $\Delta_h^m f(x) = \Delta^m_{(\lambda^{-1}h)} f_\lambda(\lambda^{-1}x)$, with the change of variables $y := \lambda^{-1}x$, $r := \lambda^{-1}t$ and $u := \lambda^{-1}h$, we get:

$$\begin{aligned} &2^{kn}\int_0^{2^{1-k}} t^{-sq} \sup_{\frac{t}{2}\leqslant|h|<t} \int_{P_{k,\nu}} |\Delta_h^m f(x)|^q\,dx\frac{dt}{t} \\ &\lesssim \lambda^{-sq}\sum_{l=0}^{1} 2^{(k+N)n}\int_0^{2^{1-(k+N)}} r^{-sq} \sup_{\frac{r}{2}\leqslant|u|<r} \int_{P_{k+N,E(2^N\lambda^{-1}\nu)+lw_0}} |\Delta_u^m f_\lambda(y)|^q dy\frac{dr}{r}. \end{aligned} \tag{17}$$

We assume that $k \in \mathbb{N}_0$ and this allows us to bound last term in (17) by

$$c\lambda^{-sq} \sup_{j\in\mathbb{N}_0} \sup_{\eta\in\mathbb{Z}^n} 2^{jn} \int_0^{2^{1-j}} r^{-sq} \sup_{r/2\leqslant|u|<r} \int_{P_{j,\eta}} |\Delta_u^m f_\lambda(y)|^q \, dy \frac{dr}{r} , \tag{18}$$

where c is independent of k. Calculating the supremum over $k \in \mathbb{N}_0$ and $\nu \in \mathbb{Z}^n$ in (17), and taking (18) into consideration, we obtain $\mathcal{N}(f) \leqslant c\lambda^{-s}\mathcal{N}(f_\lambda)$. Finally by (16), we complete the proof. □

Here our second main result is as follows.

Theorem 2. *Let $m \in \mathbb{N}$ be such that (6) is satisfied. Then $\mathcal{N}^{s,m,i}_{\infty,q}(f)$, $i = 1, 2, 3$, define equivalent quasi-seminorms in $\dot{\widetilde{F}}{}^s_{\infty,q}$.*

Proof. We consider only $\mathcal{N}^{s,m,1}_{\infty,q}(f)$, since the estimates of $\mathcal{N}^{s,m,i}_{\infty,q}(f)$, $i = 2, 3$, can be obtained in the same way. To simplify the notations, in the proof we write $\mathcal{N}(f)$ instead of $\mathcal{N}^{s,m,1}_{\infty,q}(f)$.

The proof of $\|[f]_\infty\|_{\dot{F}^s_{\infty,q}} \leqslant c\mathcal{N}(f)$, for all regular tempered distribution f obeying $\mathcal{N}(f) < \infty$ can be done as in [18, Subs. 4.1] and we omit the details.

The opposite inequality is similar to that given in [18], and we present only the needed changes. Let $f \in \dot{\widetilde{F}}{}^s_{\infty,q}$. We denote $f_k := \sum_{-k\leqslant j\leqslant k_s} Q_j f$, where $k \in \mathbb{N}_0$. We also define $k_s := 0$ as $s \in \mathbb{N}$ and $k_s = k$ as $s \in \mathbb{R}^+\backslash\mathbb{N}$. Then the function f_k belongs to $F^s_{\infty,q}$. Indeed, the inequality $\|f_k\|_\infty \leqslant c\|[f]_\infty\|_{\dot{F}^s_{\infty,q}}$ with a constant $c := c(k) > 0$, can be obtained by the assumption on s and the following estimate:

$$|Q_j f(x)| \leqslant c\, 2^{-js} \|f\|_{\dot{F}^s_{\infty,q}}, \qquad j \in \mathbb{Z}, \qquad x \in \mathbb{R}^n. \tag{19}$$

In order to prove (19), it is sufficient to employ the embedding $\dot{F}^s_{\infty,q} \hookrightarrow \dot{F}^s_{\infty,\infty} = \dot{B}^s_{\infty,\infty}$.

Now we are goin to prove that

$$\|[f_k]_\infty\|_{\dot{F}^s_{\infty,q}} \leqslant c\|[f]_\infty\|_{\dot{F}^s_{\infty,q}} \tag{20}$$

with a constant independent of f and k. We proceed as in Step 7 in the proof of Theorem 1. Then similar to (12) recalling that $Q_r Q_j f = 0$ as $|j - r| \geqslant 2$, we get

$$\begin{aligned} \|[f_k]_\infty\|_{\dot{F}^s_{\infty,q}} &= \sup_{l\in\mathbb{Z}} \sup_{\nu\in\mathbb{Z}^n} \Big(2^{ln} \int_{P_{l,\nu}} \sum_{j\geqslant l} \Big| \sum_{\substack{-k\leqslant r\leqslant k_s \\ |r-j|\leqslant 1}} Q_r Q_j f \Big|^q 2^{jsq} \, dx\Big)^{1/q} \\ &= \sup_{l\in\mathbb{Z}} \sup_{\nu\in\mathbb{Z}^n} \Big(2^{(l-N)n} \int_{P_{l-N,\nu}} \sum_{j\geqslant l-N} \Big| \sum_{\substack{-k\leqslant r\leqslant k_s \\ |r-j|\leqslant 1}} Q_r Q_j f \Big|^q 2^{jsq} \, dx\Big)^{1/q}, \end{aligned} \tag{21}$$

for all $N \in \mathbb{Z}$. Since here the supremum is taken over all $l \in \mathbb{Z}$, it is translation invariant in $\mathbb{Z}$. The last identity is trivial but is useful for the next computation. On the one hand, in the sum $\sum_{|r-j|\leqslant 1} \dots$ we have at most three terms corresponding to $r \in \{j - 1, j, j + 1\}$, and hence

$$\Big| \sum_{\substack{-k\leqslant r\leqslant k_s \\ |r-j|\leqslant 1}} Q_r Q_j f \Big|^q \leqslant 2^{2(q-1)} \sum_{\substack{-k\leqslant r\leqslant k_s \\ |r-j|\leqslant 1}} \big|Q_r Q_j f\big|^q. \tag{22}$$

On the other hand, by the following elementary inequalities

$$\begin{aligned} &\text{if} \quad -k \leqslant r \leqslant k_s \quad \text{and} \quad |r - j| \leqslant 1 \quad \Rightarrow \quad -k - 1 \leqslant j \leqslant k_s + 1, \\ &\text{if} \quad -k - 1 \leqslant j \leqslant k_s + 1 \quad \text{and} \quad |r - j| \leqslant 1 \quad \Rightarrow \quad -k - 2 \leqslant r \leqslant k_s + 2, \end{aligned}$$

by the fact that

$$\{r \in \mathbb{Z} : \, -k \leqslant r \leqslant k_s\} \subset \{r \in \mathbb{Z} : \, -k - 2 \leqslant r \leqslant k_s + 2\},$$

and by using (22), we obtain

$$\begin{aligned}\sum_{j\geqslant l-N}\Big|\sum_{\substack{-k\leqslant r\leqslant k_s\\ |r-j|\leqslant 1}} Q_rQ_jf\Big|^q 2^{jsq} &\leqslant c\sum_{j\geqslant l-N}\sum_{\substack{-k\leqslant r\leqslant k_s\\ |r-j|\leqslant 1}}\big|Q_rQ_jf\big|^q 2^{jsq}\\ &\leqslant c\sum_{\substack{j\geqslant l-N\\ -k-1\leqslant j\leqslant k_s+1}}\sum_{|r-j|\leqslant 1}\big|Q_rQ_jf\big|^q 2^{jsq}.\end{aligned}\tag{23}$$

Choosing the integer $N := N_{k,l}$ such that $-k-1 \geqslant l - N_{k,l}$, we bound the last term in (23) as follows:

$$c\sum_{j\geqslant l-N_{k,l}}\sum_{|m|\leqslant 1}\big|Q_{j+m}Q_jf\big|^q 2^{jsq} \quad \text{with} \quad m := r-j.$$

Substituting this bound into (21), letting $\ell := l - N_{k,l}$, and taking the supremum over all $\ell \in \mathbb{Z}$, we get

$$\|[f_k]_\infty\|_{\dot F^s_{\infty,q}} \leqslant c\sum_{|m|\leqslant 1}\sup_{\ell\in\mathbb{Z}}\sup_{\nu\in\mathbb{Z}^n}\Big(2^{\ell n}\int_{P_{\ell,\nu}}\sum_{j\geqslant \ell}\big|Q_{j+m}Q_jf\big|^q 2^{jsq}\,dx\Big)^{1/q}\tag{24}$$

for all $k \in \mathbb{N}_0$. We continue by letting $\widetilde{\gamma}_m := \gamma(2^{-m}(\cdot))\gamma$, and this function possesses the following properties:

$$\begin{aligned}&\operatorname{supp}\widetilde{\gamma}_0 \subset \Big\{\xi\in\mathbb{R}^n : \frac12 \leqslant |\xi| \leqslant \frac32\Big\}, && \widetilde{\gamma}_0(\xi)\geqslant 1 \quad \text{as} \quad \frac34 \leqslant |\xi| \leqslant 1,\\ &\operatorname{supp}\widetilde{\gamma}_{-1} \subset \Big\{\xi\in\mathbb{R}^n : \frac12 \leqslant |\xi| \leqslant \frac34\Big\}, && \widetilde{\gamma}_{-1}(\xi)> 0 \quad \text{as} \quad \frac{9}{16} \leqslant |\xi| \leqslant \frac{11}{16}.\end{aligned}$$

Hence,

$$\widetilde{\gamma}_{-1}(\xi) \geqslant c > 0 \quad \text{on} \quad \Big\{\xi\in\mathbb{R}^n : \frac{9}{16} \leqslant |\xi| \leqslant \frac{11}{16}\Big\}, \qquad c := \min_{\frac{9}{16}\leqslant|\eta|\leqslant\frac{11}{16}}\gamma(2\eta)\gamma(\eta).$$

The next property is

$$\operatorname{supp}\widetilde{\gamma}_1 \subset \Big\{\xi\in\mathbb{R}^n : 1 \leqslant |\xi| \leqslant \frac32\Big\}, \qquad \widetilde{\gamma}_1(\xi)>0 \quad \text{as} \quad \frac98 \leqslant |\xi| \leqslant \frac{11}{8},$$

and hence,

$$\widetilde{\gamma}_1(\xi) \geqslant c > 0 \quad \text{on} \quad \Big\{\xi\in\mathbb{R}^n : \frac98 \leqslant |\xi| \leqslant \frac{11}{8}\Big\}, \qquad c := \min_{\frac98\leqslant|\eta|\leqslant\frac{11}{8}}\gamma\Big(\frac{\eta}{2}\Big)\gamma(\eta).$$

Then we define the operators $\widetilde{Q}_{j,m}$ as $\widehat{\widetilde{Q}_{j,m}f} := \widetilde{\gamma}_m(2^{-j}(\cdot))\widehat{f}$, and as in (13), this yields

$$Q_{m+j}Q_j = \widetilde{Q}_{j,m} \qquad \text{for all} \quad j\in\mathbb{Z}.$$

We replace the operators Q_j by $\widetilde{Q}_{j,m}$ with $m \in \{-1,0,1\}$ in Definition 2 and we denote by $\|\cdot\|^{\widetilde{\gamma}_m}_{\dot F^s_{\infty,q}}$ the associated quasi-seminorms. By [12, Cor. 5.3], we have:

$$\|[f]_\infty\|^{\widetilde{\gamma}_m}_{\dot F^s_{\infty,q}} \leqslant c\,\|[f]_\infty\|_{\dot F^s_{\infty,q}},$$

where c is independent of f. But from (24), we also have

$$\|[f_k]_\infty\|_{\dot F^s_{\infty,q}} \leqslant c\sum_{m=-1}^{1}\|[f]_\infty\|^{\widetilde{\gamma}_m}_{\dot F^s_{\infty,q}} \quad \text{for all} \quad k\in\mathbb{Z}.$$

This proves estimate (20).

Applying now Lemma 10 to f_k, we obtain

$$\mathcal{N}(f_k) \leqslant c\,\|[f]_\infty\|_{\dot{F}^s_{\infty,q}} \qquad \text{for all} \qquad k \in \mathbb{N}_0, \tag{25}$$

the constant c is independent of k, see (20). On the other hand, letting

$$r_j(x) := \sum_{|\alpha|<\mu} (Q_j f)^{(\alpha)}(0)\,\frac{x^\alpha}{\alpha!}$$

and recalling that $\mu = [s]+1$, cf. (7), we obtain that the sequence $\big(f_k - \sum_{-k\leqslant j\leqslant k_s} r_j\big)_{k\geqslant 0}$ converges uniformly on each compact subset of $\mathbb{R}^n$ to a limit denoted v, see [18, (22), Subs. 2.2] for $\dot{B}^s_{\infty,\infty}$. At the same time, $\dot{F}^s_{\infty,q} \hookrightarrow \dot{B}^s_{\infty,\infty}$ cf. Lemma 3. By applying twice the Fatou lemma in (25), we get

$$\mathcal{N}(v) \leqslant c\,\|[f]_\infty\|_{\dot{F}^s_{\infty,q}}. \tag{26}$$

In case $s \in \mathbb{N}$, we add the following inequality:

$$\mathcal{N}\Big(\sum_{j\geqslant 0} Q_j f\Big) \leqslant c\,\|[f]_\infty\|_{\dot{F}^s_{\infty,q}}, \tag{27}$$

that is, $\sum_{j\geqslant 0} Q_j f \in F^s_{\infty,q}$. The latter can be obtained by Lemma 10 since we can apply (19) thanks to $s > 0$, see (6), and to obtain

$$\|\sum_{j\geqslant 0} Q_j f\|_\infty \lesssim \|[f]_\infty\|_{\dot{F}^s_{\infty,q}}$$

and similar to Step 7 in the proof of Theorem 1, we also have

$$\|\sum_{j\geqslant 0} Q_j f\|_{\dot{F}^s_{\infty,q}} \lesssim \|[f]_\infty\|_{\dot{F}^s_{\infty,q}}.$$

We let $g := v + \sum_{j\geqslant 0} Q_j f$ if $s \in \mathbb{N}$ and $g := v$ if $s \in \mathbb{R}^+\backslash\mathbb{N}$. We have $f - g \in \mathcal{P}_\mu$ and $\mathcal{N}(\mathcal{P}_\mu) = \{0\}$; recall that $\Delta^m_h(x^\alpha) = 0$ for all $|\alpha| < m$, and by assumption $m \geqslant \mu > s$. Then it follows from (26) and (27) that

$$\mathcal{N}(f) \leqslant \mathcal{N}(f-g) + \mathcal{N}(g) \lesssim \|[f]_\infty\|_{\dot{F}^s_{\infty,q}}.$$

The proof is complete. □

Remark 8. *Of course, the statement of Lemma 4 is certainly known and in particular (i) is classical, but now this can be deduced from Theorem 2 at least for $q \geqslant 1$. Indeed, the difficult part in the proof of Lemma 4 is $\|[f]_\infty\|_{\dot{F}^s_{\infty,q}} \lesssim \|f\|_{F^s_{\infty,q}}$, where now, we get*

$$\|[f]_\infty\|_{\dot{F}^s_{\infty,q}} \lesssim \mathcal{N}^{s,m,1}_{\infty,q}(f) \lesssim \mathcal{N}^{s,m,1}_{\infty,q}(f) + \|f\|_\infty \lesssim \|f\|_{F^s_{\infty,q}}$$

if $q \geqslant 1$ and $m \in \mathbb{N}$ is such that $0 < s < m$.

Conclusion

The realized spaces $\tilde{\dot{F}}^s_{\infty,q}$ of the homogeneous Triebel-Lizorkin spaces $\dot{F}^s_{\infty,q}$ are now characterized by quasi-seminorms in discrete and continuous (if $s > 0$) forms. Our next step will be the extension of the study on $\tilde{\dot{F}}^s_{\infty,q}$ to:

- the pointwise multiplication as in e.g. [2],
- the composition operators as in case of the realized homogeneous Besov spaces, see e.g. [8, Thm. 4] or [17, Thm. 5.1],
- the pseudodifferential operators as in e.g. [15].

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