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LEVI-FLAT WORLD: A SURVEY OF LOCAL THEORY

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Abstract. This expository paper concerns local properties of Levi-flat real analytic manifolds with singularities. Levi-flat manifolds arise naturally in Complex Geometry and Foliation Theory. In many cases (global) compact Levi-flat manifolds without singularities do not exist. These global obstructions make natural the study of Levi-flat objects with singularities because they always exist. The present expository paper deals with some recent results on local geometry of Levi-flat singularities. One of the main questions concerns an extension of the Levi foliation as a holomorphic foliation to a full neighborhood of singularity. It turns out that in general such extension does not exist. Nevertheless, the Levi foliation always extends as a holomorphic web (a foliation with branching) near a non-dicritical singularity. We also present an efficient criterion characterizing these singularities.

Keywords: CR structure, Levi-flat manifold.

Mathematics Subject Classification: 37F75,34M,32S,32D

1. Introduction

This expository paper paper concerns local properties of real analytic Levi-flat manifolds with singularities. Such manifolds arise naturally in the theory of holomorphic foliations and differential equations, in particular, in the study of minimal sets for foliations. They were studied recently by several authors from different points of view (see, e.g., [1, 2, 3, 4, 5, 6, 11]). In the present paper I discuss recent progress achieved in the series of our joint papers [18, 16, 17] with S. Pinchuk and R. Shafikov. Of course, they always must be considered as co-authors of the present work.

This paper is written for the special issue of Ufa Mathematical Journal dedicated to the 100th anniversary of A.F. Leontiev, the foundator of the theory function research school in Ufa. I dedicate this work to the memory of this remarkable mathematician.

2. Real analytic Levi-flat hypersurfaces in $\mathbb{C}^n$

2.1. Real analytic sets and their complexification. Let $\Omega \subset \mathbb{R}^n$ be a domain. A real analytic set $\Gamma \subset \Omega$ is a closed set locally defined as a zero locus of a finite collection of real analytic functions. In fact, we can always take just one function to locally define any real analytic set. We say that Γ is *irreducible* in Ω if it cannot be represented as the union $\Gamma = \Gamma_1 \cup \Gamma_2$ of two real analytic sets Γ_j in Ω with $\Gamma_j \setminus (\Gamma_1 \cap \Gamma_2) \neq \varnothing$, $j = 1, 2$, (this is the geometric irreducibility). In the present paper we always deal with germs of real analytic sets (without mentioning this explicitely) and assume that they are irreducible as germs. A set Γ is called a *real hypersurface* if there exists a point $q \in \Gamma$ such that near q the set Γ is a real analytic submanifold of dimension $n-1$. For a real hypersurface Γ we call such q a *regular* point. The union of all regular points form a regular locus denoted by Γ^*. Its complement $\Gamma_{sing} := \Gamma \setminus \Gamma^*$ is called the *singular locus* of Γ. Note that our convention is different from the usual definition of a regular

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point in semianalytic or subanalytic geometry, where a similar notion is less restrictive and a real analytic set is allowed to be a submanifold of *some* dimension near a regular point. By our definition, the points of a hypersurface Γ, where Γ is a submanifold of dimension smaller than $n-1$, belong to the singular locus. For that reason, Γ^* may not be dense in Γ, this can happen even if Γ is irreducible (so-called umbrellas). Note that Γ_{sing} is a closed semianalytic subset of Γ (possibly empty) of real dimension at most $n-2$.

In local questions we are interested in the geometry of a real hypersurface Γ in an arbitrarily small neighbourhood of a given point $a \in \Gamma$, i.e., of the germ at a of Γ. If the germ is irreducible at a, we may consider a sufficiently small open neighbourhood U of a and a representative of the germ which is irreducible at a, see [14] for details. In what follows we will not distinguish between the germ of Γ at a given point a and its particular representative in a suitable neighbourhood of a.

Let $\Gamma \subset \mathbb{R}^n_x$ be the germ of a real analytic set at the origin. By $\Gamma^{\mathbb{C}}$ we denote the complexification of Γ, i.e., a complex analytic germ at the origin in $\mathbb{C}^n_z = \mathbb{R}^n_x + i\mathbb{R}^n_y$, $z = x+iy$, with the property that each holomorphic function vanishing on Γ necessarily vanishes on $\Gamma^{\mathbb{C}}$. Equivalently, $\Gamma^{\mathbb{C}}$ is the smallest complex analytic germ in $\mathbb{C}^n$ that contains Γ. It is well known that the dimension of Γ is equal to the complex dimension of $\Gamma^{\mathbb{C}}$ and that the germ of $\Gamma^{\mathbb{C}}$ is irreducible at zero whenever the germ of Γ is irreducible, see Narasimhan [14] for further details and proofs. Also, given a real analytic germ $\sum_{|J|\ge 0} a_J\, x^J$, $a_J \in \mathbb{R}$, $x \in \mathbb{R}^n$, we define its complexification to be the complex analytic germ $\sum a_J\, z^J$.

While the complexification of the germ of a real analytic set is canonical and is independent of the choice of the defining function, the next lemma gives a convenient way of constructing the complexification of a real analytic hypersurface using a suitably chosen defining function. We will need the following notion of a minimal defining function for a complex hypersurface. Given a complex hypersurface $A = \{z \in \Omega : f(z) = 0\}$ in a domain $\Omega \subset \mathbb{C}^n$, f is called *minimal* if for every open subset $U \subset \Omega$ and any function g holomorphic on U and such that $g = 0$ on $A \cap U$, there exists a function h holomorphic in U such that $g = hf$. If f is a minimal defining function, then the singular locus of A coincides with the set $f = df = 0$. Locally, any irreducible complex hypersurface admits a minimal defining function, see Chirka [7].

Lemma 2.1. *Let $\Gamma \subset \mathbb{R}^n$ be an irreducible germ of a real analytic hypersurface at the origin. Then there exists a defining function $\rho(x)$ of the germ of Γ at the origin such that its complexification $\hat\rho(z)$ is a minimal defining function of the complexification $\Gamma^{\mathbb{C}}$.*

2.2. Levi-flat hypersurfaces. Let $z = (z_1, \dots, z_n)$, $z_j = x_j + iy_j$, be the standard coordinates in $\mathbb{C}^n$. Let Γ be an irreducible germ of a real analytic hypersurface at the origin defined by a function ρ provided by Lemma 2.1. In a (connected) sufficiently small neighbourhood of the origin $\Omega \subset \mathbb{C}^n$, the hypersurface Γ is a closed irreducible real analytic subset of Ω of dimension $2n-1$.

For $q \in \Gamma^*$ consider the holomorphic tangent space $H_q(\Gamma) := T_q(\Gamma) \cap JT_q(\Gamma)$. The *Levi form* of Γ is a Hermitian quadratic form defined on $H_q(\Gamma)$ by

$$L_q(v) = \sum_{k,j} \rho_{z_k \overline{z}_j}(q) v_k \overline{v}_j$$

with $v \in H_q(\Gamma)$. A real analytic hypersurface Γ is called *Levi-flat* if its Levi form vanishes on $H_q(\Gamma)$ for every regular point q of Γ. By the classical result of Elie Cartan, for every point $q \in \Gamma^*$ there exists a local biholomorphic change of coordinates centred at q such that in the new coordinates Γ in some neighbourhood U of $q = 0$ has the form $\{z \in U : z_n + \overline{z}_n = 0\}$. Hence, $\Gamma \cap U$ is locally foliated by complex hyperplanes $\{z_n = c,\ c \in i\,\mathbb{R}\}$. This foliation is called *the Levi foliation* of Γ^*, and will be denoted by $\mathcal{L}$. We denote by $\mathcal{L}_q$ the leaf of the Levi foliation through q. Note that by definition it is a connected complex hypersurface closed in Γ^*.

Let $0 \in \overline{\Gamma}^*$. We choose the neighbourhood Ω of the origin in the form of a polydisc $\Delta(\varepsilon) = \{z \in \mathbb{C}^n : |z_j| < \varepsilon\}$ of radius $\varepsilon > 0$. Then for ε small enough, the function ρ admits the Taylor expansion convergent in U:

$$\rho(z, \overline{z}) = \sum_{IJ} c_{IJ} z^I \overline{z}^J, \ c_{IJ} \in \mathbb{C}, \quad I, J \in \mathbb{N}^n. \tag{1}$$

The coefficients c_{IJ} satisfy the condition

$$\overline{c}_{IJ} = c_{JI}, \tag{2}$$

because ρ is a real-valued function. Note that in local questions we may further shrink Ω as needed.

By Lemma 2.1, the choice of the defining function ρ guarantees that the complexification of (the germ of) Γ is given by

$$\Gamma^{\mathbb{C}} = \{(z, w) \in \mathbb{C}^n \times \mathbb{C}^n : \rho(z, \overline{w}) = 0\}. \tag{3}$$

The hypersurface Γ lifts canonically to $\Gamma^{\mathbb{C}}$ as

$$\hat{\Gamma} = \Gamma^{\mathbb{C}} \cap \{w = z\}.$$

In what follows we denote by $\Gamma^{\mathbb{C}}_{sing}$ the singular locus of $\Gamma^{\mathbb{C}}$.

2.3. Segre Varieties. Our key tool is the family of *Segre varieties* associated with a real analytic hypersurface Γ. For $w \in \Delta(\varepsilon)$ consider a complex analytic hypersurface given by

$$Q_w = \{z \in \Delta(\varepsilon) : \rho(z, \overline{w}) = 0\}. \tag{4}$$

It is called the *Segre variety* of the point w. This definition uses the defining function ρ of Γ in a neighbourhood of the origin which appears in (3). We will always consider the case where the germ of Γ at the origin is irreducible and everywhere through the paper we use a defining function provided by Lemma 2.1 in a neighbourhood of the origin (the same convention is used in [18]). In general the Segre varieties Q_w also depend on the choice of ε (some irreducible components of Q_w may disappear when we shrink ε). Throughout the paper we consider only the Segre varieties Q_w defined by means of the complexification at the origin. The reader should keep this in mind. Also note that if 0 is a regular point of Γ, then the notion of the Segre variety Q_w is independent of the choice of a defining function ρ with non-vanishing gradient when w is close enough to the origin.

The following properties of Segre varieties are immediate.

Lemma 2.2. *Let Γ be a germ of an irreducible real analytic hypersurface in $\mathbb{C}^n$, $n > 1$. Then*

(a) *$z \in Q_z$ if and only if $z \in \Gamma$,*

(b) *$z \in Q_w$ if and only if $w \in Q_z$.*

We also recall the property of local biholomorphic invariance of some distinguished components of the Segre varieties near regular points. Since here we are working near a singularity, we state this property in detail using the notation introduced above. Consider a regular point $a \in \Gamma^* \cap \Delta(\varepsilon)$ and fix $\alpha > 0$ small enough with respect to ε. Consider any function ρ_a real analytic on the polydisc $\Delta(a, \alpha) = \{|z_j - a_j| < \alpha, j = 1, \ldots, n\}$ such that $\Gamma \cap \Delta(a, \alpha) = \rho_a^{-1}(0)$ and the gradient of ρ_a does not vanish on $\Delta(a, \alpha)$. Then for $w \in \Delta(a, \alpha)$ we can define the Segre variety ${}^aQ_w$ ("the Segre variety with respect to the regular point a") as

$${}^aQ_w = \{z \in \Delta(a, \alpha) : \rho_a(z, \overline{w}) = 0\},$$

(we use the Taylor series of ρ_a at a to define the complexification). For α small enough, ${}^aQ_w$ is a connected nonsingular complex submanifold of dimension $n - 1$ in $\Delta(a, \alpha)$. This definition is independent of the choice of the local defining function ρ_a satisfying the above properties.

We have the inclusion ${}^aQ_w \subset Q_w$. Note that in general Q_w can have irreducible components in $\Delta(\varepsilon)$ which do not contain ${}^aQ_w$.

Lemma 2.3. *(Invariance property) Let Γ, Γ' be irreducible germs of real analytic hypersurfaces, $a \in \Gamma^*$, $a' \in (\Gamma')^*$, and $\Delta(a,\alpha)$, $\Delta(a',\alpha')$ be small polydiscs. Let $f : \Delta(a,\alpha) \to \Delta(a',\alpha')$ be a holomorphic map such that $f(\Gamma \cap \Delta(a,\alpha)) \subset \Gamma' \cap \Delta(a',\alpha')$ and $f(a) = a'$. Then*

$$f({}^aQ_w) \subset {}^{a'}Q'_{f(w)}$$

for all $w \in \Delta(a,\alpha)$ close enough to a. In particular, if $f : \Delta(a,\alpha) \to \Delta(a',\alpha')$ is biholomorphic, then $f({}^aQ_w) = {}^{a'}Q'_{f(w)}$. Here ${}^aQ_w$ and ${}^{a'}Q'_{f(w)}$ are the Segre varieties associated with Γ and Γ' and the points a and a' respectively.

For the proof see for instance, [8]. As a simple consequence of Lemma 2.2 we have

Corollary 2.4. *Let $\Gamma \subset \mathbb{C}^n$ be an irreducible germ at the origin of a real analytic Levi-flat hypersurface. Let $a \in \Gamma^*$. Then the following holds:*

(a) *There exists a unique irreducible component S_a of Q_a containing the leaf $\mathcal{L}_a$. This is also a unique complex hypersurface through a which is contained in Γ.*
(b) *For every $a, b \in \Gamma^*$ one has $b \in S_a \iff S_a = S_b$.*
(c) *Suppose that $a \in \Gamma^*$ and $\mathcal{L}_a$ touches a point $q \in \Gamma$ such that $\dim_{\mathbb{C}} Q_q = n-1$ (the point q may be singular). Then Q_q contains S_a as an irreducible component.*

The proof is contained in [18]. Again, we emphasize that Corollary 2.4 concerns the “global” Segre varieties, i.e., those defined by (4) using the complexification at the origin.

2.4. Characterization of dicritical singularities for Levi-flat hypersurfaces. Let Γ be an irreducible germ of a real analytic Levi-flat hypersurface in $\mathbb{C}^n$ at $0 \in \overline{\Gamma^*}$. Fix a local defining function ρ chosen by Lemma 2.1 so that the complexification $\Gamma^{\mathbb{C}}$ is an irreducible germ of a complex hypersurface in $\mathbb{C}^{2n}$ given as the zero locus of the complexification of ρ. As already mentioned above, all Segre varieties which we consider are defined by means of this complexification at the origin.

Fix also $\varepsilon > 0$ small enough; all considerations are in the polydisc $\Delta(\varepsilon)$ centred at the origin. A point $q \in \overline{\Gamma^*} \cap \Delta(\varepsilon)$ is called a *dicritical* singularity if q belongs to the closure of infinitely many geometrically different leaves $\mathcal{L}_a$. Singular points in $\overline{\Gamma^*}$ which are not dicritical are called *nondicritical.*

A singular point q is called *Segre degenerate* if $\dim Q_q = n$.

Lemma 2.5. *Let Γ be a real analytic Levi-flat hypersurface. Then dicritical singular points form a complex analytic subset of Γ of complex dimension at most $n-2$, in particular, it is a discrete set if $n = 2$. If Γ is algebraic, then the set of dicritical singularities is also complex algebraic.*

We recall that the Segre degenerate singular points form a complex analytic subset of $\Delta(\varepsilon)$ of complex dimension at most $n-2$, in particular, it is a discrete set if $n = 2$. For the proof see [11, 18].

Theorem 2.6. *Let $\Gamma = \rho^{-1}(0)$ be an irreducible germ at the origin of a real analytic Levi-flat hypersurface in $\mathbb{C}^n$ and $0 \in \overline{\Gamma^*}$. Then 0 is a dicritical point if and only if it is Segre degenerate.*

This result is obtained in [16].

3. Singular webs

In this section we define singular holomorphic webs and outline the connection between webs and differential equations. This connection is transparent in dimension two, so we will discuss this case separately. For a comprehensive treatment of singular webs see, e.g., [15].

3.1. Webs in $\mathbb{C}^2$. Recall that the germ of a holomorphic codimension one foliation $\mathcal{F}$ in $\mathbb{C}^n$, $n \geq 2$, can be given by the germ of a holomorphic 1-form $\omega \in \Lambda^1(U)$ satisfying the Frobenius integrability condition $\omega \wedge d\omega = 0$. The leaves of $\mathcal{F}$ are then complex hypersurfaces L that are tangent to $\ker \omega$. The foliation $\mathcal{F}$ is *singular* if the set $\mathcal{F}^{\rm sng} = \{z : \omega(z) = 0\}$ is nonempty and of codimension at least 2.

In dimension 2 the integrability condition for ω always holds, and the above definition of a (nonsingular) foliation can be interpreted in the following way: for a suitably chosen open set U and coordinate system in $\mathbb{C}^2$ the foliation $\mathcal{F}$ is given by a *holomorphic first order ODE*

$$\frac{dz_2}{dz_1} = F(z_1, z_2) \tag{5}$$

with respect to unknown function $z_2 = z_2(z_1)$. The leaves of the foliation $\mathcal{F}$ are then the graphs of solutions of the ODE. This interpretation admits a far reaching generalization which we now describe. Our considerations are local and should be understood on the level of germs, but to simplify the discussion we will work with appropriate representatives of the germs.

Let U_1, U_2 be domains in $\mathbb{C}$ containing the origin. Set $U = U_1 \times U_2 \subset \mathbb{C}^2$, and consider a holomorphic function Φ on $U \times \mathbb{C}$. It defines a *holomorphic ordinary differential equation* on $U \times \mathbb{C}$,

$$\Phi(z_1, z_2, p) = 0 \tag{6}$$

with $z = (z_1, z_2) \in U$ and $p = \frac{dz_2}{dz_1} \in \mathbb{C}$. This is an equation for the unknown function $z_2 = z_2(z_1)$; in other words, we view z_1 and z_2 as the independent and the dependent variables respectively. For $d \in \mathbb{N}$, a *singular holomorphic d-web* $\mathcal{W}$ in U is defined by equation (6) where Φ is of the form

$$\Phi(z, p) = \sum_{j=0}^{d} \Phi_j(z) p^j. \tag{7}$$

In general, there are d families of solutions of (6) (with $\Phi(z,p)$ as in (7)), which are either unrelated to each other or may fit together along some complex curves (branching). The graphs of solutions are called the *leaves* of $\mathcal{W}$.

Example 3.1. Consider the ODE of the form $p^2 = 4z_2$ in $\mathbb{C}^2$. Its solutions form a complex one-dimensional family of curves $L_c = \{z_2 = (z_1 + c)^2\}$, $c \in \mathbb{C}$. For every point $b = (b_1, b_2) \in \mathbb{C}^2$ with $b_2 \neq 0$, there exist exactly two curves passing through this point, namely, $L_{-b_1 - \sqrt{b_2}}$ and $L_{-b_1 + \sqrt{b_2}}$ (we can take an arbitrary branch of $\sqrt{z}$). These curves meet at b transversely. But any point $(b_1, 0)$ is contained only in one curve L_{-b_1} of the family. $\diamond$

If $d = 1$, then (6) becomes resolved with respect to the derivative, so 1-webs simply coincide with holomorphic foliations (possibly singular). If (7) factors into distinct, linear in p terms, i.e., $\Phi(z,p) = \Pi_{j=1}^d (p - f_j(z))$, where $f_j(z)$ are holomorphic functions, then each ODE $p = f_j(z)$ defines a holomorphic foliation $\mathcal{F}_j$. If the leaves of $\mathcal{F}_j$ intersect in general position (resp. pairwise transversely) then the union of $\mathcal{F}_j$ is called a smooth (resp. quasi-smooth) holomorphic d-web. Thus, our definition of a singular d-web is a proper generalization of smooth webs. From this point of view one can consider singular d-webs as a “branched"version of their smooth counterparts.

3.2. Webs in $\mathbb{C}^n$, $n \geq 2$. The definition of a d-web (singular or smooth) via differential equations does not have a simple generalization to higher dimensions. There are several equivalent definitions in the literature. We will use a more geometric one that is more suitable for our purposes.

We denote by $\mathbb{P}T_n^* := \mathbb{P}T^*\mathbb{C}^n$ the projectivization of the cotangent bundle of $\mathbb{C}^n$ with the natural projection $\pi : \mathbb{P}T_n^* \to \mathbb{C}^n$. A local trivialization of $\mathbb{P}T_n^*$ is isomorphic to $U \times G(1, n)$, where $U \subset \mathbb{C}^n$ is an open set and $G(1,n) \cong \mathbb{C}P^{n-1}$ is the Grassmanian space of linear complex one dimensional subspaces in $\mathbb{C}^n$. The space $\mathbb{P}T^*_n$ has the canonical structure of a *contact manifold*, which can be described (using coordinates) as follows. Let $z = (z_1, \dots, z_n)$ be the coordinates in $\mathbb{C}^n$ and $(\tilde{p}_1, \dots, \tilde{p}_n)$ be the fibre coordinates corresponding to the basis of differentials $dz_1, \dots, dz_n$. We may view $[\tilde{p}_1, \dots, \tilde{p}_n]$ as homogeneous coordinates on $G(1,n)$. Then in the affine chart $\{\tilde{p}_n \neq 0\}$, in nonhomogeneous coordinates $p_j = \tilde{p}_j/\tilde{p}_n$, $j = 1, \dots, n-1$, the 1-form

$$\eta = dz_n + \sum_{j=1}^{n-1} p_j dz_j \tag{8}$$

is a local contact form. Considering all affine charts $\{\tilde{p}_j \neq 0\}$ we obtain a global contact structure.

Let U be a domain in $\mathbb{C}^n$. Consider a complex purely n-dimensional analytic subset W in $\pi^{-1}(U) \subset \mathbb{P}T^*_n$. Suppose that the following conditions hold:

(a) the image under π of every irreducible component of W has dimension n;
(b) a generic fibre of π intersects W in d regular (smooth) points and at every such point q the differential $d\pi(q) : T_qW \to \mathbb{C}^n$ is surjective;
(c) the restriction of the contact form η on the regular part of W is Frobenius integrable. So $\eta|_W = 0$ defines the foliation $\mathcal{F}_W$ of the regular part of W. (The leaves of the foliation $\mathcal{F}_W$ are called Legendrian submanifolds.)

Under these assumptions we define a *singular d-web* $\mathcal{W}$ in U as a triple $(W, \pi, \mathcal{F}_W)$. A leaf of the web $\mathcal{W}$ is a component of the projection of a leaf of $\mathcal{F}_W$ into U. Note that at a generic point $z \in U$ a d-web $(W, \pi, \mathcal{F}_W)$ defines in U near z exactly d families of smooth foliations.

We need first to interpret a first order PDE as a subvariety of a 1-jet bundle. Recall that two smooth functions ϕ_1 and ϕ_2 have the same k-jet at a source point $x^0 \in \mathbb{C}^n$ if $|\phi_1(x) - \phi_2(x)| = o(|x - x^0|^k)$. In other words, this simply means that their Taylor expansions of order k at x^0 coincide. The equivalence classes with respect to this relation are called k-jets at x^0.

Let $U \subset \mathbb{C}^{n-1}$ be a domain. Consider $J^1(U, \mathbb{C})$, the space of 1-jets of holomorphic functions $f : U \to \mathbb{C}$. We can view such functions as sections of the trivial line bundle $U \times \mathbb{C} \to U$. Then $J^1(U, \mathbb{C})$ can be viewed as a vector bundle

$$\pi : J^1(U, \mathbb{C}) \to U \times \mathbb{C} \tag{9}$$

of rank $(n-1)$. Let $z' = (z_1, \dots, z_{n-1})$ be the coordinates on $U \subset \mathbb{C}^{n-1}$, z_n be the coordinate in the target space, and let p_j denote the partial derivatives of z_n with respect to z_j. Then $(z, p) = (z_1, \dots, z_n, p_1, \dots, p_{n-1})$ form the coordinate system on $J^1(U, \mathbb{C})$. Note that $\dim J^1(U, \mathbb{C}) = 2n - 1$. The space $J^1(U, \mathbb{C})$ admits the structure of a contact manifold with the contact form

$$\theta = dz_n - \sum_{j=1}^{n-1} p_j dz_j. \tag{10}$$

Given a local section $f : U \to \mathbb{C}$, let $j^1 f : U \to J^1(U, \mathbb{C})$, $j^1 f : z \mapsto j^1_z f$ denote the corresponding section of the 1-jet bundle. Then a section $F : U \to J^1(U, \mathbb{C})$ locally coincides with $j_1 f$ for some section $f : U \to \mathbb{C}$ if and only if F annihilates θ. Now observe that the map $\iota : (z, p) \mapsto (z, -p)$ in the chosen coordinate systems is a biholomorphism whose pullback sends η to θ in (8), i.e., $\iota : J^1(U, \mathbb{C}) \to \mathbb{P}T^*_n$ is a contactomorphism. Using the map ι we may view the projectivized cotangent bundle $\mathbb{P}T^*_n$ as a compactification of the 1-jet bundle. Alternatively, we may compactify $J^1(U, \mathbb{C})$ in the variables p, that is, we compactify every fibre $\mathbb{C}^{n-1}_p$ to $\mathbb{C}P^{n-1}$. Since the dependence of the form θ is linear in p, the compactified bundle will be a contact complex manifold.

Any first order holomorphic PDE of the form

$$\Phi\left(z_1,\dots,z_{n-1},z_n,\frac{\partial z_n}{\partial z_1},\dots,\frac{\partial z_n}{\partial z_{n-1}}\right)=0 \tag{11}$$

with respect to the unknown function $z_n = z_n(z_1,\dots,z_{n-1})$ defines a complex hypersurface W_Φ in $J^1(U,\mathbb{C})$ given by the equation $\Phi(z,p)=0$. Any solution $z_n = f(z_1,\dots,z_{n-1})$ of (11) admits *prolongation* to $J^1(U,\mathbb{C})$, i.e., defines there an $(n-1)$-dimensional submanifold S_f given by

$$\left\{z_n = f(z_1,\dots,z_{n-1}),\quad p_j=\frac{\partial f}{\partial z_j}(z_1,\dots,z_{n-1}),\quad j=1,\dots,n-1\right\}.$$

Hence, solutions of this differential equation can be identified with holomorphic sections S_f of W_Φ annihilated by the contact form θ. As an example, for equation (5), the corresponding hypersurface $W\subset J^1(U,\mathbb{C})$, $U\subset\mathbb{C}$, is simply the graph of a holomorphic function $p=F(z)$. It is foliated by graphs of solutions, which are integral curves of the distribution defined by θ, and the corresponding foliation $\mathcal{F}$ in $\mathbb{C}^2$ is obtained by the biholomorphic projection $\pi|_W : W\to\mathbb{C}^2_z$.

Suppose now that we have several differential equations of the form (11) such that the intersection of the corresponding hypersurfaces W_Φ is a complex analytic subset W of $J^1(U,\mathbb{C})$ of pure dimension n. For example, we can have $n-1$ equations in general position. Suppose further that the compactification of W in the projectivized cotangent bundle $\mathbb{P}T^*_nU$ still forms a complex subvariety of the same dimension. This is the case, for example, if all $\Phi(z,p)$ are polynomial with respect to p with coefficients holomorphic in z. Then W satisfies the definition of a singular web given in the previous subsection. Note that we need to consider compactification of W only if the projection in (9) has fibres of positive dimension, since otherwise the projection from $J^1(U,\mathbb{C})$ gives the same web in $U\subset\mathbb{C}^n$.

Also note that for $n=2$ both definitions of a singular web agree. Indeed, given a differential equation (6), (7), the function $\Phi(z,p)$ is polynomial in p and thus it can be projectivized to define a hypersurface in $\mathbb{P}T^*_2U$. This gives the hypersurface in $\mathbb{P}T^*_2U$ that has the required properties. Conversely, let U be a neighbourhood of the origin in $\mathbb{C}^2$, let W be a complex hypersurface in $\mathbb{P}T^*_2U$ with the surjective projection $\pi: W\to U$. Without loss of generality assume that W is irreducible. If π^{-1} is discrete, then by the Weierstrass preparation theorem, in a sufficiently small neighbourhood $\tilde U$ of the origin W can be represented by a Weierstrass pseudo-polynomial in p, and we obtain the definition of the web given in Section 3.1. Suppose that $\dim\pi^{-1}(0)=1$. Let $\tau:\mathbb{C}^2_{(p_0,p_1)}\setminus\{0\}\to\mathbb{C}P^1$ be the natural projection given by $\tau(p_0,p_1)=[p_0,p_1]$. Let

$$\tilde\tau=(\mathrm{Id},\tau): U\times(\mathbb{C}^2\setminus\{0\})\to U\times\mathbb{C}P^1.$$

Then the set $\tilde W=\tilde\tau^{-1}(W)$ is complex analytic in $U\times(\mathbb{C}^2\setminus\{0\})$ of dimension 3. The set $U\times\{0\}$ is removable, and so we may assume that $\tilde W$ is complex analytic in $U\times\mathbb{C}^2$. In a neighbourhood of $(0,0)$ it can be given by an equation $\phi(z,p)=0$. But since its image is complex analytic in $U\times\mathbb{C}P^1$, the function ϕ is in fact a homogenous polynomials in p. This shows that in a neighbourhood of the origin in U, the hypersurface W can be given by an equation which is polynomial in variable p, and we again recover the definition of Section 3.1.

We also need a related notion of a *multi-valued meromorphic first integral*. Let X and Y be two complex manifolds and $\pi_X: X\times Y\to X$ and $\pi_Y: X\times Y\to Y$ be the natural projections. A d-valued *meromorphic correspondence* between X and Y is a complex analytic subset $Z\subset X\times Y$ such that the restriction $\pi_X|Z$ is a proper surjective generically d-to-one map. Hence, $\pi_Y\circ\pi_X^{-1}$ is defined generically on X (i.e., outside a proper complex analytic subset in X), and can be viewed as a d-valued map. In what follows we denote a meromorphic correspondence by a triple $(Z;X,Y)$ equipped with the canonical projections:

$$\begin{array}{ccc} & Z & \\ {\scriptstyle \pi_X} \swarrow & & \searrow {\scriptstyle \pi_Y} \\ X & & Y \end{array}$$

A *multiple-valued meromorphic first integral* of a singular d-web $\mathcal{W}$ in U is a d-valued meromorphic correspondence $(Z; U, \mathbb{C}P)$ such that level sets $\pi_X \circ \pi_Y^{-1}(c)$, $c \in \mathbb{C}P$ are invariant subsets of $\mathcal{W}$, i.e., they consist of the leaves of $\mathcal{W}$.

Definition 3.2. Let Γ be a real analytic Levi-flat hypersurface in a domain $\Omega \subset \mathbb{C}^n$. We say that a holomorphic d-web $\mathcal{W}$ in Ω is the *extension* of the Levi foliation of Γ^* if every leaf of the Levi foliation is a leaf of $\mathcal{W}$.

Although in this definition we do not require $\mathcal{W}$ to be irreducible, we suppose that at least one leaf of every component of W meets Γ^*. Clearly, under this condition the singular web extending the Levi foliation is unique.

4. Extension of the Levi foliation

Global or local extension of the Levi foliation to the ambient space is an important question, see, e.g., [2, 3, 5, 9] for recent results in this direction. Brunella [2] gave an example of a Levi-flat hypersurface in $\mathbb{C}^2$, singular at the origin, such that the Levi foliation extends to a neighbourhood of the origin as a *singular web*, but not as a foliation, see Example 4.3. The following result is obtained in [18].

Theorem 4.1. *Let $\Gamma \subset \Omega$ be an irreducible Levi-flat real analytic hypersurface in a domain $\Omega \subset \mathbb{C}^n$, $n \geq 2$, and $0 \in \overline{\Gamma^*}$. Assume that at least one of the following conditions holds:*

(a) *$0 \in \Gamma$ is not a dicritical singularity.*
(b) *Γ is a real algebraic hypersurface.*

Then there exist a neighbourhood U of the origin and a singular holomorphic d-web $\mathcal{W}$ in U such that $\mathcal{W}$ extends the Levi foliation $\mathcal{L}$. Furthermore, $\mathcal{W}$ admits a multiple-valued meromorphic first integral in U.

We note that under some additional assumptions on the singular locus of Γ, part (a) of our result was obtained recently by Fernández-Pérez [9]. Our approach is rather constructive, especially under condition (b) in Theorem 4.1. In many cases one can write down explicitly the d-web that gives the extension of the Levi foliation. The key point of our approach lies in the connection between singular webs and first order analytic partial differential equations, although we do not claim any particular originality here. Presently, the most commonly used definition of webs is through the geometry the projectivized cotangent bundle. We reconstruct the connection between geometry of singular webs and analytic PDEs through compactification of the 1-jet bundle of functions on $\mathbb{C}^{n-1}$ and its identification with the projectivized cotangent bundle of $\mathbb{C}^n$, see Section 3 for details.

The proof of Theorem 4.1 (in the Case (a)) is based on the following idea. The leaves of the Levi foliation can be identified with the components of the Segre varieties associated with Γ. It is possible to find a complex line parametrizing all Segre varieties of Γ. While for a general real analytic hypersurface Γ in $\mathbb{C}^n$, the corresponding family of Segre varieties is n-dimensional, Levi-flat hypersurfaces can be characterized as those whose Segre family is one-dimensional, and ultimately this is the reason why the Levi foliation admits extension to the ambient space. Essentially, a suitably chosen one-dimensional family of Segre varieties is the meromorphic (perhaps, multiple-valued) first integral. Its graph can be described by a system of $n-1$ first order PDEs. This system defines an n-dimensional complex analytic subvariety of the 1-jet bundle of holomorphic functions on $\mathbb{C}^{n-1}$. This subvariety can then be compactified in the projectivized cotangent bundle of $\mathbb{C}^n$, which gives the singular d-web.

The key statement is the following

Proposition 4.2. *Under the assumptions of Theorem 4.1, for a sufficiently small neighbourhood Ω of the origin there exists a complex line $A \subset \mathbb{C}^n$ with the following properties:*

(i) $A \cap Q_0 = \{0\}$;
(ii) $A \not\subset \Gamma^{sng}$;
(iii) *For every $q \in \Gamma^* \cap \Omega$, there exists a point $w \in A$ such that $\mathcal{L}_q \subset Q_w$.*

The existence of such A should be compared to the transversal of the Levi foliation in the smooth case: if Γ is given by $\{\operatorname{Re} z_n = 0\}$, then the complex line $\{z_1 = \cdots = z_{n-1} = 0\}$ intersects all Segre varieties, and can be used as a local parametrization both of the Levi foliation and its extension. The complex one-parameter family of the Segre varieties constructed in Proposition 4.2 can be viewed as a holomorphic web. The equations of this web can be explicitely constructed using tools of the local complex analytic geometry (on jet bundles). As an illustration consider the following example.

Example 4.3. This example, discovered by M. Brunella [2], shows that in general the Levi foliation of a Levi-flat hypersurface admits extension to a neighbourhood of a singular point only as a web, not as a singular foliation. Consider the Levi-flat hypersurface

$$\Gamma = \{z \in \mathbb{C}^2 : y_2^2 = 4(y_1^2 + x_2)y_1^2\}. \tag{12}$$

The singular locus of Γ is the set $\{y_1 = y_2 = 0\}$. Its subset given by $\{y_1 = y_2 = 0,\ x_2 < 0\}$ is a "stick i.e., it does not belong to the closure of smooth points of Γ. The Segre varieties of Γ are given by

$$Q_w = \{z \in \mathbb{C}^2 : (z_2 - \bar{w}_2)^2 + (z_1 - \bar{w}_1)^4 - 2(z_2 + \bar{w}_2)(z_1 - \bar{w}_1)^2 = 0\}. \tag{13}$$

We see that $Q_0 = \{z_2^2 + z_1^4 - 2z_2z_1^2 = 0\}$, and the origin is a nondicritical singularity. Following the algorithm in the proof of Theorem 4.1 we choose $A(t)$ to be given by $w_1 = 0$, $w_2 = \bar{t}$. Then (13) becomes

$$(z_2 - t)^2 + z_1^4 - 2z_1^2(z_2 + t) = 0.$$

After differentiation with respect to z_1, and using the notation $p = \frac{dz_2}{dz_1}$ we obtain

$$2(z_2 - t)p + 4z_1^3 - 4z_1(z_2 + t) - 2z_1^2 p = 0.$$

Direct calculation shows that the resultant of the two polynomials in t above vanishes (after dropping irrelevant factors) when

$$p^2 = 4z_2. \tag{14}$$

This is the 2-web that extends the Levi foliation of Γ^*. Its behaviour is described in Example 3.1. Note that the exceptional set $\{z_2 = 0\}$ intersects Γ along the line $\{z_2 = y_1 = 0\} \subset \Gamma^{sng}$.

By inspection of solutions of (14) we see that a first integral of Γ can be taken to be

$$f(z_1, z_2) = z_1 \pm \sqrt{z_2},$$

where f is understood as a multiple-valued $1-2$ map. In fact, one can immediately verify that the closure of the smooth points of Γ is given by

$$\{z \in \mathbb{C}^2 : \operatorname{Im}(z_1 \pm \sqrt{z_2}) = 0\} = \{\operatorname{Im}(z_1 + \sqrt{z_2})\} \cup \{\operatorname{Im}(z_1 - \sqrt{z_2})\}.$$

However, the points of the stick cannot be recovered from the first integral. $\diamond$

5. Levi-flat subsets, Segre varieties and Segre envelopes

A real analytic Levi-flat set M in $\mathbb{C}^N$ is a real analytic set such that its regular part is a Levi-flat CR manifold of hypersurface type. An important special case (closely related to the theory of holomorphic foliations) arises when M is a hypersurface. The main question here concerns an extension of the Levi foliation of the regular part of M as a (singular) holomorphic foliation (or, more generally, a singular holomorphic web) to a full neighbourhood of a singular point. The existence of such an extension allows one to use the holomorphic resolution theorems in order to study local geometry of singular Levi-flat hypersurfaces.

In this section we provide relevant background material on real analytic Levi-flat sets (of higher codimension) and their Segre varieties. This is quite similar to the special case of hypersurfaces considered in previous sections.

5.1. Real and complex analytic sets. Let Ω be a domain in $\mathbb{C}^N$. We denote by $z = (z_1, \dots, z_N)$ the standard complex coordinates. A closed subset $M \subset \Omega$ is called *a real (resp. complex) analytic subset* in Ω if it is locally defined by a finite collection of real analytic (resp. holomorphic) functions.

For a real analytic M this means that for every point $q \in \Omega$ there exist a neighbourhood U of q and real analytic vector function $\rho = (\rho_1, \dots, \rho_k) : U \to \mathbb{R}^k$ such that

$$M \cap U = \rho^{-1}(0) = \{z \in U : \rho_j(z, \overline{z}) = 0,\ j = 1, \dots, k\}. \tag{15}$$

In fact, one can reduce the situation to the case $k = 1$ by considering the defining function $\rho_1^2 + \dots + \rho_k^2$. Without loss of generality assume $q = 0$ and choose a neighbourhood U in (15) in the form of a polydisc $\Delta(\varepsilon) = \{z \in \mathbb{C}^N : |z_j| < \varepsilon\}$ of radius $\varepsilon > 0$. Then, for ε small enough, the (vector-valued) function ρ admits the Taylor expansion (1) convergent in U. Of course, the ($\mathbb{C}^k$-valued) coefficients c_{IJ} also satisfy the condition (2), since ρ is a real-valued ($\mathbb{R}^k$-valued) function.

An analytic subset M is called *irreducible* (as a germ) if its germ at $0 \in M$ can not be represented as a union $M = M_1 \cup M_2$ where M_j are analytic germs at 0 different from the germ of M. In what follows we always use this notion of irreducibility. A set M can be decomposed into a disjoint union $M = M_{reg} \cup M_{sing}$, the regular and the singular part respectively. Notice that here we change the notation with respect to the case of a hypersurface. The regular part M_{reg} is a nonempty and open subset of M. In the real analytic case we adopt the following convention: M is a real analytic submanifold of maximal dimension in a neighbourhood of every point of M_{reg}. This dimension is called the dimension of M and is denoted by $\dim M$. The set M_{sing} is a real semianalytic subset of Ω of dimension $< \dim M$. Unlike complex analytic sets, for a real analytic M, the set M_{sing} may contain manifolds of smaller dimension which are not in the closure of M_{reg}, as seen in the classical example of the Whitney umbrella. Therefore, in general M_{reg} is not dense in M.

Recall that the dimension of a complex analytic set A at a point $a \in A$ is defined as

$$\dim_a A := \overline{\lim_{A_{reg} \ni z \to a}} \dim_z A,$$

and that the the function $z \mapsto \dim_z A$ is upper semicontinuous. Suppose that A is an irreducible complex analytic subset of a domain Ω and let $F : A \to X$ be a holomorphic mapping into some complex manifold X. The local dimension of F at a point $z \in \Omega$ is defined as $\dim_z F = \dim A - \dim_z F^{-1}(F(z))$ and the dimension of F is set to be $\dim F = \max_{z \in A} \dim_z F$. Note that the identity $\dim_z F = \dim F$ holds on a Zariski open subset of A, and that $\dim F$ coincides with the rank of the map F.

5.2. Complexification and Segre varieties. Let M be (the germ of) an irreducible real analytic subset of $\mathbb{C}^N$ defined by (1). Denote by J the standard complex structure of $\mathbb{C}^N$ and consider the opposite structure $-J$. Consider the space $\mathbb{C}^{2N}_\bullet := (\mathbb{C}^N_z, J) \times (\mathbb{C}^N_w, -J)$ and the diagonal

$$\Delta = \left\{(z,w) \in \mathbb{C}^{2N}_\bullet : z = w\right\}.$$

The set M can be lifted to $\mathbb{C}^{2N}_\bullet$ as the real analytic subset

$$\hat{M} := \left\{(z,z) \in \mathbb{C}^{2N}_\bullet : z \in M\right\}.$$

There exists a unique irreducible complex analytic subset $M^{\mathbb{C}}$ in $\mathbb{C}^{2N}_\bullet$ of complex dimension equal to the real dimension of M such that $\hat{M} = M^{\mathbb{C}} \cap \Delta$ (see [14]). The set $M^{\mathbb{C}}$ is called the *complexification* of M. The antiholomorphic involution

$$\tau : \mathbb{C}^{2N}_\bullet \to \mathbb{C}^{2N}_\bullet, \ \ \tau : (z,w) \mapsto (w,z)$$

leaves $M^{\mathbb{C}}$ invariant and $\hat{M}$ is the set of fixed points of $\tau|_{M^{\mathbb{C}}}$.

The complexification $M^{\mathbb{C}}$ is equipped with two canonical holomorphic projections $\pi_z : (z,w) \mapsto z$ and $\pi_w : (z,w) \mapsto w$. We always suppose by convention that the domain of these projections is $M^{\mathbb{C}}$. The triple $(M^{\mathbb{C}}, \pi_z, \pi_w)$ is represented by the following diagram

$$\begin{array}{ccc} & M^{\mathbb{C}} & \\ {}^{\pi_z}\swarrow & & \searrow^{\pi_w} \\ (\mathbb{C}^N, J) & & (\mathbb{C}^N, -J) \end{array}$$

which leads to the central notion of the present paper in full generality. The *Segre variety* of a point $w \in \mathbb{C}^N$ is defined as

$$Q_w := (\pi_z \circ \pi_w^{-1})(w) = \left\{z \in \mathbb{C}^N : (z,w) \in M^{\mathbb{C}}\right\}.$$

When M is a hypersurface defined by (15) (with $k = 1$) this definition coincides with the usual definition

$$Q_w = \{z : \rho(z, \overline{w}) = 0\}.$$

The following properties of Segre varieties are well-known for hypersurfaces.

Proposition 5.1. *Let M be any real analytic subset of a domain Ω. Then*

(a) $z \in Q_z \Longleftrightarrow z \in M$.
(b) $z \in Q_w \Longleftrightarrow w \in Q_z$,
(c) *(invariance property) Let M_1, M_2 be real analytic subsets in $\mathbb{C}^N$ and $\mathbb{C}^K$ respectively, $p \in (M_1)_{reg}$, $q \in (M_2)_{reg}$, and $U_1 \ni p$, $U_2 \ni q$ be small neighbourhoods such that $M_j \cap U_j$ is a CR manifold. Let also $f : U_1 \to U_2$ be a holomorphic map such that $f(M_1 \cap U_1) \subset M_2 \cap U_2$. Then*

$$f(Q^1_w) \subset Q^2_{f(w)}$$

for all w close to p. In particular, if $f : U_1 \to U_2$ is biholomorphic, then $f(Q^1_w) = Q^2_{f(w)}$. Here Q^1_w and $Q^2_{f(w)}$ are Segre varieties associated with M_1 and M_2 respectively.

5.3. Levi-flat sets. We say that an irreducible real analytic set $M \subset \mathbb{C}^{n+m}$ is *Levi-flat* if $\dim M = 2n-1$ and M_{reg} is locally foliated by complex manifolds of complex dimension $n-1$. In particular, M_{reg} is a CR manifold of hypersurface type. The most known case arises when $m = 0$, i.e., when M is a Levi-flat hypersurface in $\mathbb{C}^n$.

We use the notation $z'' = (z_{n+1}, \ldots, z_{n+m})$, and similarly for the w variable. It follows from the Frobenius theorem and the implicit function theorem that for every point $q \in M_{reg}$ there exist an open neighbourhood U and a local biholomorphic change of coordinates $F : (U,q) \to (U',0)$ such that $F(M)$ has the form

$$\{z \in U' : z_n + \overline{z}_n = 0, \ z'' = 0\}. \tag{16}$$

The subspace $F(M)$ is foliated by complex affine subspaces $L_c = \{z_n = ic, \ z'' = 0, \ c \in \mathbb{R}\}$, which gives a foliation of $M_{reg} \cap U$ by complex submanifolds $F^{-1}(L_c)$. This defines a foliation on M_{reg} which is called *the Levi foliation* and denoted by $\mathcal{L}$. Every leaf of $\mathcal{L}$ is tangent to the complex tangent space of M_{reg}. The complex affine subspaces

$$\{z_n = c, z'' = 0\}, \quad c \in \mathbb{C} \tag{17}$$

in local coordinates given by (16) are precisely the Segre varieties of M for every complex c. Thus, the Levi foliation is closely related to Segre varieties.

For M defined by (16) its Segre varieties (17) fill the complex subspace $z'' = 0$ of $\mathbb{C}^{n+m}$. In particular, if w is not in this subspace, then Q_w is empty. We need to study some general properties of projections π_z and π_w.

Let π be one of the projections π_z or π_w. Following the discussion in the previous subsection we introduce the dimension of π by setting $\dim \pi = \max_{(z,w) \in M^{\mathbb{C}}} \dim_{(z,w)} \pi$. If M is irreducible, then so is $M^{\mathbb{C}}$ (see [14, p.92]). Hence, $(M^{\mathbb{C}})_{reg}$ is a connected complex manifold of dimension $2n-1$. Then the equality $\dim_{(z,w)} \pi = \dim \pi$ holds on a Zariski open set

$$M^{\mathbb{C}}_* := M^{\mathbb{C}} \setminus E \subset (M^{\mathbb{C}})_{reg}, \tag{18}$$

where E is a complex analytic subset of dimension $< 2n-1$. Here $\dim \pi$ coincides with the rank of π. Furthermore, $\dim(\pi | (M^{\mathbb{C}})_{sing}) \le \dim \pi$.

Lemma 5.2. (a) *We have* $\dim \pi_z = n$.
(b) *The image $\pi_z(M^{\mathbb{C}})$ is contained in the (at most) countable union of complex analytic sets of dimension $\le n$.*

Proof. (a) The image of π_z near a regular point of $\hat{M}$ is swept out by the Segre varieties (17). Hence it coincides with the subspace $\{z'' = 0\}$.

(b) This is a consequence of (a). □

Of course, the projection π_w has similar properties. Therefore, we have the following

Lemma 5.3. *For every w, one of the following holds:*
(a) *Q_w is empty.*
(b) *Q_w is a complex analytic subset of dimension $n-1$.*
(c) *Q_w is a complex analytic subset of dimension n.*
The case (b) holds for $\pi_w(M^{\mathbb{C}}_*)$.

A singular point $q \in M$ is called *Segre degenerate* if $\dim Q_q = n$. Note that the set of Segre degenerate points is contained in a complex analytic subset of dimension $n-2$. The proof is quite similar to [18] (where this claim is established for hypersurfaces) so we skip the proof.

Let $q \in M_{reg}$. Denote by $\mathcal{L}_q$ the leaf of the Levi foliation through q. Note that by definition this is a connected complex submanifold of complex dimension $n-1$ closed in M_{reg}. Denote by $M_* \subset M_{reg}$ the image of $\hat{M} \cap M^{\mathbb{C}}_*$ under the projection π, where $M^{\mathbb{C}}_*$ is defined as in (18). This set coincides with $M_{reg} \setminus A$ for some proper complex analytic subset A.

Finally, let M be an irreducible real analytic Levi-flat subset of dimension $2n-1$ in a domain Ω in $\mathbb{C}^{n+m}$. Consider the set

$$S(M) = \{z \in \Omega : z \in Q_w \text{ for some } \ w\}.$$

We call $S(M)$ *the Segre envelope* of M. Thus, we simply have

$$S(M) = \pi(M^{\mathbb{C}}).$$

This notion will play a crucial role in our approach. It follows from (16) and (17) that near every regular point of M the Segre envelope of M is a complex submanifold of dimension n containing M_{reg}. One of our goals is to describe the Segre envelope near a singular point of M.

Let M be a real analytic Levi-flat subset of dimension $2n-1$ in $\mathbb{C}^{n+m}$. A singular point $q \in M$ is called *dicritical* if q belongs to infinitely many geometrically different leaves $\mathcal{L}_a$. Singular points which are not dicritical are called *nondicritical.* We have the following efficient criterion obtained in [17].

Theorem 5.4. *Let M be an irreducible real analytic Levi-flat subset of dimension $2n-1$ in a domain $\Omega \subset \mathbb{C}^{n+m}$. Then the point $0 \in \overline{M_{reg}}$ is dicritical if and only if $\dim_{\mathbb{C}} Q_0 = n$, that is, the origin is a Segre degenerate point.*

This generalizes the case of hypersurfaces treated in Theorem 2.6.

The connection between holomorphic webs and the Levi foliation is described by the following proposition [17] generalizing Proposition 4.2 :

Proposition 5.5. *Let M be an irreducible real analytic Levi-flat subset of dimension 2n-1 in a domain $\Omega \subset \mathbb{C}^{n+m}$. Assume that $0 \in \overline{M_{reg}}$ is a nondicritical singularity for M. For a sufficiently small neighbourhood Ω of the origin there exists a complex linear map $L : \mathbb{C}^{m+1} \to \mathbb{C}^{n+m}$ with the following properties:*

(i) *$L(\mathbb{C}^{m+1}) \cap Q_0 = \{0\}$;*
(ii) *the 1-dimensional real analytic set $\gamma = L(\mathbb{C}^{m+1}) \cap M$ is not contained in M_{sing}*
(iii) *For every $q \in M_{reg} \cap \Omega$, there exists a point $w \in \gamma$ such that $\mathcal{L}_q \subset Q_w$.*

The first consequence is the following theorem obtained by Brunella [2].

Corollary 5.6. *Let M be an irreducible real analytic Levi-flat subset in $\mathbb{C}^{n+m}$.The Segre envelope $S(M)$ is an irreducible complex analytic subset of dimension n containing $\overline{M_{reg}}$.*

Proof. We have $S(M) = \pi(M^{\mathbb{C}})$. There are two cases.

Case 1. $\dim Q_0 = n-1$. Then the desired result follows from Proposition 5.5 and the Remmert Rank theorem.

Case 2. $\dim Q_0 = n$. By Theorem 5.4 for every regular point a of M we have $0 \in Q_a$, hence $a \in Q(0)$. □

The main application of Corollary 5.6 is the following result due to Brunella [2]:

Corollary 5.7. *Let M be an irreducible real analytic Levi-flat hypersurface in a complex manifold V of dimension n. Then there exist a complex manifold X of dimension n, a real analytic Levi-flat hypersurface N in X, a holomorphic foliation $\mathcal{F}$ in X extending the Levi foliation of N, and a holomorphic map $\pi : X \to V$ such that for some Zariski open subset $U \subset X$ one has:*

(i) *$\pi : N \cap U \to M_{reg}$ is an embedding;*
(ii) *$\pi : \overline{N \cap U} \to \overline{M_{reg}}$ is a proper map.*

The main idea of the proof is to consider the holomorphic tangent bundle $H(M)$ of M. This is a real analytic Levi-flat subset of the projectivization of the cotangent bundle of V, see Section 3. The second step is to apply the Hironaka desingularization theorem to the Segre envelope of $H(M)$.

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