

MINIMUM MODULUS OF LACUNARY POWER SERIES AND h -MEASURE OF EXCEPTIONAL SETS

T.M. SALO, O.B. SKASKIV

Abstract. We consider some generalizations of Fenton theorem for the entire functions represented by lacunary power series. Let $f(z) = \sum_{k=0}^{+\infty} f_k z^{n_k}$, where (n_k) is a strictly increasing sequence of non-negative integers. We denote by

$$\begin{aligned} M_f(r) &= \max\{|f(z)|: |z| = r\}, \\ m_f(r) &= \min\{|f(z)|: |z| = r\}, \\ \mu_f(r) &= \max\{|f_k| r^{n_k} : k \geq 0\} \end{aligned}$$

the maximum modulus, the minimum modulus and the maximum term of f , respectively. Let $h(r)$ be a positive continuous function increasing to infinity on $[1, +\infty)$ with a non-decreasing derivative. For a measurable set $E \subset [1, +\infty)$ we introduce h -meas $(E) = \int_E \frac{dh(r)}{r}$. In this paper we establish conditions guaranteeing that the relations

$$M_f(r) = (1 + o(1))m_f(r), \quad M_f(r) = (1 + o(1))\mu_f(r)$$

are true as $r \rightarrow +\infty$ outside some exceptional set E such that h -meas $(E) < +\infty$. For some subclasses we obtain necessary and sufficient conditions. We also provide similar results for entire Dirichlet series.

Keywords: lacunary power series, minimum modulus, maximum modulus, maximal term, entire Dirichlet series, exceptional set, h -measure

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1. INTRODUCTION

Let L be the class of positive continuous functions increasing to infinity on $[0; +\infty)$. By L^+ we denote the subclass of L consisting of the differentiable functions with a non-decreasing derivative, and L^- stands for the subclass of functions with a non-increasing derivative.

Let f be an entire function of the form

$$f(z) = \sum_{k=0}^{+\infty} f_k z^{n_k}, \tag{1}$$

where (n_k) is a strictly increasing sequence of nonnegative integers. Given $r > 0$, we denote by $M_f(r) = \max\{|f(z)|: |z| = r\}$, $m_f(r) = \min\{|f(z)|: |z| = r\}$, $\mu_f(r) = \max\{|f_k| r^{n_k} : k \geq 0\}$ the maximum modulus, the minimum modulus and the maximum term of f , respectively.

P.C. Fenton [1] (see also [2]) proved the following statement.

Theorem 1 ([1]). *If*

$$\sum_{k=0}^{+\infty} \frac{1}{n_{k+1} - n_k} < +\infty, \tag{2}$$

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then for every entire function f of the form (1) there exists a set $E \subset [1, +\infty)$ of finite logarithmic measure, i.e. $\log\text{-meas } E := \int_E d\log r < +\infty$, such that the relations

$$M_f(r) = (1 + o(1))m_f(r), \quad M_f(r) = (1 + o(1))\mu_f(r) \tag{3}$$

hold as $r \rightarrow +\infty$ ($r \notin E$).

P. Erdős and A.J. Macintyre [2] proved that condition (2) implies that (3) holds as $r = r_j \rightarrow +\infty$ for some sequence (r_j) .

Denote by $D(\Lambda)$ the class of entire (absolutely convergent in the complex plane) Dirichlet series of the form

$$F(z) = \sum_{n=0}^{+\infty} a_n e^{z\lambda_n}, \tag{4}$$

where $\Lambda = (\lambda_n)$ is a fixed sequence such that $0 = \lambda_0 < \lambda_n \uparrow +\infty$ ($1 \leq n \uparrow +\infty$).

Let us introduce some notations. Given $F \in D(\Lambda)$ and $x \in \mathbb{R}$, we denote by

$$\mu(x, F) = \max\{|a_n|e^{x\lambda_n} : n \geq 0\}$$

the maximal term of series (4), by

$$M(x, F) = \sup\{|F(x + iy)| : y \in \mathbb{R}\}$$

we denote the maximum modulus of series (4), by

$$m(x, F) = \inf\{|F(x + iy)| : y \in \mathbb{R}\}$$

we denote the minimum modulus of series (4), and

$$\nu(x, F) = \max\{n : |a_n|e^{x\lambda_n} = \mu(x, F)\}$$

stands for the central index of series (4).

In [3] (see also [4]) we find the following theorem.

Theorem 2 ([3]). *For every entire function $F \in D(\Lambda)$ the relation*

$$F(x + iy) = (1 + o(1))a_{\nu(x, F)}e^{(x+iy)\lambda_{\nu(x, F)}} \tag{5}$$

holds as $x \rightarrow +\infty$ outside some set E of finite Lebesgue measure ($\int_E dx < +\infty$) uniformly in $y \in \mathbb{R}$, if and only if

$$\sum_{n=0}^{+\infty} \frac{1}{\lambda_{n+1} - \lambda_n} < +\infty. \tag{6}$$

Note, that in the paper [5] there were proved the analogues of other statements in the paper by P.C. Fenton [1] for subclasses of functions $F \in D(\Lambda)$ defined by various restrictions on the growth rate of the maximal term $\mu(x, F)$.

The finiteness of Lebesgue measure of an exceptional set E in theorem A is the best possible description. This is implied by the next statement.

Theorem 3 ([6]). *For every sequence $\lambda = (\lambda_k)$ (including those which satisfy (6)) and for every continuously differentiable function $h : [0, +\infty) \rightarrow (0, +\infty)$ such that $h'(x) \nearrow +\infty$ ($x \rightarrow +\infty$) there exist an entire Dirichlet series $F \in D(\lambda)$, a constant $\beta > 0$ and a measurable set $E_1 \subset [0, +\infty)$ of infinite h -measure ($h\text{-meas}(E_1) \stackrel{\text{def}}{=} \int_{E_1} dh(x) = +\infty$) such that*

$$(\forall x \in E_1) : M(x, F) > (1 + \beta)\mu(x, F), \quad M(x, F) > (1 + \beta)m(x, F). \tag{7}$$

Recently, Ya.V. Mykytyuk remarked that in Theorem 3, it is sufficient to assume that a positive non-decreasing function h is such that

$$\frac{h(x)}{x} \rightarrow +\infty \quad \text{as } x \rightarrow +\infty.$$

It follows from Theorem 3 that the finiteness of logarithmic measure of an exceptional set E in Fenton's Theorem 1 is also the best possible description.

It is easy to see that the relation

$$F(x + iy) = (1 + o(1))a_{\nu(x,F)}e^{(x+iy)\lambda_{\nu(x,F)}}$$

holds as $x \rightarrow +\infty$ ($x \notin E$) uniformly in $y \in \mathbb{R}$ if and only if

$$M(x, F) \sim \mu(x, F) \quad \text{and} \quad M(x, F) \sim m(x, F) \quad (x \rightarrow +\infty, x \notin E). \tag{8}$$

In view of Theorem 3, the natural question arises: *what conditions should an entire Dirichlet series satisfy in order to relation (5) be true as $x \rightarrow +\infty$ outside some set E_2 of finite h -measure, i.e.,*

$$h - \text{meas}(E_2) < +\infty?$$

In this paper we provide the answer to this question as $h \in L^+$.

2. h -MEASURE WITH NON-DECREASING DENSITY

According to Theorem 3, in the case $h \in L^+$, condition (6) must be fulfilled. Therefore, in the subclass

$$D(\Lambda, \Phi) = \{F \in D(\Lambda) : \ln \mu(x, F) \geq x\Phi(x) \ (x > x_0)\}, \quad \Phi \in L,$$

it should be strengthened. The following theorem indicates this.

Theorem 4. *Let $\Phi \in L$, $h \in L^+$ and φ be the inverse function for the function Φ . If*

$$(\forall b > 0) : \sum_{k=0}^{+\infty} \frac{1}{\lambda_{k+1} - \lambda_k} h' \left(\varphi(\lambda_k) + \frac{b}{\lambda_{k+1} - \lambda_k} \right) < +\infty, \tag{9}$$

then for all $F \in D(\Lambda, \Phi)$ identity (5) is true as $x \rightarrow +\infty$ outside some set E of a finite h -measure uniformly in $y \in \mathbb{R}$.

Before proving this theorem, we need additional notations and an auxiliary lemma.

Denote $\Delta_0 = 0$ and

$$\Delta_n = \sum_{j=0}^{n-1} (\lambda_{j+1} - \lambda_j) \sum_{m=j+1}^{\infty} \left(\frac{1}{\lambda_m - \lambda_{m-1}} + \frac{1}{\lambda_{m+1} - \lambda_m} \right).$$

for $n \geq 1$. The next lemma is similar to Lemma 1 in [8].

Lemma 1. *For all $n \geq 0$ and $k \geq 1$, the inequality*

$$\frac{\alpha_n}{\alpha_k} e^{\tau_k(\lambda_n - \lambda_k)} \leq e^{-q|n-k|}, \tag{10}$$

is true, where $\alpha_n = e^{q\Delta_n}$, $q > 0$, and

$$\tau_k = \tau_k(q) = qx_k + \frac{q}{\lambda_k - \lambda_{k-1}}, \quad x_k = \frac{\Delta_{k-1} - \Delta_k}{\lambda_k - \lambda_{k-1}}.$$

Proof. Since

$$\ln \alpha_n - \ln \alpha_{n-1} = q(\Delta_n - \Delta_{n-1}) = -qx_n(\lambda_n - \lambda_{n-1}),$$

for $n \geq k + 1$ we have

$$\begin{aligned} \ln \frac{\alpha_n}{\alpha_k} + \tau_k(\lambda_n - \lambda_k) &= -q \sum_{j=k+1}^n x_j(\lambda_j - \lambda_{j-1}) + \tau_k \sum_{j=k+1}^n (\lambda_j - \lambda_{j-1}) \\ &= - \sum_{j=k+1}^n (qx_j - \tau_k)(\lambda_j - \lambda_{j-1}) \\ &\leq - \sum_{j=k+1}^n (qx_j - \tau_{j-1})(\lambda_j - \lambda_{j-1}) \\ &= -q \sum_{j=k+1}^n 1 = -q(n - k). \end{aligned}$$

Similarly, for $n \leq k - 1$ we obtain

$$\begin{aligned} \ln \frac{\alpha_n}{\alpha_k} + \tau_k(\lambda_n - \lambda_k) &= - \ln \frac{\alpha_k}{\alpha_n} - \tau_k(\lambda_k - \lambda_n) \\ &= q \sum_{j=n+1}^k x_j(\lambda_j - \lambda_{j-1}) - \tau_k \sum_{j=n+1}^k (\lambda_j - \lambda_{j-1}) \\ &= - \sum_{j=n+1}^k (\tau_k - qx_j)(\lambda_j - \lambda_{j-1}) \\ &\leq - \sum_{j=n+1}^k (\tau_j - qx_j)(\lambda_j - \lambda_{j-1}) = -q \sum_{j=n+1}^k 1 = -q(k - n), \end{aligned}$$

and this completes the proof. □

Proof of Theorem 4. We first note that condition (9) implies the convergence of series (6). We consider the function

$$f_q(z) = \sum_{n=0}^{+\infty} \frac{a_n}{\alpha_n} e^{z\lambda_n}.$$

Since $\Delta_n \geq 0$, we have $f_q \in D(\Lambda)$ and $\nu(x, f_q) \rightarrow +\infty$ ($x \rightarrow +\infty$).

Let J be the range of the central index $\nu(x, f_q)$. Denote by (R_k) the sequence of the jump points of central index, numbered in such a way that $\nu(x, f_q) = k$ for all $x \in [R_k, R_{k+1})$ and $R_k < R_{k+1}$. Then for all $x \in [R_k, R_{k+1})$ and $n \geq 0$ we have

$$\frac{a_n}{\alpha_n} e^{x\lambda_n} \leq \frac{a_k}{\alpha_k} e^{x\lambda_k}.$$

According to Lemma 1, for $x \in [R_k + \tau_k, R_{k+1} + \tau_k)$ we obtain

$$\frac{a_n e^{x\lambda_n}}{a_k e^{x\lambda_k}} \leq \frac{\alpha_n}{\alpha_k} e^{\tau_k(\lambda_n - \lambda_k)} \leq e^{-q|n-k|} \quad (n \geq 0).$$

Therefore,

$$\nu(x, F) = k, \quad \mu(x, F) = a_k e^{x\lambda_k} \quad (x \in [R_k + \tau_k, R_{k+1} + \tau_k)) \tag{11}$$

and

$$\begin{aligned} |F(x + iy) - a_{\nu(x, F)} e^{(x+iy)\lambda_{\nu(x, F)}}| &\leq \sum_{n \neq \nu(x, F)} \mu(x, F) e^{-q|n - \nu(x, F)|} \\ &\leq 2 \frac{e^{-q}}{1 - e^{-q}} \mu(x, F) \end{aligned} \tag{12}$$

for all $x \in [R_k + \tau_k, R_{k+1} + \tau_k)$ and $k \in J$. Thus, inequality (12) holds for all $x \notin E_1(q) \stackrel{\text{def}}{=} \bigcup_{k=0}^{+\infty} [R_{k+1} + \tau_k, R_{k+1} + \tau_{k+1})$.

Since

$$\tau_{k+1} - \tau_k = \frac{2q}{\lambda_{k+1} - \lambda_k},$$

and by the Lagrange theorem

$$h(R_{k+1} + \tau_{k+1}) - h(R_{k+1} + \tau_k) = (\tau_{k+1} - \tau_k)h'(R_{k+1} + \tau_k + \theta_k(\tau_{k+1} - \tau_k)),$$

where $\theta_k \in (0; 1)$, for each $q > 0$ we have

$$\begin{aligned} h - \text{meas}(E_1(q)) &= \sum_{k=0}^{+\infty} \int_{R_{k+1} + \tau_k}^{R_{k+1} + \tau_{k+1}} dh(x) \\ &= \sum_{k=0}^{+\infty} (h(R_{k+1} + \tau_{k+1}) - h(R_{k+1} + \tau_k)) \\ &\leq 2q \sum_{k=0}^{+\infty} \frac{1}{\lambda_{k+1} - \lambda_k} h' \left(R_{k+1} + \tau_k + 2q \frac{1}{\lambda_{k+1} - \lambda_k} \right). \end{aligned} \quad (13)$$

Here we have employed the condition $h \in L^+$.

For $F \in D(\Lambda, \Phi)$ and $x > \max\{x_0, 1\}$ we have

$$x\Phi(x) \leq \ln \mu(x, F) = \ln \mu(1, F) + \int_1^x \lambda_{\nu(x, f)} dx \leq \ln \mu(1, F) + (x - 1)\lambda_{\nu(x-0, F)}.$$

This implies

$$x\Phi(x) \leq x\lambda_{\nu(x-0, F)} \quad (14)$$

for all $x \geq x_1 \geq x_0$, i.e.

$$x \leq \varphi(\lambda_{\nu(x-0, F)}) \quad (x \geq x_1).$$

Thus, according to (11), for $k \geq k_0$ we obtain

$$R_{k+1} + \tau_k \leq \varphi(\lambda_{\nu(R_{k+1} + \tau_k - 0, F)}) = \varphi(\lambda_k).$$

Applying this inequality to inequality (13), by the condition $h \in L^+$ we have

$$h - \text{meas}(E_1(q)) \leq 2q \sum_{k=0}^{+\infty} \frac{1}{\lambda_{k+1} - \lambda_k} h' \left(\varphi(\lambda_k) + 2q \frac{1}{\lambda_{k+1} - \lambda_k} \right). \quad (15)$$

Therefore, using (9) we conclude that $h - \text{meas}(E_1(q)) < +\infty$.

Let $q_k = k$. Since $h - \text{meas}(E_1(q_k)) < +\infty$, we have

$$h - \text{meas}(E_1(q_k) \cap [x, +\infty)) = o(1) \quad (x \rightarrow +\infty),$$

hence, it is possible to choose an increasing to $+\infty$ sequence (x_k) such that

$$h - \text{meas}(E_1(q_k) \cap [x_k; +\infty)) \leq \frac{1}{k^2}$$

for all $k \geq 1$. Denote $E_1 = \bigcup_{k=1}^{+\infty} (E_1(q_k) \cap [x_k; x_{k+1}))$. Then

$$h - \text{meas}(E_1) = \sum_{k=1}^{+\infty} h - \text{meas}(E_1(q_k) \cap [x_k; x_{k+1})) \leq \sum_{k=1}^{+\infty} \frac{1}{k^2} < +\infty,$$

On the other hand, by inequality (12), for $x \in [x_k; x_{k+1}) \setminus E_1$ we get

$$|F(x + iy) - a_{\nu(x,F)} e^{(x+iy)\lambda_{\nu(x,F)}}| \leq 2 \frac{e^{-qk}}{1 - e^{-qk}} \mu(x, F),$$

and therefore, as $x \rightarrow +\infty$ ($x \notin E_1$), we obtain (5). The proof is complete. □

We observe that if $h(x) \equiv x$, then condition (9) becomes condition (6), and h -measure of the set E is its Lebesgue measure.

Let $\Phi \in L$. Consider the classes

$$D_0(\Lambda, \Phi) = \{F \in D(\Lambda) : (\exists K > 0)[\ln \mu(x, \Phi) \geq Kx\Phi(x) \ (x > x_0)]\},$$

$$D_1(\Lambda, \Phi) = \{F \in D(\Lambda) : (\exists K_1, K_2 > 0)[\ln \mu(x, \Phi) \geq K_1x\Phi(K_2x) \ (x > x_0)]\}.$$

Theorem 5. *Let $\Phi_0 \in L$, $h \in L^+$ and φ_0 be the inverse function for the function Φ_0 . If*

$$(\forall b > 0) : \sum_{n=0}^{+\infty} \frac{1}{\lambda_{n+1} - \lambda_n} h' \left(\varphi_0(b\lambda_n) + \frac{b}{\lambda_{n+1} - \lambda_n} \right) < +\infty, \tag{16}$$

then for each function $F \in D_0(\Lambda, \Phi_0)$ relation (5) holds as $x \rightarrow +\infty$ outside some set E of finite h -measure uniformly in $y \in \mathbb{R}$.

Theorem 6. *Let $\Phi_1 \in L$, $h \in L^+$, and φ_1 be the inverse function to the function Φ_1 . If*

$$(\forall b > 0) : \sum_{n=0}^{+\infty} \frac{h'(b\varphi_1(b\lambda_n))}{\lambda_{n+1} - \lambda_n} < +\infty, \tag{17}$$

then for every function $F \in D_1(\Lambda, \Phi_1)$ relation (5) holds as $x \rightarrow +\infty$ outside some set E of finite h -measure uniformly in $y \in \mathbb{R}$.

Proof of Theorems 5 and 6. Theorems 5 and 6 are implied immediately by Theorem 4.

Indeed, if $F \in D_0(\Lambda, \Phi_0)$, then $F \in D(\Lambda, \Phi)$ as $\Phi(x) = K\Phi_0(x)$. But in this case $\varphi(x) = \varphi_0(x/K)$ and condition (9) follows condition (16). Then it remains to apply Theorem 4.

In the same way, if $F \in D_1(\Lambda, \Phi_1)$, then $F \in D(\Lambda, \Phi)$ as $\Phi(x) = K_1\Phi_1(K_2x)$. But in this case $\varphi(x) = \varphi_1(x/K_1)/K_2$ and hence, condition (9) follows condition (17). It remains to employ Theorem 4 once again. □

Remark 1. *It is easy to see that for each fixed functions $h \in L^+$ and $\Phi \in L$ there exists a sequence Λ such that conditions (9), (16) and (17) hold.*

The next theorem shows that condition (17) is necessary for relations (5), (8) to hold for each $F \in D_1(\Lambda, \Phi_1)$ as $x \rightarrow +\infty$ outside a set of a finite h -measure. Here we assume that condition (6) is satisfied.

Theorem 7. *Let $\Phi_1 \in L$, $h \in L^+$, and φ_1 be the inverse function for the function Φ_1 . For each sequence Λ such that*

$$(\exists b > 0) : \sum_{n=0}^{+\infty} \frac{h'(b\varphi_1(b\lambda_n))}{\lambda_{n+1} - \lambda_n} = +\infty, \tag{18}$$

there exist a function $F \in D_1(\Lambda, \Phi_1)$, a set $E \subset [0, +\infty)$ and a constant $\beta > 0$ such that inequalities (7) hold for all $x \in E$ and $h - \text{meas}(E) = +\infty$.

Proof. We denote $\varkappa_1 = \varkappa_2 = 1$, $\varkappa_n = \sum_{k=1}^{n-2} r_k$, ($n \geq 3$), where

$$r_1 = \max \left\{ b\varphi_1(b\lambda_2), \frac{1}{\lambda_2 - \lambda_1} \right\},$$

$$r_k = \max \left\{ b\varphi_1(b\lambda_{k+1}) - b\varphi_1(b\lambda_k), \frac{1}{\lambda_{k+1} - \lambda_k} \right\} \quad (k \geq 2),$$

and we also choose

$$a_0 = 1, \quad a_n = \exp \left\{ - \sum_{k=1}^n \varkappa_k (\lambda_k - \lambda_{k-1}) \right\} \quad (n \geq 1).$$

We prove that the function F defined by series (4) with the above defined coefficients (a_n) and the exponents (λ_n) belongs to the class $D_1(\Lambda, \Phi_1)$.

Since the condition

$$\sum_{n=0}^{+\infty} \frac{1}{\lambda_{n+1} - \lambda_n} < +\infty$$

implies $n^2 = o(\lambda_n)$ ($n \rightarrow +\infty$), we have $\frac{\ln n}{\lambda_n} \rightarrow 0$ ($n \rightarrow +\infty$). By the construction,

$$\varkappa_n = \frac{\ln a_{n-1} - \ln a_n}{\lambda_n - \lambda_{n-1}} \quad (n \geq 1)$$

and $\varkappa_n \uparrow +\infty$ ($n \rightarrow +\infty$). Therefore Stolz theorem yields that $-\frac{\ln a_n}{\lambda_n} \rightarrow +\infty$ ($n \rightarrow +\infty$) and by Valiron formula [9] the abscissa of the absolute convergence of series (4) is equal to $+\infty$, i.e., $F \in D(\Lambda)$.

Moreover, it is known that in the case $\varkappa_n \uparrow +\infty$ ($n \rightarrow +\infty$) we have

$$\forall x \in [\varkappa_n, \varkappa_{n+1}) : \quad \mu(x, F) = a_n e^{x\lambda_n}, \quad \nu(x, F) = n. \tag{19}$$

Since by the construction

$$\varkappa_n \leq b\varphi_1(b\lambda_{n-1}) + \sum_{k=1}^{n-2} \frac{1}{\lambda_{k+1} - \lambda_k} \leq 2b\varphi_1(b\lambda_{n-1}) \quad (n > n_0),$$

for sufficiently large n for all $x \in [\varkappa_n, \varkappa_{n+1})$ we have

$$\begin{aligned} \ln \mu(2x, F) &= \ln \mu(x, F) + \int_x^{2x} \lambda_{\nu(t)} dt \geq x\lambda_{\nu(x)} \\ &= x\lambda_n \geq \frac{x}{b} \Phi_1 \left(\frac{\varkappa_{n+1}}{2b} \right) \geq \frac{x}{b} \Phi_1 \left(\frac{x}{2b} \right). \end{aligned}$$

Hence, for $x \geq x_0$ we have

$$\ln \mu(x, F) \geq \frac{1}{2b} x \Phi_1 \left(\frac{x}{4b} \right),$$

and thus $F \in D_1(\Lambda, \Phi_1)$.

We observe that

$$\varkappa_{n+1} - \varkappa_n = r_{n-1} \geq \frac{1}{\lambda_n - \lambda_{n-1}} \quad (n \geq 1).$$

For $x \in \left[\varkappa_n, \varkappa_n + \frac{1}{\lambda_n - \lambda_{n-1}} \right]$ we have

$$\frac{a_{n-1} e^{x\lambda_{n-1}}}{\mu(x, F)} = \frac{a_{n-1} e^{x\lambda_{n-1}}}{a_n e^{x\lambda_n}} = \exp\{(\lambda_n - \lambda_{n-1})(\varkappa_n - x)\} \geq e^{-1} := \beta, \tag{20}$$

and, therefore, for $x \in E = \bigcup_{n=1}^{\infty} \left[\varkappa_n, \varkappa_n + \frac{1}{\lambda_n - \lambda_{n-1}} \right]$, by choosing $n = \nu(x, F)$ we get

$$F(x) \geq a_{n-1}e^{x\lambda_{n-1}} + a_n e^{x\lambda_n} = \mu(x, F) \left(1 + \frac{a_{n-1}e^{x\lambda_{n-1}}}{a_n e^{x\lambda_n}} \right) \geq (1 + \beta)\mu(x, F).$$

Hence, inequalities (7) are true.

Now we prove that $h - \text{meas}(E) = +\infty$. By the construction of (\varkappa_n) for all $n \geq 1$ we have

$$\varkappa_n \geq b\varphi_1(b\lambda_{n-1}). \quad (21)$$

Taking into consideration the Lagrange theorem, the condition $h \in L^+$ and inequality (21), we obtain

$$\begin{aligned} h - \text{meas}(E) &= \sum_{n=1}^{+\infty} \int_{\varkappa_n}^{\varkappa_n + \frac{1}{\lambda_n - \lambda_{n-1}}} dh(x) = \sum_{n=1}^{+\infty} \left(h\left(\varkappa_n + \frac{1}{\lambda_n - \lambda_{n-1}}\right) - h(\varkappa_n) \right) \\ &\geq \sum_{n=1}^{+\infty} \frac{h'(\varkappa_n)}{\lambda_n - \lambda_{n-1}} \geq \sum_{n=1}^{+\infty} \frac{h'(b\varphi_1(b\lambda_{n-1}))}{\lambda_n - \lambda_{n-1}} = +\infty. \end{aligned}$$

The proof is complete. \square

The next criterion is implied immediately by Theorems 6 and 7.

Theorem 8. *Let $\Phi_1 \in L$, $h \in L^+$ and φ_1 be the inverse function for the function Φ_1 . For each entire function $F \in D_1(\Lambda, \Phi_1)$ relation (5) holds as $x \rightarrow +\infty$ outside some set E of a finite h -measure uniformly in $y \in \mathbb{R}$ if and only if (17) is true.*

It is worth noting that if condition (16) of Theorem 5 is not fulfilled, that is

$$(\exists b_1 > 0) : \sum_{n=0}^{+\infty} \frac{1}{\lambda_{n+1} - \lambda_n} h' \left(\varphi_0(b_1\lambda_n) + \frac{b_1}{\lambda_{n+1} - \lambda_n} \right) = +\infty,$$

then for $b = \max\{b_1; 2\}$ we have

$$\sum_{n=0}^{+\infty} \frac{h'(b\varphi_0(b\lambda_n))}{\lambda_{n+1} - \lambda_n} = +\infty.$$

Therefore, condition (18) holds and according Theorem 7, there exist a function $F \in D_1(\Lambda, \Phi_0)$, a set $E \subset [0, +\infty)$ and a constant $\beta > 0$ such that inequalities (7) hold for all $x \in E$ and $h - \text{meas}(E) = +\infty$.

Since for $\Phi_0(x) = x^\alpha$, $\alpha > 0$, we have $D_0(\Lambda, \Phi_0) = D_1(\Lambda, \Phi_0)$, from Theorem 5 and 7 we obtain the following theorem.

Theorem 9. *Let $\Phi_0(x) = x^\alpha$ ($\alpha > 0$), $h \in L^+$. For each entire function $F \in D_0(\Lambda, \Phi_0)$ relation (5) holds as $x \rightarrow +\infty$ outside some set E of a finite h -measure uniformly in $y \in \mathbb{R}$ if and only if*

$$(\forall b > 0) : \sum_{n=0}^{+\infty} \frac{1}{\lambda_{n+1} - \lambda_n} h' \left(b(\lambda_n)^{1/\alpha} + \frac{b}{\lambda_{n+1} - \lambda_n} \right) < +\infty,$$

is true.

3. h -MEASURE WITH A NON-INCREASING DENSITY

We note that for each differentiable function $h: \mathbb{R}_+ \rightarrow \mathbb{R}_+$ with a bounded derivative $h'(x) \leq c < +\infty$ ($x > 0$) we have

$$\int_E dh(x) = \int_E h'(x)dx \leq c \int_E dx.$$

Hence, the finiteness of Lebesgue measure of a set $E \subset \mathbb{R}_+$ implies $h - \text{meas}(E) < +\infty$. Therefore, according Theorem A, condition (6) provides that the exceptional set E is of a finite h -measure. However, we conjecture that for $h \in L^-$ in the subclass

$$D_\varphi(\Lambda) = \{F \in D(\Lambda) : (\exists n_0)(\forall n \geq n_0)[|a_n| \leq \exp\{-\lambda_n \varphi(\lambda_n)\}]\}, \quad \varphi \in L,$$

condition (6) can be weakened significantly. The following conjecture seems to be true.

Conjecture 1. *Let $\varphi \in L$, $h \in L^-$. If*

$$\sum_{n=0}^{+\infty} \frac{h'(\varphi(\lambda_n))}{\lambda_{n+1} - \lambda_n} < +\infty,$$

then for all $F \in D_\varphi(\Lambda)$ relation (5) is true as $x \rightarrow +\infty$ outside some set E of finite h -measure uniformly in $y \in \mathbb{R}$.

4. h -MEASURE AND LACUNARY POWER SERIES

The important corollaries for entire functions represented by a lacunary power series of the form (1) are implied by the proven theorems.

For an entire function f of the form (1) we let $F(z) = f(e^z)$, $z \in \mathbb{C}$.

We observe that as $x = \ln r$, $y = \varphi$,

$$F(x + iy) = F(\ln r + i\varphi) = f(re^{i\varphi})$$

and $M(x, F) = M_f(r)$, $m(x, F) = m_f(r)$, $\mu(x, F) = \mu_f(r)$, $\nu(x, F) = \nu_f(r)$. In addition, for $E_2 \stackrel{\text{def}}{=} \{r \in \mathbb{R} : \ln r \in E_1\}$ and h_1 such that $h'_1(x) = h'(e^x)$ we have

$$h - \log - \text{meas}(E_2) \stackrel{\text{def}}{=} \int_{E_2} \frac{dh(r)}{r} = \int_{E_1} \frac{dh(e^x)}{e^x} = \int_{E_1} dh_1(x) = h_1 - \text{meas}(E_1).$$

The next corollary is implied by Theorem B.

Corollary 1. *For each sequence (n_k) such that condition (6) holds and for each function $h \in L^+$ there exist an entire function f of the form (1), a constant $\beta > 0$ and a set E_2 of an infinite h -log-measure, i.e. $(\int_{E_2} \frac{dh(r)}{r} = +\infty)$ such that*

$$(\forall r \in E_2) : M_f(r) \geq (1 + \beta)\mu_f(r), \quad M_f(r) \geq (1 + \beta)m_f(r). \tag{22}$$

By Theorem 4 we obtain the following corollary.

Corollary 2. *Let $\Phi \in L$, $h \in L^+$ and φ be the inverse function for the function Φ . If for an entire function f of the form (1)*

$$\ln \mu_f(r) \geq \ln r \Phi(\ln r) \quad (r \geq r_0) \tag{23}$$

and

$$(\forall b > 0) : \sum_{k=0}^{+\infty} \frac{1}{n_{k+1} - n_k} h' \left(\exp \left\{ \varphi(n_k) + \frac{b}{n_{k+1} - n_k} \right\} \right) < +\infty, \tag{24}$$

then the relation

$$f(re^{i\varphi}) = (1 + o(1))a_{\nu_f(r)} r^{\nu_f(r)} e^{i\varphi n_{\nu_f(r)}} \tag{25}$$

holds as $r \rightarrow +\infty$ outside some set E_2 of finite h -log-measure uniformly in $\varphi \in [0, 2\pi]$.

In fact, it follows from condition (23) that $F \in D(\Lambda, \Phi)$ with $\Lambda = (n_k)$ and it remains to apply Theorem 4 with the function h_1 .

Denote by \mathcal{E} the class of entire functions of positive lower order, i.e.

$$\lambda_f := \varliminf_{r \rightarrow +\infty} \ln \ln M_f(r) / \ln r > 0.$$

By Theorem 8 we obtain the following corollary.

Corollary 3. *Let $h \in L^+$. In order the relations (3) hold for each function $f \in \mathcal{E}$ of the form (1) as $r \rightarrow +\infty$ outside a set of a finite h -log-measure, it is necessary and sufficient to have*

$$(\forall b > 0) : \sum_{k=0}^{+\infty} \frac{1}{n_{k+1} - n_k} h'((n_k)^b) < +\infty.$$

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