

NEVANLINNA'S FIVE-VALUE THEOREM FOR ALGEBROID FUNCTIONS

ASHOK RATHOD

Abstract. By using the second main theorem of the algebroid function, we study the following problem. Let $W_1(z)$ and $W_2(z)$ be two ν -valued non-constant algebroid functions, a_j ($j = 1, 2, \dots, q$) be $q \geq 4\nu + 1$ distinct complex numbers or ∞ . Suppose that $k_1 \geq k_2 \geq \dots \geq k_q, m$ are positive integers or ∞ , $1 \leq m \leq q$ and $\delta_j \geq 0$, $j = 1, 2, \dots, q$, are such that

$$\left(1 + \frac{1}{k_m}\right) \sum_{j=m}^q \frac{1}{1+k_j} + 3\nu + \sum_{j=1}^q \delta_j < (q-m-1) \left(1 + \frac{1}{k_m}\right) + m.$$

Let $B_j = \overline{E}_{k_j}(a_j, f) \setminus \overline{E}_{k_j}(a_j, g)$ for $j = 1, 2, \dots, q$. If

$$\overline{N}_{B_j}\left(r, \frac{1}{W_1 - a_j}\right) \leq \delta_j T(r, W_1)$$

and

$$\liminf_{r \rightarrow \infty} \frac{\sum_{j=1}^q \overline{N}_{k_j}\left(r, \frac{1}{W_1 - a_j}\right)}{\sum_{j=1}^q \overline{N}_{k_j}\left(r, \frac{1}{W_2 - a_j}\right)} > \frac{\nu k_m}{(1+k_m) \sum_{j=1}^q \frac{k_j}{k_j+1} - 2\nu(1+k_m) + (m-2\nu - \sum_{j=1}^q \delta_j)k_m},$$

then $W_1(z) \equiv W_2(z)$. This result improves and generalizes the previous results given by Xuan and Gao.

Keywords: value distribution theory, Nevanlinna theory, algebroid functions, uniqueness.

Subject Classification: 30D35

1. INTRODUCTION

The value distribution theory of meromorphic functions was extended to the corresponding theory of algebroid functions by Ullarich [1] and Valiron [2] around 1930, and important results on uniqueness for algebroid functions were obtained. It is well known that Valiron obtained a famous $(4\nu + 1)$ -valued theorem. The uniqueness theory of algebroid functions is an interesting problem in the value distribution theory. Many researchers like Valiron [2], Baganas [3], He [4] and others ([6],[7],[9-27]) made lot of work in this area. In this article, we extend a result by Indrajit Lahiri and Rupa Pal [5] in the Nevanlinna's value distribution theory of meromorphic functions on Nevanlinna's five values theorem to algebroid functions

Let $A_\nu(z), A_{\nu-1}(z), \dots, A_0(z)$ be analytic functions with no common zeros in the complex plane and consider the equation

$$A_\nu(z)W^\nu + A_{\nu-1}(z)W^{\nu-1} + \dots + A_1(z)W + A_0(z) = 0. \quad (1)$$

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This equation defines a ν -valued algebroid function $W(z)$ [8].

It is well known [8] that on the complex plane with the projection of the critical points of the function W cut out, the Nevanlinna characteristic $T(r, W)$ is defined as

$$T(r, W) = m(r, W) + N(r, W),$$

where

$$m(r, W) = \frac{1}{2\pi\nu} \sum_{j=1}^{\nu} \int_0^{2\pi} \log^+ |w_j(re^{i\theta})| d\theta,$$

$$N(r, W) = \frac{1}{\nu} \int_0^r \frac{n(t, W) - n(0, W)}{t} dt + \frac{n(0, W)}{\nu} \log r.$$

Let $w_i(z)$ and $m_j(z)$ be one-valued branches of two (μ -valued and ν -valued) algebroid functions. Following Prokopovich [15], we consider their quotient in the domain of the complex plane with the projection of the critical points of both functions cut out. The one-valued branches of the function W/M ($W \cdot M$) are defined as w_i/m_j ($w_i \cdot m_j$), where $1 \leq i \leq m$, $1 \leq j \leq n$. The Nevanlinna's characteristic $T(r, W/M)$ is defined by $T(r, W) + T(r, M)$.

Lemma 1 (8). *Let $W(z)$ be a ν -valued algebroid function and $\{a_j\}_{j=1}^q \subset \overline{\mathbb{C}}$ be q distinct complex numbers and let $\{k_j\}_{j=1}^q \subset \mathbb{N}$ be q positive integers. Then*

$$(q - 2\nu)T(r, W) \leq \sum_{k=1}^q \frac{k_j}{k_j + 1} \overline{N}_{k_j}(r, W = a_j) + \sum_{k=1}^q \frac{1}{k_j + 1} N(r, W = a_j) + S(r, W),$$

$$\left(q - 2\nu - \sum_{k=1}^q \frac{1}{k_j + 1} \right) T(r, W) \leq \sum_{k=1}^q \frac{k_j}{k_j + 1} \overline{N}_{k_j}(r, W = a_j) + N(r, W = a_j) + S(r, W).$$

In 2006 Zu-Xing Xuan and Zong-Sheng Gao [18] improved this statement as follows.

Theorem 1. *Let $W(z)$ and $M(z)$ be two ν -valued non-constant algebroid functions, let a_j ($j = 1, 2, \dots, 4\nu + 1$) be $4\nu + 1$ distinct complex numbers in $\overline{\mathbb{C}}$. If*

$$\overline{E}_{2\nu+1}(a_j, W) = \overline{E}_{2\nu+1}(a_j, M), \quad j = 1, 2, \dots, 2\nu + 1$$

and

$$\overline{E}_{2\nu}(a_j, W) = \overline{E}_{2\nu}(a_j, M), \quad j = 1, 2, \dots, 4\nu + 1,$$

then $W(z) = M(z)$

2. MAIN RESULTS

Let $W(z)$ be a ν -valued algebroid function and $a \in \overline{\mathbb{C}}$ be a complex number. The symbol $\overline{E}_k(W = a)$ denotes the set of zeros of $W(z) - a$, whose multiplicities are not greater than k . The symbol $\overline{n}_k(W = a)$ stands for the number of distinct zeros of $W(z) - a$ in $|z| \leq r$, whose multiplicities do not exceed k and are counted only once. Similarly, we define the functions $\overline{n}_{(k+1)}(r, W = a)$, $\overline{N}_k(r, W = a)$ and $\overline{N}_{(k+1)}(r, W = a)$.

In this paper, we study the problem on the Nevanlinna's five value theorem for algebroid functions. To state our main theorem, we first introduce the following definition.

Definition 1. *For $B \subset \mathbb{A}$ and $a \in \overline{\mathbb{C}}$, we denote by $\overline{N}_B(r, \frac{1}{f-a})$ the reduced counting function of the zeros of $f - a$ on \mathbb{A} belonging to the set B .*

Theorem 2. *Let $W_1(z)$ and $W_2(z)$ be two ν -valued non-constant algebroid functions, let a_j ($j = 1, 2, \dots, q$) be $q \geq 4\nu + 1$ distinct complex numbers or ∞ . Suppose that*

$k_1 \geq k_2 \geq \dots \geq k_q$, m are positive integers or ∞ ; $1 \leq m \leq q$ and $\delta_j \geq 0$, $j = 1, 2, \dots, q$, are such that

$$\left(1 + \frac{1}{k_m}\right) \sum_{j=m}^q \frac{1}{1+k_j} + 3\nu + \sum_{j=1}^q \delta_j < (q-m-1) \left(1 + \frac{1}{k_m}\right) + m. \quad (2)$$

Let $B_j = \overline{E}_{k_j}(a_j, f) \setminus \overline{E}_{k_j}(a_j, g)$ for $j = 1, 2, \dots, q$. If

$$\overline{N}_{B_j}\left(r, \frac{1}{W_1 - a_j}\right) \leq \delta_j T(r, W_1) \quad (3)$$

and

$$\liminf_{r \rightarrow \infty} \frac{\sum_{j=1}^q \overline{N}_{k_j}\left(r, \frac{1}{W_1 - a_j}\right)}{\sum_{j=1}^q \overline{N}_{k_j}\left(r, \frac{1}{W_2 - a_j}\right)} > \frac{\nu k_m}{(1+k_m) \sum_{j=1}^q \frac{k_j}{k_{j+1}} - 2\nu(1+k_m) + (m-2\nu - \sum_{j=1}^q \delta_j)k_m} \quad (4)$$

then $W_1(z) \equiv W_2(z)$.

Proof. Suppose that $W_1(z) \neq W_2(z)$. The by Lemma 1 for each integer m , $1 \leq m \leq q$, we have

$$\begin{aligned} (q-2\nu)T(r, W_1) &\leq \sum_{j=1}^q \overline{N}\left(r, \frac{1}{W_1 - a_j}\right) + S(r, W_1) \\ &\leq \sum_{j=1}^q \left\{ \overline{N}_{k_j}\left(r, \frac{1}{W_1 - a_j}\right) + \overline{N}_{(k_{j+1})}\left(r, \frac{1}{W_1 - a_j}\right) \right\} + S(r, W_1) \\ &\leq \sum_{j=1}^q \left\{ \overline{N}_{k_j}\left(r, \frac{1}{W_1 - a_j}\right) + \frac{1}{1+k_j} N_{(k_{j+1})}\left(r, \frac{1}{W_1 - a_j}\right) \right\} + S(r, W_1) \\ &\leq \sum_{j=1}^q \left\{ \frac{k_j}{1+k_j} \overline{N}_{k_j}\left(r, \frac{1}{W_1 - a_j}\right) + \frac{1}{1+k_j} N\left(r, \frac{1}{W_1 - a_j}\right) \right\} + S(r, W_1) \\ &\leq \sum_{j=1}^q \frac{k_j}{1+k_j} \overline{N}_{k_j}\left(r, \frac{1}{W_1 - a_j}\right) + \sum_{j=1}^q \frac{1}{1+k_j} T(r, W_1) + S(r, W_1) \\ &\leq \sum_{j=1}^q \left(\frac{k_j}{1+k_j} - \frac{k_m}{1+k_m} \right) \overline{N}_{k_j}\left(r, \frac{1}{W_1 - a_j}\right) + \sum_{j=1}^q \frac{1}{1+k_j} T(r, W_1) \\ &\quad + \sum_{j=1}^q \frac{k_m}{1+k_m} \overline{N}_{k_j}\left(r, \frac{1}{W_1 - a_j}\right) + S(r, W_1) \\ &\leq \sum_{j=1}^q \frac{k_m}{1+k_m} \overline{N}_{k_j}\left(r, \frac{1}{W_1 - a_j}\right) \\ &\quad + \left(m-1 - \frac{(m-1)k_m}{k_m+1} + \sum_{j=1}^q \frac{k_j}{1+k_j} \right) T(r, W_1) + S(r, W_1) \end{aligned}$$

Therefore

$$\left(\sum_{j=m}^q \frac{k_j}{k_{j+1}} - 2\nu + \frac{(m-1)k_m}{k_m+1} \right) T(r, W_1) \leq \sum_{j=1}^q \frac{k_m}{k_m+1} \overline{N}_{k_j}\left(r, \frac{1}{W_1 - a_j}\right) + S(r, W_1). \quad (5)$$

Similarly,

$$\left(\sum_{j=m}^q \frac{k_j}{k_j+1} - 2\nu + \frac{(m-1)k_m}{k_m+1} \right) T(r, W_2) \leq \sum_{j=1}^q \frac{k_m}{k_m+1} \bar{N}_{k_j} \left(r, \frac{1}{W_2 - a_j} \right) + S(r, W_2). \quad (6)$$

Since $B_j = \bar{E}_{k_j}(a_j, W_1) - \bar{E}_{k_j}(a_j, W_2)$, let $D_j = \bar{E}_{k_j}(a_j, W_1) - B_j$ for $j = 1, 2, \dots, q$. Thus, by (5) and (6), for a sequence of values of r tending to ∞ we get:

$$\begin{aligned} \sum_{j=m}^q \bar{N}_{k_j} \left(r, \frac{1}{W_1 - a_j} \right) &= \sum_{j=m}^q \bar{N}_{B_j} \left(r, \frac{1}{W_1 - a_j} \right) + \sum_{j=m}^q \bar{N}_{D_j} \left(r, \frac{1}{W_1 - a_j} \right) \\ &\leq \sum_{j=m}^q \delta_j T(r, W_1) + \nu N \left(r, \frac{1}{W_1 - W_2} \right) \\ &\leq \left(\nu + \sum_{j=m}^q \delta_j \right) T(r, W_1) + \nu T(r, W_2) + O(1) \end{aligned}$$

Therefore,

$$\begin{aligned} &\left(\sum_{j=m}^q \frac{k_j}{k_j+1} - 2\nu + \frac{(m-1)k_m}{k_m+1} + O(1) \right) \bar{N}_{k_j} \left(r, \frac{1}{W_1 - a_j} \right) \\ &\leq \left(\nu + \sum_{j=m}^q \delta_j \right) \sum_{j=1}^q \frac{k_m}{k_m+1} \bar{N}_{k_j} \left(r, \frac{1}{W_1 - a_j} \right) \\ &\quad + (\nu + O(1)) \sum_{j=1}^q \frac{k_m}{k_m+1} \bar{N}_{k_j} \left(r, \frac{1}{W_2 - a_j} \right). \end{aligned} \quad (7)$$

Since

$$1 \geq \frac{k_1}{k_1+1} \geq \dots \geq \frac{k_q}{k_q+1} \geq \frac{1}{2},$$

by (7), for a sequence of values of r tending to $+\infty$, we get

$$\begin{aligned} &\left(\sum_{j=m}^q \frac{k_j}{k_j+1} - 2\nu + \frac{(m-1)k_m}{k_m+1} - \frac{k_m}{k_m+1} \left(\nu + \sum_{j=m}^q \delta_j \right) + O(1) \right) \bar{N}_{k_j} \left(r, \frac{1}{W_1 - a_j} \right) \\ &\leq (\nu + O(1)) \frac{k_m}{k_m+1} \sum_{j=1}^q \bar{N}_{k_j} \left(r, \frac{1}{W_2 - a_j} \right). \end{aligned}$$

This implies that

$$\begin{aligned} \liminf_{r \rightarrow \infty} \frac{\bar{N}_{k_j} \left(r, \frac{1}{W_1 - a_j} \right)}{\bar{N}_{k_j} \left(r, \frac{1}{W_2 - a_j} \right)} &\leq \frac{\nu \frac{k_m}{k_m+1}}{\left(\sum_{j=m}^q \frac{k_j}{k_j+1} - 2\nu + \frac{k_m}{k_m+1} \left(m - 2\nu - \sum_{j=m}^q \delta_j \right) \right)}, \\ &\leq \frac{\nu k_m}{(1 + k_m) \left(\sum_{j=m}^q \frac{k_j}{k_j+1} - 2\nu(1 + k_m) + k_m \left(m - 2\nu - \sum_{j=m}^q \delta_j \right) \right)}. \end{aligned}$$

This contradicts equation (2). Thus, we have $f(z) \not\equiv g(z)$. The proof is complete. \square

Theorem 2 yield the following corollaries.

Corollary 1. Let $m = 1$, $k_j = \infty$ for $j = 1, 2, 3, \dots, q$ and

$$\gamma = \liminf_{r \rightarrow \infty} \frac{\overline{N}_{k_j} \left(r, \frac{1}{W_1 - a_j} \right)}{\overline{N}_{k_j} \left(r, \frac{1}{W_2 - a_j} \right)} > \frac{1}{q - 2\nu + 1}$$

If $\overline{N}_{B_j} \left(r, \frac{1}{W_1 - a_j} \right) \leq \delta_j T(r, W_1)$, where $\delta \geq 0$ satisfies

$$0 \leq \sum_{j=1}^q \delta_j < k - (2\nu + 1) - \frac{1}{\gamma},$$

then $f(z) \equiv g(z)$

If we take $q = 4\nu + 1$ and $\overline{E}(a_j, f) \subseteq \overline{E}(a_j, g)$, then $B_j = \emptyset$ for $j = 1, 2, \dots, 4\nu + 1$. Therefore, if we choose $\delta_j = 0$ for $j = 1, 2, \dots, 4\nu + 1$ and take any constant γ obeying $0 \leq 2\nu - \frac{1}{\gamma}$ in Corollary 1, we can get that $f \equiv g$. Moreover, if $q = 4\nu + 1$ and $\overline{E}(a_j, f) = \overline{E}(a_j, g)$, then $\gamma = 1$ and $\delta_j = 0$ for $j = 1, 2, \dots, 4\nu + 1$; this implies $f \equiv g$. Then Corollary 1 is an improvement of Theorem 1.

Corollary 2. Let $W_1(z)$ and $W_2(z)$ be two ν -valued non-constant algebroid functions, let a_j , $j = 1, 2, \dots, q$, be $q \geq 5$ distinct complex numbers or ∞ . Suppose that k_1, k_2, \dots, k_q are positive integers or ∞ with $k_1 \geq k_2 \geq \dots \geq k_q$ if $\overline{E}_{k_j}(a_j, f) \subseteq \overline{E}_{k_j}(a_j, g)$ and

$$\sum_{j=2\nu}^q \frac{k_j}{k_j + 1} - \frac{k_1}{\gamma(k_1 + 1)} - 2\nu > 0,$$

where γ is as stated in Corollary 1. Then $f(z) \equiv g(z)$.

Corollary 3. Under the assumptions of Corollary 2, we have $\overline{E}_{k_j}(a_j, W_1) = \overline{E}_{k_j}(a_j, W_2)$ and

$$\sum_{j=2\nu}^q \frac{k_j}{k_j + 1} - \frac{k_1}{(k_1 + 1)} - 2\nu > 0,$$

Corollary 4. Let $W_1(z)$ and $W_2(z)$ be two ν -valued non-constant algebroid functions, let a_j , $j = 1, 2, \dots, q$, be $q \geq 5$ distinct complex numbers or ∞ . Suppose that k_1, k_2, \dots, k_q are positive integers or ∞ with $k_1 \geq k_2 \geq \dots \geq k_q$ if $\overline{E}_{k_j}(a_j, f) \subseteq \overline{E}_{k_j}(a_j, g)$ and

$$\sum_{j=2\nu}^q \frac{k_j}{k_j + 1} - 2\nu + \frac{(m - 2\nu - \frac{1}{\gamma})k_m}{\gamma(k_m + 1)} - 2\nu > 0,$$

where γ is as stated a in Corollary 1. Then $f(z) \equiv g(z)$.

СПИСОК ЛИТЕРАТУРЫ

1. E. Ullrich. *Über den Einfluß der verzweigthheit einer algebloide auf ihre wertvertellung* // J. Reine Angew. Math. **1932**:167, 198–220 (1932).
2. G. Valiron. *Sur quelques propriétés des fonctions algébroides* // Compt. Rend. Math. **189**, 824–826 (1929).
3. N. Baganas. *Sur les valeurs algébriques dune fonctions algebroides et les intégrales pseudo-abelinnes* // Annales Ecole Norm. Sup. Ser. 3. **66**, 161–208 (1949).
4. H.S. Gopalkrishna, S.S. Bhooanurmth. *Uniqueness theorems for meromorphic functions* // Math. Scand. **39**, 125–130 (1976).
5. Lahiri Indrajit, Pal Rupa. *A note on Nevanlinna's five value theorerm* // Bull. Korean Math. Soc. **52**:2, 345–350 (2015).

6. Yu-Zan He and Ye-Zhou Li. *Some results on algebroid functions* // Comp. Variab. Ellip. Equat. **43**:3-4, 299–313 (2001).
7. S. Daochun, G. Zongsheng. *On the operation of algebroid functions* // Acta Math. Sci. **30**:1, 247–256 (2010).
8. S. Daochun, G. Zongsheng. *Value distribution theory of algebroid functions*. Science Press, Beijing (2014).
9. Yu-Zan He and Xiu-Zhi Xiao. *Algebroid functions and Ordinary Difference Equations*. Science Press, Beijing (1988).
10. S. Daochun, G. Zongsheng. *Theorems for algebroid functions* // Acta Math. Sinica. **49**:5, 1–6 (2006).
11. Y. Hongxun. *On the multiple values and uniqueness of algebroid functions* // Chinese J. Eng. Math. **8**, 1–8 (1991).
12. W.K. Hayman. *Meromorphic functions*. Oxford University Press, Oxford (1964).
13. F. Minglang. *Unicity theorem for algebroid functions* // Acta. Math. Sinica. **3**:6, 217–222 (1993).
14. Pingyuan Zhang, Peichu Hu. *On uniqueness for algebroid functions of finite order* // Acta. Math. Sinica. **35**:3, 630–638 (2015).
15. G.S. Prokoporich. *Fix-points of meromorphic or entire functions* // Ukrainian Math. J. **25**:2, 248–260 (1973).
16. Z. Qingcai. *Uniqueness of algebroid functions* // Math. Pract. Theory. **43**:1, 183–187 (2003).
17. Cao Tingbin, Yi Hongxun. *On the uniqueness theory of algebroid functions* // Southeast Asian Bull. Math. **33**:1, 25–39 (2009).
18. Zu-Xing Xuan, ZongG-Sheng Gao. *Uniqueness theorems for algebroid functions* // Compl. Variab. Ellipt. Equat. **51**:7, 701–712 (2006).
19. C.C. Yang, H.X. Yi. *Uniqueness theory of meromorphic functions*. Math. Appl. **557**. Kluwer Academic Publishers, Dordrecht (2003).
20. H.X. Yi, *The multiple values of meromorphic functions and uniqueness* // Chinese Ann. Math. Ser. A. **10**:4, 421–427 (1989).
21. R.S. Dyavanal, Ashok Rathod. *Some generalisation of Nevanlinna's five-value theorem algebroid functions on annuli* // Asian J. Math. Comp. Resear. **20**:2, 85–95 (2017).
22. R.S. Dyavanal, Ashok Rathod. *Nevanlinna's five-value theorem for derivatives of meromorphic functions sharing values on annuli* // Asian J. Math. Comp. Resear. **20**:1, 13–21 (2017).
23. R.S. Dyavanal, Ashok Rathod, *Unicity theorem for algebroid functions related to multiple values and derivatives on annuli* // Int. J. Fuzzy Math. Arch. **13**:1, 25–39 (2017).
24. Ashok Rathod. *Several uniqueness theorems for algebroid functions* // J. Anal. **25**:2, 203–213 (2017).
25. Ashok Rathod. *The multiple values of algebroid functions and uniqueness* // Asian J. Math. Comp. Resear. **14**:2, 150–157 (2016).
26. Ashok Rathod. *The multiple values of algebroid functions and uniqueness on annuli* // Konoralf J. Math. **5**:2, 216–227 (2017).
27. Ashok Rathod. *On the deficiencies of algebroid functions and their differential polynomials* // J. Basic Appl. Resear. Int. **1**:1, 1–11 (2016).

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