ABSTRACTS

F.G. Avkhadiev, R.G. Nasibullin, I.K. Shafigullin

Conformal invariants of hyperbolic planar domains

Abstract. We consider planar domains of hyperbolic type and conformally invariant functionals defined as sharp constants for Hardy type inequalities. We study relationships between these functionals and optimal constants in hyperbolic isoperimetric inequalities. The studied Hardy type inequalities involve weigh functions depending on a hyperbolic radius of a domain and are conformally invariant. We prove that the positivity of Hardy constants is connected with existence of some hyperbolic isoperimetric inequalities of special kind. We also prove a comparison theorem for Hardy constants with different numerical parameters and we study the relationships between the linear hyperbolic isoperimetric inequality in a domain and Euclidean maximum modulus of this domain. In the proofs, an essential role is played by characteristics of domains with uniformly perfect boundary. In addition, we generalize certain results from the papers *J.L. Fernández, J.M. Rodríguez, "The exponent of convergence of Riemann surfaces, bass Riemann surfaces", Ann. Acad. Sci. Fenn. Series A. I. Mathematica.* **15**, *165–183 (1990); V. Alvarez, D. Pestana, J.M. Rodríguez, "Isoperimetric inequalities in Riemann surfaces of infinite type", Revista Matemática Iberoamericana,* **15***:2, 353–425 (1999).*

Keywords: Poincaré metric, hyperbolic isoperimetric inequality, uniformly perfect set, Hardy type inequality.

A.K. Bazzaev, I.D. Tsopanov

Difference schemes for partial differential equations of fractional order

Abstract. Nowadays, fractional differential equations arise while describing physical systems with such properties as power nonlocality, long-term memory and fractal property. The order of the fractional derivative is determined by the dimension of the fractal. Fractional mathematical calculus in the theory of fractals and physical systems with memory and non-locality becomes as important as classical analysis in continuum mechanics.

In this paper we consider higher order difference schemes of approximation for differential equations with fractional-order derivatives with respect to both spatial and time variables. Using the maximum principle, we obtain apriori estimates and prove the stability and the uniform convergence of difference schemes.

Keywords: initial-boundary value problem, fractional differential equations, Caputo fractional derivative, stability, slow diffusion equation, difference scheme, maximum principle, stability, uniform convergence, apriori estimate, heat capacity concentrated at the boundary.

M.Kh. Beshtokov

Boundary value problems for degenerate and degenerate fractional order differential equations with non-local linear source and difference methods for their numerical implementation

Abstract. In the paper we study non-local boundary value problems for differential and partial differential equations of fractional order with a non-local linear source being mathematical models of the transfer of moisture and salts in soils with fractal organization. Apart of the Cartesian case, the paper considers one-dimensional cases with cylindrical and spherical symmetry. By the method of energy inequalities, we obtain apriori estimates of solutions to nonlocal boundary value problems in differential form. We construct difference schemes and for these schemes, we prove analogs of apriori estimates in the difference form and provide estimates for errors assuming a sufficient smoothness of solutions to the equations. By the obtained apriori estimates, we get the uniqueness and stability of the solution with respect to the the initial data and the right par, as well as the convergence of the solution of the difference problem to the solution of the corresponding differential problem with the rate of $O(h^2 + \tau^2)$.

Keywords: boundary value problem, apriori estimate, the equation of moisture transfer, the differential equation of fractional order, Gerasimov-Caputo fractional derivative.

D.I. Borisov, M.N. Konyrkulzhaeva

Simplest graphs with small edges: asymptotics for resolvents and holomorphic dependence of the spectrum

Abstract. In the work we consider a simples graph formed by two finite edges and a small edge coupled at a common vertex. The length of the small edge serves as a small parameter characterizing the perturbation. On such graph, we consider the Schrödinger operator with the Kirchoff condition at the internal vertex, the Dirichlet condition on the boundary vertices of finite edges and the Dirichlet or Neumann condition on the boundary vertice of the small edge. We show that such operators converge to a Schrödinger operator on the graph without the small edge in the norm resolvent sense; at the internal vertex one has to impose the Dirichlet condition if the same was on the boundary vertex of the small edge. If the boundary vertex was subject to the Neumann condition, the internal vertex keeps the Kirchoff condition but the coupling constant can change. The main obtained result for the resolvents is the two-terms asymptotics for their resolvents and an estimate for the error term.

The second part of the work is devoted to studying the dependence of the eigenvalues on the small parameter. Despite the graph is perturbed singularly, the eigenvalues are holomorphic in the small parameter and are represented by convergent series. It is found out that under the perturbation, there can be stable eigenvalues independent of the parameter. We provide a criterion determining the existence of such eigenvalues. For varying eigenvalues we find the leading terms of their Taylor series.

Keywords: Graph, small edge, spectrum, asymptotics

S.Kh. Gekkieva, M.A. Kerefov

Dirichlet boundary value problem for Aller–Lykov moisture transfer equation with time fractional derivative

Abstract. The heat-moisture transfer in soils is a fundamental base in addressing many problems of hydrology, agrophysics, building physics and other fields of science. The researchers focus on possibility of reflecting specific features of the studied arrays in the equations as well as their structure, physical properties, the processes going on in them, etc. In view of this, there arises a new class of fractional differential equations of state and transport being the base for most mathematical models describing a wide class of physical and chemical processes in media with a fractal structure and memory.

This paper studies the Dirichlet boundary value problem for the Aller–Lykov moisture transfer equation with the Riemann–Liouville fractional derivative in time. The considered equation is a generalization of the Aller-Lykov equation obtained by means of introducing the concept of the fractal rate of humidity change, which accounts the presence of flows moving against the moisture potential.

The existence of the solution to the Dirichlet boundary value problem is proved by the Fourier method. By means of energy inequalities method, for the solution we obtain an apriori estimate in terms of fractional Riemann-Liouville derivative, which implies the uniqueness of the solution.

Keywords: Aller–Lykov moisture transfer equation, Riemann–Liouville fractional derivative, Fourier method, apriori estimate.

A.R. Danilin, A.A. Shaburov

Asymptotic expansion of solution to singularly perturbed optimal control problem with a convex quality criterion whose terminal part depends on slow and fast variables

Abstract. We consider an optimal control problem with a convex quality criterion for a linear system with fast and slow variables in the class of piecewise continuous controls with smooth constraints on the control

$$\begin{cases} \dot{x}_\varepsilon = A_{11}x_\varepsilon + A_{12}y_\varepsilon + B_1u, \quad t \in [0,T], \quad \|u\| \leqslant 1, \\ \varepsilon\dot{y}_\varepsilon = A_{22}y_\varepsilon + B_2u, \quad x_\varepsilon(0) = x^0, \quad y_\varepsilon(0) = y^0, \quad \nabla\varphi_2(0) = 0, \\ J(u) := \varphi_1\left(x_\varepsilon(T)\right) + \varphi_2\left(y_\varepsilon(T)\right) + \int\limits_0^T \|u(t)\|^2\,dt \to \min, \end{cases}$$

where $x \in \mathbb{R}^n$, $y \in \mathbb{R}^m$, $u \in \mathbb{R}^r$; A_{ij} and B_i, $i,j = 1,2$, are constant matrices of corresponding dimension, and the functions $\varphi_1(\cdot), \varphi_2(\cdot)$ are continuously differentiable in $\mathbb{R}^n, \mathbb{R}^m$, strictly convex, and cofinite in the sense of the convex analysis. In the general case, for such problem, the Pontryagin maximum principle is a necessary and sufficient optimality condition and there exist unique vectors l_ε and ρ_ε determining an optimal control by the formula

$$u_\varepsilon(T-t) := \frac{C^*_{1,\varepsilon}(t)l_\varepsilon + C^*_{2,\varepsilon}(t)\rho_\varepsilon}{S\left(\|C^*_{1,\varepsilon}(t)l_\varepsilon + C^*_{2,\varepsilon}(t)\rho_\varepsilon\|\right)},$$

where

$$C^*_{1,\varepsilon}(t) := B^*_1 e^{A^*_{11}t} + \varepsilon^{-1} B^*_2 \mathcal{W}^*{}_\varepsilon(t), \quad C^*_{2,\varepsilon}(t) := \varepsilon^{-1} B^*_2 e^{A^*_{22}t/\varepsilon},$$

$$\mathcal{W}_\varepsilon(t) := e^{A_{11}t} \int_0^t e^{-A_{11}\tau} A_{12} e^{A_{22}\tau/\varepsilon}\, d\tau, \quad S(\xi) := \begin{cases} 2, & 0 \leqslant \xi \leqslant 2, \\ \xi, & \xi > 2. \end{cases}$$

The main difference of our problem from the previous papers is that the terminal part of quality criterion depends on the slow and fast variables and the controlled system is a more general form. We prove that in the case of a finite number of control change points, a power asymptotic expansion can be constructed for the initial vector of dual state $\lambda_\varepsilon = (l^*_\varepsilon\ \rho^*_\varepsilon)^*$, which determines the type of the optimal control.

Keywords: optimal control, singularly perturbed problems, asymptotic expansion, small parameter.

K.G. Malyutin, T.I. Malyutina, T.V. Shevtsova

Azarin limiting sets of functions and asymptotic representation of integrals

Abstract. In the paper we consider integrals of form

$$\int_a^b f(t) \exp[i\varphi(rt) \ln(rt)]\, dt\,,$$

where $\varphi(r)$ is a smooth increasing function on the semi-axis $[0, \infty)$ such that $\lim_{r\to+\infty} \varphi(r) = \infty$. We obtain precise information on their asymptotic behavior and we prove an analogue of Riemann-Lebesgue lemma for trigonometric integrals. By applying this lemma, we succeed to obtain the asymptotic formulae for integrals with an absolutely continuous function. The proposed method of obtaining asymptotic formulae differs from classical method like Laplace method, applications of residua, saddle-point method, etc. To make the presentation more solid, we mostly restrict ourselves by the kernels $\exp[i \ln^p(rt)]$. Appropriate smoothness conditions for the function $f(t)$ allow us to obtain many-terms formulae. The properties of the integrals and the methods of obtaining asymptotic estimates differ for the cases $p \in (0, 1)$, $p = 1$, $p > 1$. As $p \in (0, 1)$, the asymptotic expansions are obtained by another method, namely, by expanding the kernel into a series. We consider the cases, when as an absolutely continuous function $f(t)$ we take a product of a power function t^ρ and the Poisson kernel or the conjugate Poisson kernel for the half-plane and as the integration set, the imaginary semi-axis serves. The real and imaginary parts of these integrals are harmonic functions in the complex plane cut along the positive semi-axis. We find the Azarin limiting sets for such functions.

Keywords: Riemann-Lebesgue lemma, trigonometric integral, asymptotic formula, Poisson kernel, harmonic function, Azarin limiting set.

R.G. Salakhudinov

Some properties of functionals on level sets

Abstract. In the paper we consider special functionals on a planar domain G constructed by means of the distance to the boundary ∂G and a classical warping function. The functionals depending on the distance function are considered for simply-connected domains. We also study the functionals depending on the warping function for a finite-connected domain. We prove that the property of isoperimetric monotonicity with respect to a free parameter gives rise to another monotonicity, namely, the monotonicity of the functionals considered as the functions of the sets defined on subsets of the domain. Some partial cases of the inequality were earlier obtained by Payne. We note that the inequalities were successfully applied for justifying new estimates for the torsional rigidity of simply-connected and multiply-connected domains. In particular, we construct new functionals of domains monotone in both its variables. Moreover, we find sharp estimates of variation rate of the functions, that is, we obtain sharp estimates of their derivatives.

Keywords: distance to boundary, warping function, Payne type inequality, isoperimetric inequality, isoperimetric monotonicity.