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EXHAUSTION BY BALLS AND ENTIRE FUNCTIONS OF BOUNDED L-INDEX IN JOINT VARIABLES

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Abstract. For entire functions of several complex variables, we prove criteria of boundedness of $\mathbf{L}$-index in joint variables. Here $\mathbf{L} : \mathbb{C}^n \to \mathbb{R}^n_+$ is a continuous vector function. The criteria describe local behavior of partial derivatives of entire function on sphere in an n-dimensional complex space. Our main result provides an upper bound for maximal absolute value of partial derivatives of entire function on the sphere in terms of the absolute value of the function at the center of the sphere multiplied by some constant. This constant depends only on the radius of sphere and is independent of the location of its center. Some of the obtained results are new even for entire functions with a bounded index in joint variables, i.e., $\mathbf{L}(z) \equiv 1$, because we use an exhaustion of $\mathbb{C}^n$ by balls instead an exhaustion of $\mathbb{C}^n$ by polydiscs. The ball exhaustion is based on Cauchy's integral formula for a ball. Also we weaken sufficient conditions of index boundedness in our main result by replacing an universal quantifier by an existential quantifier. The polydisc analogues of the obtained results are fundamental in theory of entire functions of bounded index in joint variables. They are used for estimating the maximal absolute value by the minimal absolute value, for estimating partial logarithmic derivatives and distribution of zeroes.

Keywords: entire function, ball, bounded $\mathbf{L}$-index in joint variables, maximum modulus, partial derivative, Cauchy's integral formula, geometric exhaustion.

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1. Introduction

Recently, we published two papers [4], [3] devoted to the properties of entire functions of bounded $\mathbf{L}$-index in joint variables. Our studies employed a more general concept of an entire function with bounded $\mathbf{L}$-index in joint variables than in [10], [12]. Unlike these papers, here we suppose that $\mathbf{L}(z) = (l_1(z), \ldots, l_n(z))$ and $z = (z_1, z_2, \ldots, z_n) \in \mathbb{C}^n$, while M.T. Bordulyak and M.M. Sheremeta [10], [12] assumed that $\mathbf{L}(z) = (l_1(|z_1|), \ldots, l_n(|z_n|))$.

The main method we employ to study this subclass of entire functions consists in applying the integral Cauchy's formula and exhaustion of $\mathbb{C}^n$ by polydiscs and its skeletons. This turns out to be very flexible and convenient. This helped us to establish the criteria of the boundedness of $\mathbf{L}$-index in joint variables and this describes a local behavior of partial derivatives and the maximal absolute value of an entire function in a polydisc [4], [3].

Besides a polydisc, an important geometric object in $\mathbb{C}^n$ is a ball. Obviously, $\mathbb{C}^n$ can be exhausted by balls, too. There are two well-known monographs [25], [21] on the spaces of holomorphic functions in the unit ball of $\mathbb{C}^n$: Bergman spaces, Hardy spaces, Besov spaces, Lipschitz spaces, the Bloch space, etc. Authors of these books chose the unit ball because most of the results can be proved here using straightforward formulae without much technicalities. However, in the theory of entire and analytic functions in polydisc of bounded $\mathbf{L}$-index in joint

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variables or of bounded index ($\mathbf{L} \equiv 1$) the situation is different. Many papers were devoted to the polydisc properties, see [10], [12], [16], [17], [18], [22], [13], [14], [15], [5], [6]. In view of this, it is natural to pose the question: *What are the ball properties of entire functions of bounded* $\mathbf{L}$*-index in joint variables?* This problem is considered in the present paper.

We note that for each entire function F with bounded multiplicities of zero points [11], [9], there exists a positive continuous function $\mathbf{L} : \mathbb{C}^n \to \mathbb{R}^n_+$ such that F is of bounded L-index in joint variables. Thus, the concept of bounded L-index allows one to study the properties of a very wide class of entire functions.

In addition to $\mathbf{L}$-index in joint variables, there is another approach to introduce a bounded index in $\mathbb{C}^n$, a so-called L-index bounded in a direction (see more in [1], [2]). This concept uses slice function to study properties of entire functions in $\mathbb{C}^n$.

It should be noted that the concepts of bounded L-index in a direction and bounded $\mathbf{L}$-index in joint variables have few advantages in the comparison with traditional approaches to study properties of entire solutions of differential equations. In particular, if an entire solution has a bounded index [8], [7], [19], this implies immediately estimates for its growth rate, an uniform distribution of its zeros, a certain regular behavior of the solution, etc. A full bibliography on applications in theory of ordinary and partial differential equations can be found in [1], [23].

2. Main definitions and notations

We recall some standard notations. We denote

$$\mathbb{R}_+ = [0, +\infty), \qquad \mathbf{0} = (0, \dots, 0) \in \mathbb{R}^n_+, \qquad \mathbf{e} = (1, \dots, 1) \in \mathbb{R}^n_+,$$

$$R = (r_1, \dots, r_n) \in \mathbb{R}^n_+, \qquad z = (z_1, \dots, z_n) \in \mathbb{C}^n, \qquad |z| = \sqrt{\sum_{j=1}^n |z_j|^2}.$$

For $A = (a_1, \dots, a_n) \in \mathbb{R}^n$, $B = (b_1, \dots, b_n) \in \mathbb{R}^n$, we will use the formal notations

$$AB = (a_1 b_1, \cdots, a_n b_n), \quad A/B = (a_1/b_1, \dots, a_n/b_n),$$
$$A^B = a_1^{b_1} a_2^{b_2} \cdot \ldots \cdot a_n^{b_n}, \quad \|A\| = a_1 + \cdots + a_n,$$

and the notation $A < B$ means that $a_j < b_j$, $j \in \{1, \dots, n\}$; the relation $A \leqslant B$ is defined in a similar way. Given $K = (k_1, \dots, k_n) \in \mathbb{Z}^n_+$, we let $K! = k_1! \cdot \ldots \cdot k_n!$. The summation, scalar multiplication, and conjugation on $\mathbb{C}^n$ are defined componentwise. For $z \in \mathbb{C}^n$ and $w \in \mathbb{C}^n$, we define

$$\langle z, w\rangle = z_1 \overline{w}_1 + \cdots + z_n \overline{w}_n,$$

where w_k is the complex conjugate of w_k. The polydisc $\{z \in \mathbb{C}^n : \ |z_j - z_j^0| < r_j, \ j = 1, \dots, n\}$ is denoted by $\mathbb{D}^n(z^0, R)$, its skeleton $\{z \in \mathbb{C}^n : \ |z_j - z_j^0| = r_j, \ j = 1, \dots, n\}$ is denoted by $\mathbb{T}^n(z^0, R)$, and the closed polydisc $\{z \in \mathbb{C}^n : \ |z_j - z_j^0| \leqslant r_j, \ j = 1, \dots, n\}$ is denoted by $\mathbb{D}^n[z^0, R]$, $\mathbb{D}^n = \mathbb{D}^n(\mathbf{0}, \mathbf{1})$, $\mathbb{D} = \{z \in \mathbb{C} : \ |z| < 1\}$. The open ball $\{z \in \mathbb{C}^n : \ |z - z^0| < r\}$ is denoted by $\mathbb{B}^n(z^0, r)$, its boundary is the sphere $\mathbb{S}^n(z^0, r) = \{z \in \mathbb{C}^n : \ |z - z^0| = r\}$, the closed ball $\{z \in \mathbb{C}^n : \ |z - z^0| \leqslant r\}$ is denoted by $\mathbb{B}^n[z^0, r]$, $\mathbb{B}^n = \mathbb{B}^n(\mathbf{0}, 1)$, $\mathbb{D} = \mathbb{B}^1 = \{z \in \mathbb{C} : \ |z| < 1\}$.

For $K = (k_1, \dots, k_n) \in \mathbb{Z}^n_+$ and the partial derivatives of an entire function $F(z) = F(z_1, \dots, z_n)$ we use the notation

$$F^{(K)}(z) = \frac{\partial^{\|K\|} F}{\partial z^K} = \frac{\partial^{k_1 + \cdots + k_n} f}{\partial z_1^{k_1} \dots \partial z_n^{k_n}}.$$

Let $\mathbf{L}(z) = (l_1(z), \dots, l_n(z))$, where $l_j(z) : \mathbb{C}^n \to \mathbb{R}_+$ is a positive continuous function. We denote

$$\ell(z) = \min_{1 \leqslant j \leqslant n} l_j(z), \qquad \mathcal{L}(z) = \max_{1 \leqslant j \leqslant n} l_j(z).$$

It is obvious that $\ell(z) \leqslant \mathcal{L}(z)$.

An entire function $F\colon \mathbb{C}^n \to \mathbb{C}$ is said to be of a *bounded* **L**-*index in joint variables* if there exists $n_0 \in \mathbb{Z}_+$ such that for all $z \in \mathbb{C}^n$ and for all $J \in \mathbb{Z}^n_+$ the inequality

$$\frac{|F^{(J)}(z)|}{J!\mathbf{L}^J(z)} \leqslant \max\left\{\frac{|F^{(K)}(z)|}{K!\mathbf{L}^K(z)}:\ K \in \mathbb{Z}^n_+,\ \|K\| \leqslant n_0\right\} \tag{1}$$

holds true. The least among such integers n_0 is called the **L**-*index in joint variables of the function* F and is denoted by $N(F, \mathbf{L})$ (see [4]– [12]).

By $Q^n_{\mathbb{B}}$ we denote the class of functions $\mathbf{L}$ obeying the condition

$$(\forall r > 0,\ j \in \{1, \dots, n\}):\ 0 < \lambda_{1,j}(r) \leqslant \lambda_{2,j}(r) < \infty, \tag{2}$$

where

$$\lambda_{1,j}(r) = \inf_{z^0\in\mathbb{C}^n} \inf\left\{\frac{l_j(z)}{l_j(z^0)} : z \in \mathbb{B}^n\left[z^0, r/\ell(z^0)\right]\right\}, \tag{3}$$

$$\lambda_{2,j}(r) = \sup_{z^0\in\mathbb{C}^n} \sup\left\{\frac{l_j(z)}{l_j(z^0)} : z \in \mathbb{B}^n\left[z^0, r/\ell(z^0)\right]\right\}. \tag{4}$$

$$\Lambda_1(r) = (\lambda_{1,1}(r), \dots, \lambda_{1,n}(r)), \qquad \Lambda_2(r) = (\lambda_{2,1}(r), \dots, \lambda_{2,n}(r)).$$

It is not difficult to confirm that the class $Q^n_{\mathbb{B}}$ can be defined as following:

$$\sup_{z,w\in\mathbb{C}^n}\left\{\frac{l_j(z)}{l_j(w)}\colon |z-w| \leqslant \frac{\eta}{\min\{\ell(z), \ell(w)\}}\right\} < \infty \quad \text{for all} \quad j \in \{1, \dots, n\}, \tag{5}$$

i. e. conditions (2) and (5) are equivalent. Moreover, $Q^n_{\mathbb{B}} \subset Q^n$ (see the definition of Q^n in [4], [1]).

We also need the following lemma, see [25, Lm. 1.11].

Lemma 1. *Suppose $m = (m_1, \dots, m_n)$ is a multi-index of nonnegative integers. Then*

$$\int_{\mathbb{S}_n} |\xi^m|^2 d\sigma(\xi) = \frac{(n-1)!m!}{(n-1+\|m\|)!} = \frac{\Gamma(n)\prod_{j=1}^n \Gamma(m_j+1)}{\Gamma(n+\|m\|)}.$$

Using the idea of the proof of Lemma 1 and the definition of the Gamma function, it is easy to prove a more general formula

$$\int_{\mathbb{S}_n} |\xi^m| d\sigma(\xi) = \frac{\Gamma(n)\prod_{j=1}^n \Gamma(m_j/2+1)}{\Gamma(n+\|m\|/2)} \quad \text{for each} \quad m \in \mathbb{Z}^n_+. \tag{6}$$

Indeed, we identify $\mathbb{C}^n$ with $\mathbb{R}^{2n}$ and let dV be the usual Lebesgue measure on $\mathbb{C}^n$, dv be the normalized volume measure on $\mathbb{B}^n$ so that $v(\mathbb{B}^n = 1$, c_n is the Euclidean volume of $\mathbb{B}^n$, i.e., $c_n dv = dV$.

We find the integral $I = \int_{\mathbb{C}^n} |z^m| e^{-|z|^2} dV(z)$ in two steps. First, by employing Fubini's theorem and the substitution

$$x = \sqrt{r}\cos\varphi, \qquad y = \sqrt{r}\sin\varphi, \qquad r \in [0, \infty), \varphi \in [0, 2\pi]$$

give

$$I = \prod_{k=1}^n \int_{\mathbb{R}^2} (x^2+y^2)^{m_k/2} e^{-(x^2+y^2)} dxdy = \pi^n \prod_{k=1}^n \int_0^\infty r^{m_k/2} e^{-r} dr = \pi^n \prod_{j=1}^n \Gamma(m_j/2+1).$$

Then we pass to the polar coordinates and we obtain

$$\begin{aligned} I =& 2nc_n \int_0^\infty r^{2n-1+\|m\|} e^{-r^2} dr \int_{\mathbb{S}^n} |\xi^m| d\sigma(\xi) \\ =& nc_n \int_0^\infty t^{n-1+\|m\|/2} e^{-t} dt \int_{\mathbb{S}^n} |\xi^m| d\sigma(\xi) \end{aligned}$$

$$=nc_n\Gamma(n+\|m\|/2)\int_{\mathbb{S}^n}|\xi^m|d\sigma(\xi).$$

Two obtained identities yield

$$\int_{\mathbb{S}^n}|\xi^m|d\sigma(\xi)=\frac{\pi^n\prod_{j=1}^n\Gamma(m_j/2+1)}{nc_n\Gamma(n+\|m\|/2)}. \tag{7}$$

Let $m=\mathbf{0}=(0,\dots,0)$. Then $1=\frac{\pi^n}{nc_n(n-1)!}$, i.e., $c_n=\frac{\pi^n}{n!}$. Substituting c_n in (7) we arrive at (6).

3. Local behavior of derivatives of function of bounded L-index in joint variables

The following theorem is basic in theory of functions of bounded index. It is motivated by the need to prove a more efficient criteria of index boundedness, which describes a behavior of maximal absolute value on a disc or a behavior of the logarithmic derivative (see [1], [23], [24]).

Theorem 1. *Let $\mathbf{L}\in Q^n_{\mathbb{B}}$. If an entire in $\mathbb{C}^n$ function F is of bounded $\mathbf{L}$-index in joint variables then for each $r>0$ there exist $n_0\in\mathbb{Z}_+$, $p_0>0$ such that for each $z^0\in\mathbb{C}^n$ there exists $K^0\in\mathbb{Z}^n_+$, $\|K^0\|\leqslant n_0$, such that*

$$\max\left\{\frac{|F^{(K)}(z)|}{K!\mathbf{L}^K(z)}\colon \|K\|\leqslant n_0,\ z\in\mathbb{B}_n\left[z^0,r/\mathcal{L}(z^0)\right]\right\}\leqslant p_0\frac{|F^{(K^0)}(z^0)|}{K^0!\mathbf{L}^{K^0}(z^0)}. \tag{8}$$

If for each $r>0$ there exist $n_0\in\mathbb{Z}_+$, $p_0>0$ such that for every $z^0\in\mathbb{C}^n$ there exists $K^0\in\mathbb{Z}^n_+$, $\|K^0\|\leqslant n_0$, such that

$$\max\left\{\frac{|F^{(K)}(z)|}{K!\mathbf{L}^K(z)}\colon \|K\|\leqslant n_0,\ z\in\mathbb{B}_n\left[z^0,r/\ell(z^0)\right]\right\}\leqslant p_0\frac{|F^{(K^0)}(z^0)|}{K^0!\mathbf{L}^{K^0}(z^0)}, \tag{9}$$

then the entire in $\mathbb{C}^n$ function F is of bounded $\mathbf{L}$-index in joint variables.

Proof. Let F be of bounded $\mathbf{L}$-index in joint variables with $N=N(F,\mathbf{L})<\infty$. For each $r>0$ we let

$$q=q(r)=[2(N+1)r\sqrt{n}\prod_{j=1}^n(\lambda_{1,j}(r))^{-N}(\lambda_{2,j}(r))^{N+1}]+1$$

where $[x]$ is the integer part of the real number x. For $p\in\{0,\dots,q\}$ and $z^0\in\mathbb{C}^n$ we denote

$$S_p(z^0,r)=\max\left\{\frac{|F^{(K)}(z)|}{K!\mathbf{L}^K(z)}\colon\ \|K\|\leqslant N,\ z\in\mathbb{B}^n\left[z^0,\frac{pr}{q\mathcal{L}(z^0)}\right]\right\},$$
$$S^*_p(z^0,r)=\max\left\{\frac{|F^{(K)}(z)|}{K!\mathbf{L}^K(z^0)}\colon \|K\|\leqslant N,\ z\in\mathbb{B}^n\left[z^0,\frac{pr}{q\mathcal{L}(z^0)}\right]\right\}.$$

By (3) and

$$\mathbb{B}^n\left[z^0,\frac{pr}{q\ell(z^0)}\right]\subset\mathbb{B}^n\left[z^0,\frac{r}{\mathcal{L}(z^0)}\right]$$

we have

$$\begin{aligned}S_p(z^0,r)=&\max\left\{\frac{|F^{(K)}(z)|}{K!\mathbf{L}^K(z)}\frac{\mathbf{L}^K(z^0)}{\mathbf{L}^K(z^0)}\colon \|K\|\leqslant N, z\in\mathbb{B}^n\left[z^0,\frac{pr}{q\mathcal{L}(z^0)}\right]\right\}\\ \leqslant& S^*_p(z^0,r)\max\left\{\prod_{j=1}^n\frac{l_j^N(z^0)}{l_j^N(z)}\colon z\in\mathbb{B}^n\left[z^0,\frac{pr}{q\mathcal{L}(z^0)}\right]\right\}\\ \leqslant& S^*_p(z^0,r)\prod_{j=1}^n(\lambda_{1,j}(r))^{-N}.\end{aligned}$$

Using (4), we obtain

$$\begin{aligned} S_p^*(z^0,r) =&\max\left\{\frac{|F^{(K)}(z)|}{K!\mathbf{L}^K(z)}\frac{\mathbf{L}^K(z)}{\mathbf{L}^K(z^0)}: \|K\|\leqslant N, z\in\mathbb{B}^n\left[z^0,\frac{pr}{q\mathcal{L}(z^0)}\right]\right\} \\ \leqslant&\max\left\{\frac{|F^{(K)}(z)|}{K!\mathbf{L}^K(z)}(\Lambda_2(r))^K: \|K\|\leqslant N, z\in\mathbb{B}^n\left[z^0,\frac{pr}{q\mathcal{L}(z^0)}\right]\right\} \\ \leqslant& S_p(z^0,r)\prod_{j=1}^n(\lambda_{2,j}(r))^N. \end{aligned} \tag{10}$$

Let $K^{(p)}$ with $\|K^{(p)}\|\leqslant N$ and $z^{(p)}\in\mathbb{B}^n\left[z^0,\frac{pr}{q\mathcal{L}(z^0)}\right]$ be such that

$$S_p^*(z^0,r)=\frac{|F^{(K^{(p)})}(z^{(p)})|}{K^{(p)}!\mathbf{L}^{K^{(p)}}(z^0)} \tag{11}$$

Since by the maximum principle $z^{(p)}\in\mathbb{S}^n(z^0,\frac{pr}{q\mathcal{L}(z^0)})$, we have $z^{(p)}\neq z^0$. We choose

$$\widetilde{z}_j^{(p)}=z_j^0+\frac{p-1}{p}(z_j^{(p)}-z_j^0),\qquad j\in\{1,\dots,n\}.$$

Hence,

$$|\widetilde{z}^{(p)}-z^0|=\frac{p-1}{p}|z^{(p)}-z^0|=\frac{p-1}{p}\frac{pr}{q\mathcal{L}(z^0)}, \tag{12}$$

$$|\widetilde{z}^{(p)}-z^{(p)}|=|z^0+\frac{p-1}{p}(z^{(p)}-z^0)-z^{(p)}|=\frac{1}{p}|z^0-z^{(p)}|=\frac{r}{q\mathcal{L}(z^0)}. \tag{13}$$

By (12) we obtain $\widetilde{z}^{(p)}\in\mathbb{B}^n\left[z^0,\frac{(p-1)r}{q\mathcal{L}(z^0)}\right]$ and

$$S_{p-1}^*(z^0,r)\geqslant\frac{|F^{(K^{(p)})}(\widetilde{z}^{(p)})|}{K^{(p)}!\mathbf{L}^{K^{(p)}}(z^0)}.$$

It follows from (11) that

$$\begin{aligned} 0\leqslant& S_p^*(z^0,r)-S_{p-1}^*(z^0,r)\leqslant\frac{|F^{(K^{(p)})}(z^{(p)})|-|F^{(K^{(p)})}(\widetilde{z}^{(p)})|}{K^{(p)}!\mathbf{L}^{K^{(p)}}(z^0)} \\ =&\frac{1}{K^{(p)}!\mathbf{L}^{K^{(p)}}(z^0)}\int_0^1\frac{d}{dt}|F^{(K^{(p)})}(\widetilde{z}^{(p)}+t(z^{(p)}-\widetilde{z}^{(p)}))|dt \\ \leqslant&\frac{1}{K^{(p)}!\mathbf{L}^{K^{(p)}}(z^0)}\int_0^1\sum_{j=1}^n|z_j^{(p)}-\widetilde{z}_j^{(p)}||F^{K^{(p)}+\mathbf{1}_j}(\widetilde{z}^{(p)}+t(z^{(p)}-\widetilde{z}^{(p)}))|dt \\ =&\frac{1}{K^{(p)}!\mathbf{L}^{K^{(p)}}(z^0)}\sum_{j=1}^n|z_j^{(p)}-\widetilde{z}_j^{(p)}||F^{K^{(p)}+\mathbf{1}_j}(\widetilde{z}^{(p)}+t^*(z^{(p)}-\widetilde{z}^{(p)}))|, \end{aligned} \tag{14}$$

where

$$0\leqslant t^*\leqslant 1,\qquad \widetilde{z}^{(p)}+t^*(z^{(p)}-\widetilde{z}^{(p)})\in\mathbb{B}^n(z^0,\frac{pr}{q\mathcal{L}(z^0)}).$$

For $z\in\mathbb{B}^n(z^0,\frac{pr}{q\mathcal{L}(z^0)})$ and $J\in\mathbb{Z}_+^n$, $\|J\|\leqslant N+1$ we have

$$\begin{aligned} \frac{|F^{(J)}(z)|\mathbf{L}^J(z)}{J!\mathbf{L}^J(z^0)\mathbf{L}^J(z)}\leqslant&(\Lambda_2(r))^J\max\left\{\frac{|F^{(K)}(z)|}{K!\mathbf{L}^K(z)}:\|K\|\leqslant N\right\} \\ \leqslant&\prod_{j=1}^n(\lambda_{2,j}(r))^{N+1}(\lambda_{1,j}(r))^{-N}\max\left\{\frac{|F^{(K)}(z)|}{K!\mathbf{L}^K(z^0)}:\|K\|\leqslant N\right\} \end{aligned}$$

$$\leqslant \prod_{j=1}^{n} (\lambda_{2,j}(r))^{N+1} (\lambda_{1,j}(r))^{-N} S_p^*(z^0, r).$$

By (14) and (13) we obtain

$$\begin{aligned} 0 \leqslant & S_p^*(z^0, r) - S_{p-1}^*(z^0, r) \\ \leqslant & \prod_{j=1}^{n} (\lambda_{2,j}(r))^{N+1} (\lambda_{1,j}(r))^{-N} S_p^*(z^0, r) \sum_{j=1}^{n} (k_j^{(p)} + 1) l_j(z^0) |z_j^{(p)} - \widetilde{z}_j^{(p)}| \\ = & \prod_{j=1}^{n} (\lambda_{2,j}(r))^{N+1} (\lambda_{1,j}(r))^{-N} S_p^*(z^0, r)(N+1) \sum_{j=1}^{n} l_j(z^0) |z_j^{(p)} - \widetilde{z}_j^{(p)}| \\ \leqslant & \prod_{j=1}^{n} (\lambda_{2,j}(r))^{N+1} (\lambda_{1,j}(r))^{-N} (N+1) S_p^*(z^0, R) \sqrt{n} \mathcal{L}(z^0) |z^{(p)} - \widetilde{z}^{(p)}| \\ = & \prod_{j=1}^{n} (\lambda_{2,j}(r))^{N+1} (\lambda_{1,j}(r))^{-N} \sqrt{n} \frac{(N+1)r}{q(r)} S_p^*(z^0, R) \leqslant \frac{1}{2} S_p^*(z^0, R). \end{aligned}$$

This inequality implies $S_p^*(z^0, r) \leqslant 2S_{p-1}^*(z^0, r)$, and in view of inequalities (10) and (11) we get:

$$S_p(z^0, r) \leqslant 2 \prod_{j=1}^{n} (\lambda_{1,j}(r))^{-N} S_{p-1}^*(z^0, r) \leqslant 2 \prod_{j=1}^{n} (\lambda_{1,j}(r))^{-N} (\lambda_{2,j}(r))^N S_{p-1}(z^0, r).$$

Therefore,

$$\begin{aligned} & \max \left\{ \frac{|F^{(K)}(z)|}{K! \mathbf{L}^K(z)} : \|K\| \leqslant N, z \in \mathbb{B}_n \left[z^0, \frac{pr}{q\mathcal{L}(z^0)} \right] \right\} \\ & \quad = S_q(z^0, r) \leqslant 2 \prod_{j=1}^{n} (\lambda_{1,j}(r))^{-N} (\lambda_{2,j}(r))^N S_{q-1}(z^0, r) \leqslant \ldots \\ & \quad \leqslant (2 \prod_{j=1}^{n} (\lambda_{1,j}(r))^{-N} (\lambda_{2,j}(r))^N)^q S_0(z^0, r) \\ & \quad = (2 \prod_{j=1}^{n} (\lambda_{1,j}(r))^{-N} (\lambda_{2,j}(r))^N)^q \max \left\{ \frac{|F^{(K)}(z^0)|}{K! \mathbf{L}^K(z^0)} : \|K\| \leqslant N \right\}. \end{aligned} \tag{15}$$

The above inequalities imply (8) with $p_0 = (2 \prod_{j=1}^{n} (\lambda_{1,j}(r))^{-N} (\lambda_{2,j}(r))^N)^q$ and some K^0 with $\|K^0\| \leqslant N$. The necessity of condition (8) is proved.

We proceed to proving the sufficient condition. Assume that for each $r > 0$ there exist $n_0 \in \mathbb{Z}_+$, $p_0 > 1$ such that for all $z_0 \in \mathbb{C}^n$ and some $K^0 \in \mathbb{Z}_+^n$, $\|K^0\| \leqslant n_0$, inequality (9) holds true. We write Cauchy formula for a ball (see [25, p. 109] or [21, p. 349]) as follows: for all $z^0 \in \mathbb{C}^n$, $K \in \mathbb{Z}_+^n$, $S \in \mathbb{Z}_+^n$, $z \in \mathbb{B}^n(z^0, r/\ell(z^0))$ the identity

$$F^{(K+S)}(z) = \frac{(n + \|S\| - 1)!}{(n-1)!} \int\limits_{\mathbb{S}^n(z^0, r/\ell(z^0))} \frac{|\xi - z^0| (\overline{\xi - z^0})^S F^{(K)}(\xi)}{(|\xi - z^0|^2 - \langle z - z^0, \xi - z^0 \rangle)^{n + \|S\|}} d\sigma(\xi)$$

holds true, where $d\sigma(\xi)$ is the normalized surface measure on $\mathbb{S}_n$ so that $\sigma(\mathbb{S}_n(\mathbf{0}, 1)) = 1$. We choose $z = z^0$ to obtain:

$$F^{(K+S)}(z^0) = \frac{(n + \|S\| - 1)!}{(n-1)!} \int_{\mathbb{S}^n(z^0, r/\ell(z^0))} \frac{(\overline{\xi - z^0})^S F^{(K)}(\xi)}{|\xi - z^0|^{2(n + \|S\|) - 1}} d\sigma(\xi) \tag{16}$$

Therefore, thanks to (9) and (6), we get

$$
\begin{aligned}
|F^{(K+S)}(z^0)| \leqslant & \frac{(n+\|S\|-1)!}{(n-1)!}\int_{\mathbb{S}^n(z^0,r/\ell(z^0))}\frac{|(\xi-z^0)^S||F^{(K)}(\xi)|}{|\xi-z^0|^{2(n+\|S\|)-1}}d\sigma(\xi)\\
\leqslant & \left(\frac{\ell(z^0)}{r}\right)^{2(n+\|S\|)-1}\frac{(n+\|S\|-1)!}{(n-1)!}\\
& \cdot\int_{\mathbb{S}^n(z^0,r/\ell(z^0))}\frac{|(\xi-z^0)^S||F^{(K)}(\xi)|K!\mathbf{L}^K(\xi)}{K!\mathbf{L}^K(\xi)}d\sigma(\xi)\\
\leqslant & p_0\left(\frac{\ell(z^0)}{r}\right)^{2(n+\|S\|)-1}\frac{(n+\|S\|-1)!}{(n-1)!}\\
& \cdot\int_{\mathbb{S}^n(z^0,r/\ell(z^0))}\frac{|(\xi-z^0)^S||F^{(K^0)}(z^0)|K!\mathbf{L}^K(z)}{K^0!\mathbf{L}^{K^0}(z^0)}d\sigma(\xi)\\
\leqslant & p_0\left(\frac{\ell(z^0)}{r}\right)^{2(n+\|S\|)-1}\frac{(n+\|S\|-1)!}{(n-1)!}\frac{|F^{(K^0)}(z^0)|K!\prod_{j=1}^n\lambda_{2,j}^{n_0}(r)\mathbf{L}^K(z^0)}{K^0!\mathbf{L}^{K^0}(z^0)}\\
& \cdot\int_{\mathbb{S}^n(z^0,r/\ell(z^0))}|(\xi-z^0)^S|d\sigma(\xi)\leqslant \qquad (17)\\
\leqslant & p_0\left(\frac{\ell(z^0)}{r}\right)^{\|S\|}\frac{(n+\|S\|-1)!}{(n-1)!}\frac{|F^{(K^0)}(z^0)|K!\prod_{j=1}^n\lambda_{2,j}^{n_0}(r)\mathbf{L}^K(z^0)}{K^0!\mathbf{L}^{K^0}(z^0)}\\
& \cdot\int_{\mathbb{S}^n(z^0,r/\ell(z^0))}\frac{|(\xi-z^0)^S|}{(r/\ell(z^0))^{\|S\|}}d\sigma\left(\frac{\xi-z^0}{r/\ell(z^0)}\right)\\
\leqslant & p_0\left(\frac{\ell(z^0)}{r}\right)^{\|S\|}\frac{(n+\|S\|-1)!}{(n-1)!}\frac{|F^{(K^0)}(z^0)|K!\prod_{j=1}^n\lambda_{2,j}^{n_0}(r)\mathbf{L}^K(z^0)}{K^0!\mathbf{L}^{K^0}(z^0)}\\
& \cdot\int_{\mathbb{S}^n(\mathbf{0},1)}|\xi^S|d\sigma(\xi)\\
= & p_0\left(\frac{\ell(z^0)}{r}\right)^{\|S\|}\frac{(n+\|S\|-1)!}{(n-1)!}\\
& \cdot\frac{|F^{(K^0)}(z^0)|K!\prod_{j=1}^n\lambda_{2,j}^{n_0}(r)\mathbf{L}^K(z^0)}{K^0!\mathbf{L}^{K^0}(z^0)}\frac{\Gamma(n)\prod_{j=1}^n\Gamma(s_j/2+1)}{\Gamma(n+\|S\|/2)}
\end{aligned}
$$

This implies

$$
\begin{aligned}
\frac{|F^{(K+S)}(z^0)|}{(K+S)!\mathbf{L}^{K+S}(z^0)} \leqslant & \frac{|F^{(K^0)}(z^0)|}{K^0!\mathbf{L}^{K^0}(z^0)}p_0\left(\frac{\ell(z^0)}{r}\right)^{\|S\|}\\
& \cdot\frac{K!\prod_{j=1}^n\lambda_{2,j}^{n_0}(r)(n+\|S\|-1)!\prod_{j=1}^n\Gamma(s_j/2+1)}{(K+S)!\Gamma(n+\|S\|/2)\mathbf{L}^S(z^0)}\\
\leqslant & \frac{|F^{(K^0)}(z^0)|}{K^0!\mathbf{L}^{K^0}(z^0)}p_0\frac{K!\prod_{j=1}^n\lambda_{2,j}^{n_0}(r)(n+\|S\|-1)!\prod_{j=1}^n\Gamma(s_j/2+1)}{(K+S)!\Gamma(n+\|S\|/2)r^{\|S\|}}
\end{aligned} \qquad (18)
$$

We choose $r>1$. Since $\|K\|\leqslant n_0$, the quantity $p_0K!\prod_{j=1}^n\lambda_{2,j}^{n_0}(R)$ is independent of S. Hence, there exists n_1 such that

$$
\frac{p_0K!\prod_{j=1}^n\lambda_{2,j}^{n_0}(r)}{r^{\|S\|}}\leqslant 1 \quad \text{for all} \quad \|S\|\geqslant n_1. \qquad (19)
$$

The asymptotic behavior of $\frac{(n+\|S\|-1)!\prod_{j=1}^n\Gamma(s_j/2+1)}{(K+S)!\Gamma(n+\|S\|/2)r^{\|S\|}}$ is more complicated as $\|S\|\to+\infty$. Using the Stirling formula

$$\Gamma(m+1)=\sqrt{2\pi m}\left(\frac{m}{e}\right)^m(1+\frac{\theta}{12m}),$$

where $\theta=\theta(m)\in[0,1]$, we obtain

$$\begin{aligned}\frac{(n+\|S\|-1)!\prod_{j=1}^n\Gamma(s_j/2+1)}{(K+S)!\Gamma(n+\|S\|/2)r^{\|S\|}} &\leqslant \frac{(n+\|S\|-1)!\prod_{j=1}^n\Gamma(s_j/2+1)}{S!\Gamma(n+\|S\|/2)r^{\|S\|}}\\ &= \frac{\sqrt{2\pi(n+\|S\|-1)}(\frac{n+\|S\|-1}{e})^{n+\|S\|-1}\prod_{j=1}^n\sqrt{2\pi s_j/2}(\frac{s_j}{2e})^{s_j/2}}{\prod_{j=1}^n\sqrt{2\pi s_j}(\frac{s_j}{e})^{s_j}\sqrt{2\pi(n+\|S\|/2-1)}(\frac{n+\|S\|/2-1}{e})^{n+\|S\|/2-1}r^{\|S\|}}\\ &\cdot\frac{(1+\frac{\theta(n+\|S\|-1)}{12(n+\|S\|-1)})\prod_{j=1}^n(1+\frac{\theta(s_j/2)}{12s_j/2})}{(1+\frac{\theta(n+\|S\|/2)}{12(n+\|S\|/2)})\prod_{j=1}^n(1+\frac{\theta(s_j)}{12s_j})}.\end{aligned}$$

We denote

$$\Theta(S)=\frac{(1+\frac{\theta(n+\|S\|-1)}{12(n+\|S\|-1)})\prod_{j=1}^n(1+\frac{\theta(s_j/2)}{12s_j/2})}{(1+\frac{\theta(n+\|S\|/2)}{12(n+\|S\|/2)})\prod_{j=1}^n(1+\frac{\theta(s_j)}{12s_j})}$$

and simplify the previous inequality to obtain

$$\begin{aligned}&\frac{(n+\|S\|-1)!\prod_{j=1}^n\Gamma(s_j/2+1)}{(K+S)!\Gamma(n+\|S\|/2)r^{\|S\|}}\\ &\leqslant\Theta(S)\frac{2^{(1-n)/2}e^{-\|S\|/2}}{r^{\|S\|}}\left(\frac{n-1+\|S\|}{n-1+\|S\|/2}\right)^{n-1+\|S\|/2}(n-1+\|S\|)^{\|S\|/2}\prod_{j=1}^n(\frac{e}{2s_j})^{s_j/2}\\ &\leqslant\Theta(S)\frac{2^{(n-1+\|S\|)/2}e^{-\|S\|/2}}{r^{\|S\|}}(n-1+\|S\|)^{\|S\|/2}\prod_{j=1}^n(\frac{e}{2s_j})^{s_j/2}\\ &=\Theta(S)\frac{2^{(n-1)/2}}{r^{\|S\|}}\left(1+\frac{n-1}{\|S\|}\right)^{\frac{\|S\|}{n-1}\cdot\frac{n-1}{2}}\cdot\|S\|^{\|S\|/2}\prod_{j=1}^n\frac{1}{s_j^{s_j/2}}\\ &\leqslant\Theta(S)(2e)^{(n-1)/2}\left(\frac{1}{r}\prod_{j=1}^n\left(\frac{\|S\|}{s_j}\right)^{\frac{s_j}{2\|S\|}}\right)^{\|S\|}\quad\text{as}\quad s_j\to\infty.\end{aligned}\tag{20}$$

We denote $x_j=\frac{\|S\|}{s_j}\in(1,+\infty)$, $x=(x_1,\dots,x_n)$. It is clear that $\Theta(S)\to1$ as $s_j\to\infty$, $j\in\{1,\dots,n\}$. Then (20) implies the constrained optimization problem

$$H(x):=\prod_{j=1}^n x_j^{\frac{1}{2x_j}}\to\max,\qquad\sum_{j=1}^n\frac{1}{x_j}=1,\quad x_j\in(1,+\infty).\tag{21}$$

If this problem is solvable, then $H(x)$ does not exceed some H^* and we choose $r>H^*$ in (20).

We introduce a Lagrange multiplier λ and we study the Lagrange function $\mathcal{L}(x,\lambda)$ defined by

$$\mathcal{L}(x,\lambda)=\prod_{j=1}^n x_j^{\frac{1}{2x_j}}+\lambda\left(\sum_{j=1}^n\frac{1}{x_j}-1\right).$$

A necessary condition for optimality in constrained problems yields

$$\frac{\partial\mathcal{L}}{\partial x_j}=\frac{1-\ln x_j}{2x_j^2}\prod_{k=1}^n x_k^{\frac{1}{2x_k}}-\frac{\lambda}{x_j^2}=0$$

or

$$\frac{1-\ln x_j}{2} = \frac{\lambda}{\prod_{k=1}^{n} x_k^{\frac{1}{2x_k}}}$$

Hence,

$$x_j = \exp\left(1 - 2\frac{\lambda}{\prod_{k=1}^{n} x_k^{\frac{1}{2x_k}}}\right),$$

i.e., $x_1 = x_2 = \ldots = x_n$. Constraint (21) implies

$$\sum_{j=1}^{n} \frac{1}{x_j} = \frac{n}{x_1} = 1$$

or $x_j = n$ for each $j \in \{1, \ldots, n\}$. Hence, $H(x) \leqslant \prod_{j=1}^{n} n^{1/(2n)} = \sqrt{n}$.

We choose $r \geqslant \sqrt{n}$. For this r we have

$$\frac{1}{r}\prod_{j=1}^{n}\left(\frac{\|S\|}{s_j}\right)^{\frac{s_j}{2\|S\|}} \leqslant 1.$$

In view of (20), this implies the existence of n_2 such that

$$\frac{(n+\|S\|-1)!\prod_{j=1}^{n}\Gamma(s_j/2+1)}{(K+S)!\Gamma(n+\|S\|/2)r^{\|S\|}} \leqslant 1 \tag{22}$$

for all $\|S\| \geqslant n_2$.

The asymptotic behavior of the right hand side in (18) for other S can be studied in the same way. Taking into consideration (18), (19) and (22), for all $\|S\| \geqslant n_1 + n_2$ we get

$$\frac{|F^{(K+S)}(z^0)|}{(K+S)!\mathbf{L}^{S+K}(z^0)} \leqslant \frac{|F^{(K^0)}(z^0)|}{K^0!\mathbf{L}^{K^0}(z^0)}.$$

This means that

$$\frac{|F^{(J)}(z^0)|}{J!\mathbf{L}^{J}(z^0)} \leqslant \max\left\{\frac{|F^{(K)}(z^0)|}{K!\mathbf{L}^{K}(z^0)} : \|K\| \leqslant n_0 + n_1 + n_2\right\}$$

for each $J \in \mathbb{Z}_+^n$, where n_0, n_1, n_2 are independent of z_0. Therefore, the function F has bounded $\mathbf{L}$-index in joint variables with $N(F, \mathbf{L}) \leqslant n_0 + n_1 + n_2$. The proof is complete. □

Imposing an additional constraint for the function $\mathbf{L}$, by Theorem 1 we arrive at the following criterion.

Theorem 2. *Let $\mathbf{L} \in Q_{\mathbb{B}}^n$ be such that*

$$\sup_{z\in\mathbb{C}^n} \frac{\mathcal{L}(z)}{\ell(z)} = C < \infty.$$

An entire function F has bounded $\mathbf{L}$-index in joint variables if and only if for each $r > 0$ there exist $n_0 \in \mathbb{Z}_+$, $p_0 > 0$ such that for each $z^0 \in \mathbb{C}^n$ there exists $K^0 \in \mathbb{Z}_+^n$, $\|K^0\| \leqslant n_0$ such that inequality (9) *is satisfied.*

Proof. The sufficiency was proved in Theorem 1. To prove the necessity, we choose

$$q = q(R) = [2(N+1)Cr\sqrt{n}\prod_{j=1}^{n}(\lambda_{1,j}(r))^{-N}(\lambda_{2,j}(r))^{N+1}] + 1$$

and we replace $\mathcal{L}(z^0)$ by $\ell(z^0)$ in the proof of Theorem 1. No other changes are needed and this proves this theorem. □

Theorem 3. *Let* $\mathbf{L} \in Q_{\mathbb{B}}^n$. *If an entire function* F *has bounded* $\mathbf{L}$*-index in joint variables, then for all* $r > 0$ *there exist* $n_0 \in \mathbb{Z}_+$, $p \geqslant 1$ *such that for all* $z^0 \in \mathbb{C}^n$ *there exists* $K^0 \in \mathbb{Z}_+^n$ *obeying* $\|K^0\| \leqslant n_0$ *and*

$$\max\left\{|F^{(K^0)}(z)| : z \in \mathbb{B}^n\left[z^0, r/\mathcal{L}(z^0)\right]\right\} \leqslant p|F^{(K^0)}(z^0)|. \tag{23}$$

If for all $r > 0$ *there exists* $n_0 \in \mathbb{Z}_+$, $p \geqslant 1$ *such that for all* $z^0 \in \mathbb{C}^n$, $j \in \{1, \ldots, n\}$ *there exists* $K_j^0 = (0, \ldots, 0, \underbrace{k_j^0}_{j\text{-th place}}, 0, \ldots, 0)$ *such that* $k_j^0 \leqslant n_0$ *and*

$$\max\left\{|F^{(K_j^0)}(z)| : z \in \mathbb{B}^n\left[z^0, r/\ell(z^0)\right]\right\} \leqslant p|F^{(K_j^0)}(z^0)| \qquad \text{for all} \quad j \in \{1, \ldots, n\}, \tag{24}$$

then the entire function F *has bounded* $\mathbf{L}$*-index in joint variables.*

Proof. Proof of Theorem 1 implies that inequality (8) is true for some K^0. Therefore, we have

$$\begin{aligned}
\frac{p_0}{K^0!}\frac{|F^{(K^0)}(z^0)|}{\mathbf{L}^{K^0}(z^0)} &\geqslant \max\left\{\frac{|F^{(K^0)}(z)|}{K^0!\mathbf{L}^{K^0}(z)} : z \in \mathbb{B}^n\left[z^0, r/\mathcal{L}(z^0)\right]\right\} \\
&= \max\left\{\frac{|F^{(K^0)}(z)|}{K^0!}\frac{\mathbf{L}^{K^0}(z^0)}{\mathbf{L}^{K^0}(z^0)\mathbf{L}^{K^0}(z)} : z \in \mathbb{B}^n\left[z^0, r/\mathcal{L}(z^0)\right]\right\} \\
&\geqslant \max\left\{\frac{|F^{(K^0)}(z)|}{K^0!}\frac{\prod_{j=1}^n(\lambda_{2,j}(r))^{-n_0}}{\mathbf{L}^{K^0}(z^0)} : z \in \mathbb{B}^n\left[z^0, r/\mathcal{L}(z^0)\right]\right\}.
\end{aligned}$$

This inequality implies

$$\frac{p_0\prod_{j=1}^n(\lambda_{2,j}(r))^{n_0}}{K^0!}\frac{|F^{(K^0)}(z^0)|}{\mathbf{L}^{K^0}(z^0)} \geqslant \max\left\{\frac{|F^{(K^0)}(z)|}{K^0!\mathbf{L}^{K^0}(z^0)} : z \in \mathbb{B}^n\left[z^0, r/\mathcal{L}(z^0)\right]\right\}. \tag{25}$$

Thanks to the obtained inequality we arrive to inequality (23) with $p = p_0\prod_{j=1}^n(\lambda_{2,j}(r))^{n_0}$. The necessity of condition (23) is proved.

We proceed to proving the sufficiency of (24). Assume that for each $r > 0$ there exists $n_0 \in \mathbb{Z}_+$, $p > 1$ such that for all $z_0 \in \mathbb{C}^n$ and some $K_j^0 \in \mathbb{Z}_+^n$ with $k_j^0 \leqslant n_0$ inequality (24) is satisfied.

In view of (16), we write Cauchy formula as follows: for all $z^0 \in \mathbb{C}^n$, $S \in \mathbb{Z}_+^n$ we have

$$F^{(K_j^0+S)}(z^0) = \frac{(n+\|S\|-1)!}{(n-1)!}\int_{\mathbb{S}^n(z^0,r/\ell(z^0))}\frac{(\overline{\xi - z^0})^S F^{(K_j^0)}(\xi)}{|\xi - z^0|^{2(n+\|S\|)-1}}d\sigma(\xi).$$

As in (17), this yields

$$\begin{aligned}
|F^{(K_j^0+S)}(z^0)| \leqslant& \frac{(n+\|S\|-1)!}{(n-1)!}\left(\frac{\ell(z^0)}{r}\right)^{2(n+\|S\|)-1} \\
&\cdot\max\left\{|F^{(K_j^0)}(z)|\colon z \in \mathbb{B}^n\left[z^0, r/\ell(z^0)\right]\right\}\int_{\mathbb{S}^n(z^0,r/\ell(z^0))}|(\overline{\xi - z^0})^S|d\sigma(\xi) \\
\leqslant& \frac{(n+\|S\|-1)!}{(n-1)!}\left(\frac{\ell(z^0)}{r}\right)^{\|S\|}\max\left\{|F^{(K_j^0)}(z)|\colon z \in \mathbb{B}^n\left[z^0, r/\ell(z^0)\right]\right\} \\
&\cdot\int_{\mathbb{S}^n(z^0,r/\ell(z^0))}\frac{|(\xi - z^0)^S|}{(r/\ell(z^0))^{\|S\|}}d\sigma\left(\frac{\xi - z^0}{r/\ell(z^0)}\right) \\
\leqslant& \frac{(n+\|S\|-1)!}{(n-1)!}\left(\frac{\ell(z^0)}{r}\right)^{\|S\|}
\end{aligned}$$

$$
\begin{aligned}
&\cdot \max\left\{|F^{(K_j^0)}(z)|\colon z\in\mathbb{B}^n\left[z^0, r/\ell(z^0)\right]\right\}\int_{\mathbb{S}^n(\mathbf{0},1)}|\xi^S|d\sigma(\xi)\\
=&\frac{(n+\|S\|-1)!}{(n-1)!}\left(\frac{\ell(z^0)}{r}\right)^{\|S\|}\max\{|F^{(K_j^0)}(z)|\colon z\in\mathbb{B}^n\left[z^0, r/\ell(z^0)\right]\}\\
&\cdot\frac{\Gamma(n)\prod_{j=1}^n\Gamma(s_j/2+1)}{\Gamma(n+\|S\|/2)}.
\end{aligned}
$$

We let $r=\beta>\sqrt{n}$ and by (24) we obtain

$$
\begin{aligned}
|F^{(K_j^0+S)}(z^0)|\leqslant&\left(\frac{\ell(z^0)}{\beta}\right)^{\|S\|}\frac{(n+\|S\|-1)!\prod_{j=1}^n\Gamma(s_j/2+1)}{\Gamma(n+\|S\|/2)}\\
&\cdot\max\{|F^{(K_j^0)}(z)|\colon z\in\mathbb{B}^n\left[z^0,\beta/\ell(z^0)\right]\}\\
\leqslant& p\left(\frac{\ell(z^0)}{\beta}\right)^{\|S\|}\frac{(n+\|S\|-1)!\prod_{j=1}^n\Gamma(s_j/2+1)}{\Gamma(n+\|S\|/2)}|F^{(K_j^0)}(z^0)|.
\end{aligned}\tag{26}
$$

The above inequalities imply that for all $j\in\{1,\ldots,n\}$ and $k_j^0\leqslant n_0$ the inequalities

$$
\begin{aligned}
\frac{|F^{(K_j^0+S)}(z^0)|}{\mathbf{L}^{K_j^0+S}(z^0)(K_j^0+S)!}\leqslant& p\frac{K_j^0!(n+\|S\|-1)!\prod_{j=1}^n\Gamma(s_j/2+1)}{\beta^{\|S\|}(K_j^0+S)!\Gamma(n+\|S\|/2)}\frac{|F^{(K_j^0)}(z^0)|}{\mathbf{L}^{K_j^0}(z^0)K_j^0!}\\
\leqslant& pn_0!\frac{(n+\|S\|-1)!\prod_{j=1}^n\Gamma(s_j/2+1)}{\beta^{\|S\|}S!\Gamma(n+\|S\|/2)}\frac{|F^{(K_j^0)}(z^0)|}{\mathbf{L}^{K_j^0}(z^0)K_j^0!}
\end{aligned}
$$

hold true. In view of (22) there exists n_1 such that for all $\|S\|\geqslant n_1$ we have

$$
\frac{(n+\|S\|-1)!\prod_{j=1}^n\Gamma(s_j/2+1)}{\beta^{\|S\|}S!\Gamma(n+\|S\|/2)}\leqslant 1.
$$

It is obvious that there exists n_2 such that for all $\|S\|\geqslant n_2$ we have the inequaliuty

$$
\frac{pn_0!}{\beta^{\|S\|}}\leqslant 1.
$$

Thus, we have

$$
\frac{|F^{(K_j^0+S)}(z^0)|}{\mathbf{L}^{K_j^0+S}(z^0)(K_j^0+S)!}\leqslant\frac{|F^{(K_j^0)}(z^0)|}{\mathbf{L}^{K_j^0}(z^0)K_j^0!}\ \text{for all } \|S\|\geqslant n_1+n_2
$$

i.e., $N(F,\mathbf{L})\leqslant n_0+n_1+n_2$. This complete the proof. □

Remark 1. *Inequality* (23) *is a boundedness criterion for l-index of functions of one variable [23], [24]. But it is unknown whether this condition is sufficient for boundedness of* **L***-index in joint variables. Our restrictions* (24) *are corresponding multidimensional sufficient conditions.*

Lemma 2. *Let $\mathbf{L}_1$, $\mathbf{L}_2$ be positive continuous functions and for each $z\in\mathbb{C}^n$ the inequality $\mathbf{L}_1(z)\leqslant\mathbf{L}_2(z)$ holds. If an entire function F has bounded $\mathbf{L}_1$-index in joint variables, F is of bounded $\mathbf{L}_2$-index in joint variables. If, in addition, for each $z\in\mathbb{C}^n$ we have $\mathcal{L}_1(z)\leqslant\ell_2(z)$ then $N(F,\mathbf{L}_2)\leqslant N(F,\mathbf{L}_1)$.*

Proof. Let $N(F,\mathbf{L}_1)=n_0$. By (1) we get

$$\begin{aligned}\frac{|F^{(J)}(z)|}{J!\mathbf{L}_2^J(z)} =&\frac{\mathbf{L}_1^J(z)}{\mathbf{L}_2^J(z)}\frac{|F^{(J)}(z)|}{J!\mathbf{L}_1^J(z)} \leqslant \frac{\mathbf{L}_1^J(z)}{\mathbf{L}_2^J(z)}\max\left\{\frac{|F^{(K)}(z)|}{K!\mathbf{L}_1^K(z)}:\ K\in\mathbb{Z}_+^n,\ \|K\|\leqslant n_0\right\}\\ \leqslant&\frac{\mathbf{L}_1^J(z)}{\mathbf{L}_2^J(z)}\max\left\{\frac{\mathbf{L}_2^K(z)}{\mathbf{L}_1^K(z)}\frac{|F^{(K)}(z)|}{K!\mathbf{L}_2^K(z)}:\ K\in\mathbb{Z}_+^n,\ \|K\|\leqslant n_0\right\}\\ \leqslant& \max_{\|K\|\leqslant n_0}\left(\frac{\mathbf{L}_1(z)}{\mathbf{L}_2(z)}\right)^{J-K}\max\left\{\frac{|F^{(K)}(z)|}{K!\mathbf{L}_2^K(z)}:\ K\in\mathbb{Z}_+^n,\ \|K\|\leqslant n_0\right\}.\end{aligned}\tag{27}$$

Since $\mathbf{L}_1(z)\leqslant\mathbf{L}_2(z)$, this implies that for all $\|J\|\geqslant nn_0$ the inequality

$$\frac{|F^{(J)}(z)|}{J!\mathbf{L}_2^J(z)}\leqslant\max\left\{\frac{|F^{(K)}(z)|}{K!\mathbf{L}_2^K(z)}:\ K\in\mathbb{Z}_+^n,\ \|K\|\leqslant n_0\right\}$$

holds true. Thus, F has bounded $\mathbf{L}_2$-index in joint variables.

If, in addition, for each $z\in\mathbb{C}^n$ $\mathcal{L}_1(z)\leqslant\ell_2(z)$, then by (27) for all $\|J\|\geqslant n_0$ we get

$$\begin{aligned}\frac{|F^{(J)}(z)|}{J!\mathbf{L}_2^J(z)}\leqslant& \max_{\|K\|\leqslant n_0}\left(\frac{\mathcal{L}_1(z)}{\ell_2(z)}\right)^{\|J-K\|}\max\left\{\frac{|F^{(K)}(z)|}{K!\mathbf{L}_2^K(z)}:\ K\in\mathbb{Z}_+^n,\ \|K\|\leqslant n_0\right\}\\ \leqslant&\max\left\{\frac{|F^{(K)}(z)|}{K!\mathbf{L}_2^K(z)}:\ K\in\mathbb{Z}_+^n,\ \|K\|\leqslant n_0\right\}\end{aligned}$$

and $N(F,\mathbf{L}_2)\leqslant N(F,\mathbf{L}_1)$. The proof is complete. □

We denote $\widetilde{\mathbf{L}}(z)=(\widetilde{l}_1(z),\ldots,\widetilde{l}_n(z))$. The notation $\mathbf{L}\asymp\widetilde{\mathbf{L}}$ means that there exist $\Theta_1=(\theta_{1,j},\ldots,\theta_{1,n})\in\mathbb{R}_+^n$, $\Theta_2=(\theta_{2,j},\ldots,\theta_{2,n})\in\mathbb{R}_+^n$ such that $\theta_{1,j}\widetilde{l}_j(z)\leqslant l_j(z)\leqslant\theta_{2,j}\widetilde{l}_j(z)$ for each $j\in\{1,\ldots,n\}$ and for all $z\in\mathbb{C}^n$.

Theorem 4. *Let $\mathbf{L}\in Q_{\mathbb{B}}^n$, $\mathbf{L}\asymp\widetilde{\mathbf{L}}$, $\sup_{z\in\mathbb{C}^n}\frac{\mathcal{L}(z)}{\ell(z)}=C<\infty$. An entire function F has bounded $\widetilde{\mathbf{L}}$-index in joint variables if and only if it has bounded $\mathbf{L}$-index.*

Proof. It is easy to prove that if $\mathbf{L}\in Q_{\mathbb{B}}^n$ and $\mathbf{L}\asymp\widetilde{\mathbf{L}}$, then $\widetilde{\mathbf{L}}\in Q_{\mathbb{B}}^n$.

Let $N(F,\widetilde{\mathbf{L}})=\widetilde{n}_0<+\infty$. Then by Theorem 1, for each $\widetilde{r}>0$ there exists $\widetilde{p}\geqslant1$ such that for each $z^0\in\mathbb{C}^n$ and some K^0 with $\|K^0\|\leqslant\widetilde{n}_0$, inequality (8) holds with $\widetilde{\mathbf{L}}$ and $\widetilde{r}$ instead of $\mathbf{L}$ and r. Hence,

$$\begin{aligned}\frac{\widetilde{p}}{K^0!}\frac{|F^{(K^0)}(z^0)|}{\mathbf{L}^{K^0}(z^0)} =&\frac{\widetilde{p}}{K^0!}\frac{\Theta_2^{K^0}|F^{(K^0)}(z^0)|}{\Theta_2^{K^0}\mathbf{L}^{K^0}(z^0)}\geqslant\frac{\widetilde{p}}{K^0!}\frac{|F^{(K^0)}(z^0)|}{\Theta_2^{K^0}\widetilde{\mathbf{L}}^{K^0}(z^0)}\\ \geqslant&\frac{1}{\Theta_2^{K^0}}\max\left\{\frac{|F^{(K)}(z)|}{K!\widetilde{\mathbf{L}}^K(z)}:\|K\|\leqslant\widetilde{n}_0, z\in\mathbb{B}^n\left[z^0,\widetilde{r}/\widetilde{\mathcal{L}}(z)\right]\right\}\\ \geqslant&\frac{1}{\Theta_2^{K^0}}\max\left\{\frac{\Theta_1^K|F^{(K)}(z)|}{K!\mathbf{L}^K(z)}:\|K\|\leqslant\widetilde{n}_0, z\in\mathbb{B}^n\left[z^0,\min_{1\leqslant j\leqslant n}\Theta_{1,j}\widetilde{r}/\mathcal{L}(z)\right]\right\}\\ \geqslant&\frac{\min\limits_{0\leqslant\|K\|\leqslant n_0}\{\Theta_1^K\}}{\Theta_2^{K^0}}\max\left\{\frac{|F^{(K)}(z)|}{K!\mathbf{L}^K(z)}:\|K\|\leqslant\widetilde{n}_0, z\in\mathbb{B}^n\left[z^0,\min_{1\leqslant j\leqslant n}\Theta_{1,j}\widetilde{r}/\mathcal{L}(z)\right]\right\}\\ \geqslant&\frac{\min\limits_{0\leqslant\|K\|\leqslant n_0}\{\Theta_1^K\}}{\Theta_2^{K^0}}\max\left\{\frac{|F^{(K)}(z)|}{K!\mathbf{L}^K(z)}:\|K\|\leqslant\widetilde{n}_0, z\in\mathbb{B}^n\left[z^0,\frac{\widetilde{r}\min_{1\leqslant j\leqslant n}\Theta_{1,j}}{C\ell(z)}\right]\right\}.\end{aligned}$$

Thanks to Theorem 1, now we conclude that the function F has bounded $\mathbf{L}$-index. □

Theorem 5. *Let* $\mathbf{L} \in Q^n_{\mathbb{B}}$, F *be an entire function. If there exist* $r > 0$, $n_0 \in \mathbb{Z}_+$, $p_0 > 1$ *such that for each* $z^0 \in \mathbb{C}^n$ *and for some* $K^0 \in \mathbb{Z}^n_+$ *with* $\|K^0\| \leqslant n_0$ *inequality* (9) *holds true, then* F *has bounded* **L**-*index in joint variables.*

Proof. The proof of the sufficiency in Theorem 1 with $r \geqslant \sqrt{n}$ implies that $N(F, \mathbf{L}) < +\infty$.

Let $\mathbf{L}^*(z) = \frac{\sqrt{n}\mathbf{L}(z)}{r}$, $\ell^*(z) = \frac{\sqrt{n}\ell(z)}{r}$, and r is radius for which (9) is true. In a general case, by (9) for F and $\mathbf{L}$ with $r < \beta$ we obtain

$$\max\left\{\frac{|F^{(K)}(z)|}{K!(\mathbf{L}^*(z))^K} : \|K\| \leqslant n_0, z \in \mathbb{B}_n\left[z^0, \sqrt{n}/\ell^*(z^0)\right]\right\}$$
$$\leqslant \max\left\{\frac{|F^{(K)}(z)|}{K!(\sqrt{n}\mathbf{L}(z)/r)^K} : \|K\| \leqslant n_0, z \in \mathbb{B}_n\left[z^0, \sqrt{n}/(\sqrt{n}\ell(z^0)/r)\right]\right\}$$
$$\leqslant \max\left\{\frac{|F^{(K)}(z)|}{K!\mathbf{L}^K(z)} : \|K\| \leqslant n_0, z \in \mathbb{B}_n\left[z^0, r/\ell(z^0)\right]\right\}$$
$$\leqslant \frac{p_0}{K^0!}\frac{|F^{(K^0)}(z^0)|}{\mathbf{L}^{K^0}(z^0)} = \frac{\beta^{\|K^0\|}p_0}{r^{\|K^0\|}K^0!}\frac{|F^{(K^0)}(z)|}{(\sqrt{n}\mathbf{L}(z)/r)^{K^0}} = \frac{p_0\beta^{n_0}}{r^{n_0}}\frac{|F^{(K^0)}(z)|}{K^0!(\mathbf{L}^*(z))^{K^0}}.$$

i.e., (9) holds for F, $\mathbf{L}^*$ and $r = \sqrt{n}$. As above, now we apply Theorem 1 to the function $F(z)$ and $\mathbf{L}^*(z) = \sqrt{n}\mathbf{L}(z)/r$. This gives that F is of bounded $\mathbf{L}^*$-index in joint variables. Therefore, by Lemma 2, the function F has bounded **L**-index in joint variables. ☐

Remark 2. *We note that counterparts of Theorems 1-5 were obtained in [1], [3], [4], [10], [5] for polydiscs. But for balls these theorems are new even if* $\mathbf{L}(z) \equiv 1$, *that is, in the case of entire functions of bounded index in joint variables (see results in [13], [14], [15], [16], [22], [20]).*

REFERENCES

1. A. Bandura, O. Skaskiv. *Entire functions of several variables of bounded index.* Publisher I. E. Chyzhykov, Chyslo, Lviv (2016).
2. A.I. Bandura. *Some improvements of criteria of L-index boundedness in direction* // Mat. Stud. **47**:(1), 27–32 (2017).
3. A. Bandura. *New criteria of boundedness of* **L**-*index in joint variables for entire functions* // Math. Bull. Shevchenko Sci. Soc. **13**, 58–67 (2016). (in Ukrainian).
4. A.I. Bandura, M.T. Bordulyak, O.B. Skaskiv. *Sufficient conditions of boundedness of* **L**-*index in joint variables* // Mat. Stud. **45**:1, 12–26 (2016).
5. A.I. Bandura, N.V. Petrechko, O.B. Skaskiv. *Analytic functions in a polydisc of bounded* **L**-*index in joint variables* // Mat. Stud. **46**:1, 72–80 (2016).
6. A. Bandura, O. Skaskiv. *Sufficient conditions of boundedness of* **L**-*index and analog of Hayman's Theorem for analytic functions in a ball* // Stud. Univ. Babeş-Bolyai Math. **63**:4, 483–501 (2018).
7. A.I. Bandura, O.B. Skaskiv. *Boundedness of the L-index in a direction of entire solutions of second order partial differential equation* // Acta Comment. Univ. Tartu. Math. **22**:2, 223–234 (2018).
8. A. Bandura, O. Skaskiv, P. Filevych. *Properties of entire solutions of some linear PDE's* // J. Appl. Math. Comput. Mech. **16**:2, 17–28 (2017).
9. A.I. Bandura, O.B. Skaskiv. *Iyer's metric space, existence theorem and entire functions of bounded* **L**-*index in joint variables* // Bukovyn. Mat. Zh. **5**:3-4, 8–14 (2017). (in Ukrainian).
10. M.T. Bordulyak, M.M. Sheremeta. *Boundedness of the* **L**-*index of an entire function of several variables* // Dopov. Akad. Nauk Ukr. **9**, 10–13 (1993). (in Ukrainian).
11. M.T. Bordulyak. *A proof of Sheremeta conjecture concerning entire function of bounded l-index* // Mat. Stud. **11**:2, 108–110 (1999).
12. M.T. Bordulyak. *The space of entire in* $\mathbb{C}^n$ *functions of bounded* **L**-*index* // Mat. Stud. **4**, 53–58 (1995). (in Ukrainian)

13. B.C. Chakraborty, R. Chanda. *A class of entire functions of bounded index in several variables* // J. Pure Math. **12**, 16–21 (1995).
14. B.C. Chakraborty, T.K. Samanta. *On entire functions of bounded index in several variables* // J. Pure Math. **17**, 53–71 (2000).
15. B.C. Chakraborty, T.K. Samanta. *On entire functions of L-bounded index* // J. Pure Math. **18**, 53–64 (2001).
16. G.J. Krishna, S.M. Shah. *Functions of bounded indices in one and several complex variables* // In: "Mathematical essays dedicated to A.J. Macintyre", Ohio Univ. Press, Athens, Ohio, 223–235 (1970).
17. F. Nuray, R.F. Patterson. *Entire bivariate functions of exponential type* // Bull. Math. Sci. **5**:2, 171–177 (2015).
18. F. Nuray, R.F. Patterson. *Multivalence of bivariate functions of bounded index* // Le Matematiche. **70**:2, 225–233 (2015).
19. F. Nuray, R. F. Patterson. *Vector-valued bivariate entire functions of bounded index satisfying a system of differential equations* // Mat. Stud. **49**:1, 67–74 (2018).
20. R. Patterson, F. Nuray. *A characterization of holomorphic bivariate functions of bounded index* // Math. Slovaca **67**:3, 731–736 (2017).
21. W. Rudin. *Function Theory in the unit ball on* $\mathbb{C}^n$. Springer-Verlag, Berlin (2008).
22. M. Salmassi. *Functions of bounded indices in several variables* // Indian J. Math. **31**:3, 249–257 (1989).
23. M. Sheremeta. *Analytic functions of bounded index.* VNTL Publishers, Lviv (1999).
24. M.N. Sheremeta, A.D. Kuzyk. *Logarithmic derivative and zeros of an entire function of bounded l-index* // Siber. Math. J. **33**: 2, 304–312 (1992).
25. K. Zhu. *Spaces of holomorphic functions in the unit ball.* Springer, New York (2005).

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