ISSN 2074-1863 Уфимский математический журнал. Том 7. № 4 (2015). С. 160-171.

УДК 517.5

ABOUT A CONJECTURE REGARDING PLURISUBHARMONIC FUNCTIONS

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Abstract. In this work we present Khabibullin's conjecture in its different equivalent forms. Applying the concept of the integral operator, we transform the original conjecture into a new form which proves to be helpful in studying it by means of the Laplace transform. Using Laplace transform of integral inequalities, we are able to show the uniqueness of a solution that satisfies both inequalities with identity. Furthermore we provide a new proof of Khabibullin's theorem by methods of the Laplace transform and contour integration from complex analysis. However, this method of transform fails to prove the conjecture and a brief reasoning is provided.

Keywords: Khabibullin's hypothesis, integral inequalities, plurisubharmonic function, Laplace transform, complex analysis, contour integration, sharp estimate.

Mathematics Subject Classification:31A05, 31A10, 31C10, 44A10

1. Introduction

A general class of functions from the space of complex numbers to the extended real line is the class of plurisubharmonic functions. These functions play an important role in complex analysis because they provide useful insights about extremal value problems for the entire functions of several variables. A function u with values in $[-\infty, \infty)$ defined in an open set $X \subset \mathbb{C}^n$ is called plurisubharmonic if

- u is upper semi-continuous
- for arbitrary z and $w \in \mathbb{C}^n$ the function

$$\tau \mapsto u(z + \tau w)$$

  is subharmonic in the open subset $\mathbb{C}$ where it is defined.

For example, if $f(z)$ is an holomorphic function on some domain $U \subseteq \mathbb{C}$, then the function $u = \log |f(z)|$ can be shown to be a plurisubharmonic function. The set of these type of functions satisfy certain properties, which are stated in the following theorem without proof.

Theorem 1. [1] *If u is plurisubharmonic and $0 < c$ where $c \in \mathbb{R}$, then cu is plurisubharmonic. If $u_1, u_2, \dots, u_n$ are plurisubharmonic then $u_1 + u_2 + \ldots + u_n$ and $\max\{u_1, u_2, \dots, u_n\}$ are also plurisubharmonic. If $u_1, u_2, \dots$ is a decreasing sequence of plurisubharmonic functions, then $u^* = \lim_{n\to\infty} u_n$ is also plurisubharmonic.*

The above theorem states that the set of plurisubharmonic functions forms a convex cone in the vector space of semi-continuous functions and that a monotonically decreasing sequence of plurisubharmonic functions has a limiting function, which is also plurisubharmonic. A particular investigation in the theory of complex valued functions was made for value distributions and growth rate of entire and meromorphic functions. And a similar study was made for the set of plurisubharmonic functions. The following result on the growth rate of a plurisubharmonic function is due to Khabibullin.

Theorem 2 (Khabibullin's Theorem). [2] *Let u be a plurisubharmonic function in $\mathbb{C}^n$ of finite lower order ρ then* [1]

$$\liminf_{r\to\infty} \frac{M(u,r)}{T(u,r)} \leqslant P(\rho) \prod_{k=1}^{n-1} \left(1+\frac{\rho}{2k}\right) \tag{1}$$

for $\rho \leqslant 1$ and this estimate is best possible. Here $M(u,r) = \max\{u(z) : |z| = r\}$ and $T(u,r)$ is the Nevanlinna characteristic of the function u. The real-valued function $P(\rho)$ is a piece-wise function defined as follows,

$$P(\rho) = \begin{cases} \dfrac{\pi\rho}{\sin(\pi\rho)} & : 0 \leqslant \rho \leqslant \dfrac{1}{2} \\ \pi\rho & : \rho > \dfrac{1}{2} \end{cases}$$

While theorem 2 provides an upper bound for all non-negative parameter values ρ not greater than one, the problem whether the same upper bound holds for parameter values strictly greater than one remains an open problem. The conjecture that the same upper bound holds for any parameter value greater than one is what is called "Khabibullin's conjecture". We formally state the conjecture below.

Conjecture 1. [2] *Let u be a plurisubharmonic function in $\mathbb{C}^n$ of finite lower order ρ. Then*

$$\liminf_{r\to\infty} \frac{M(u,r)}{T(u,r)} \leqslant P(\rho) \prod_{k=1}^{n-1} \left(1+\frac{\rho}{2k}\right) \tag{2}$$

for $\rho > 1$.

A sufficient condition for the above hypothesis to hold is the confirmation of the following,

Conjecture 2. [2] *Let S be a non-negative increasing function on $\mathbb{R}^+, S(0) = 0$, and the function $S(t)$ is convex with respect to $\log t$, i.e. $S(e^x)$ is convex on $[-\infty, \infty)$. Further, let $\lambda > 1, n \in \mathbb{N}, n \geqslant 2$. If*

$$\int_0^1 S(tx)(1-x^2)^{n-2}\, dx \leqslant t^\lambda, \quad 0 \leqslant t < \infty, \tag{3}$$

then

$$\int_0^\infty S(t) \frac{t^{2\lambda-1}}{(1+t^{2\lambda})^2}\, dt \leqslant \frac{\pi(n-1)}{2\lambda} \prod_{k=1}^{n-1} \left(1+\frac{\lambda}{2k}\right). \tag{4}$$

Therefore, if Conjecture 2 holds, it follows that Conjecture 1 is true.

Lemma 3. [3] *A real valued function $S = S(x)$ on the interval $[0,\infty)$ such that $S(0) = 0$ is an increasing logarithmic convex function if and only if there is an increasing function $s(t)$ on the interval $[0,\infty)$ such that S can be represented as*

$$S(x) = \int_0^x \frac{s(t)}{t}\, dt \tag{5}$$

Proposition 1.1. [3] *Conjecture 2 is equivalent to the following condition: let $\alpha > 1/2$. For any increasing function $h(t)$ on the interval $[0,\infty)$ and for any $n \geqslant 2$ if*

$$\int_0^1 \frac{h(tx)}{x}(1-x)^{(n-1)}\, dx \leqslant t^\alpha, \quad 0 \leqslant t < \infty, \tag{6}$$

[1] We say a function f is of finite lower order ρ if $\rho = \lim_{r\to\infty} \inf \dfrac{\log\log M(f,r)}{\log r}$.

then

$$\int_0^\infty \frac{h(t)}{t}\frac{dt}{1+t^{2\alpha}} \leqslant \frac{\pi}{2}\prod_{k=1}^{n-1}\Big(1+\frac{\alpha}{k}\Big). \tag{7}$$

Proof. Using lemma 3, we can rewrite the left side of the inequality in (3) as follows,

$$\begin{aligned}\int_0^1 S(tx)(1-x^2)^{(n-2)}x\,dx &= \int_0^1 \Big(\int_0^{tx}\frac{s(u)}{u}\,du\Big)(1-x^2)^{(n-2)}x\,dx\\ &= -\frac{1}{2(n-1)}\int_0^1 \Big(\int_0^{tx}\frac{s(u)}{u}\,du\Big)\,d(1-x^2)^{n-1}\\ &= \frac{1}{2(n-1)}\int_0^1 \frac{s(tx)}{x}(1-x^2)^{n-1}\,dx. \end{aligned}\tag{8}$$

Since $s(x)$ is assumed to be an increasing function, without loss of generality we can replace it with another function $h(x)$ defined as

$$h(x^2) = \frac{1}{4(n-1)}s(x).$$

Then the expression in (8) becomes

$$2\int_0^1 \frac{h(t^2x^2)}{x}(1-x^2)^{n-1}\,dx.$$

Upon the substitution $x \equiv x^2$ we have that inequality (3) in Conjecture 2 becomes

$$\int_0^1 \frac{h(t^2x)}{x}(1-x)^{n-1}\,dx \leqslant t^\lambda.$$

We let $t^2 \equiv t$ and $\frac{\lambda}{2} = \alpha$ to arrive at the exact expression in inequality (6). Using the above lemma and a similar method, one can arrive at inequality (7). □

2. Reformulating the conjecture

Proposition in the last section gives an equivalent form of Conjecture 2 that we state below.

Conjecture 3. *Let $\alpha > 1/2$. For any increasing function $h(t)$ on the interval $[0,\infty)$ and for any $n \geqslant 2$ if*

$$\int_0^1 \frac{h(tx)}{x}(1-x)^{(n-1)}\,dx \leqslant t^\alpha, \quad 0 \leqslant t < \infty, \tag{9}$$

then

$$\int_0^\infty \frac{h(t)}{t}\frac{dt}{1+t^{2\alpha}} \leqslant \frac{\pi}{2}\prod_{k=1}^{n-1}\Big(1+\frac{\alpha}{k}\Big). \tag{10}$$

Using the integral operator defined as

$$(\mathcal{J}f)(t) = \int_0^t f(x)\,dx, \quad t > 0, \tag{11}$$

we can actually to transform the premise inequality in Conjecture 3 in a more compact form. A repeated application of the operator above yields the Cauchy formula for repeated integration of a real valued function $f(x)$ as

$$(\mathcal{J}^n f)(t) = \frac{1}{(n-1)!}\int_0^t (t-x)^{n-1}f(x)\,dx, \quad t > 0. \tag{12}$$

A change of variable $tx \equiv x$ in the left hand side of (9) leads us to the following equivalent form:

$$\int_0^1 \frac{h(tx)}{x}(1-x)^{n-1}\,dx \mapsto \frac{1}{t^{n-1}}\int_0^t \frac{h(x)}{x}(t-x)^{n-1}\,dx.$$

Using the formula in (12) and substituting $h(x)/x$ with $f(x)$, we can write the last expression as follows

$$\frac{1}{t^{n-1}}\int_0^t f(x)(t-x)^{n-1}\,dx = \frac{(n-1)!}{t^{n-1}}(\mathcal{J}^n(f))(t).$$

With this at hand we can reformulate Conjecture 3 in our final form that we state below.

Conjecture 4 (Reformulated)**.** *Let $\alpha > 1/2$. For any real valued function $f(x)$ such that $xf(x)$ is increasing on the interval $[0,\infty)$ and for any $n \geqslant 2$ if*

$$(\mathcal{J}^n f)(t) \leqslant \frac{t^{\alpha+n-1}}{(n-1)!}, \quad 0 \leqslant t < \infty, \tag{13}$$

then

$$\int_0^\infty f(t)\frac{dt}{1+t^{2\alpha}} \leqslant \frac{\pi}{2}\prod_{k=1}^{n-1}\left(1+\frac{\alpha}{k}\right). \tag{14}$$

3. Some preliminary results

3.1. Laplace transform.

Definition 3.1. *Let $f(t)$ be a real valued function. The unilateral Laplace transform of $f(t)$ is denoted by $(\mathcal{L}f)(s)$ and is defined as*

$$(\mathcal{L}f)(s) = \int_0^\infty f(t)e^{-st}\,dt \tag{15}$$

for Re(s) *sufficiently large such that the integral in* (16) *exists*[1].

Below we present a relation between the Laplace integral transform and the integral operator introduced in the previous section.

Proposition 3.1. *Let $f(t)$ be a real valued function defined on the non-negative real line. Assume further that $(\mathcal{L}f)(s)$ exists then for any $n > 1$ and $n \in \mathbb{N}$*

$$(\mathcal{L}\{\mathcal{J}^n(f)\})(s) = \frac{1}{s^n}(\mathcal{L}f)(s) \tag{16}$$

Proof. The proof of this statement is straightforward by induction. For $n = 1$ we get

$$(\mathcal{L}\{\mathcal{J}(f)\})(s) = \int_0^\infty (\mathcal{J}f)(t)e^{-st}\,dt = \int_0^\infty \Big(\int_0^t f(x)\,dx\Big)e^{-st}\,dt$$

Upon integration by parts the last integral expression becomes

$$\int_0^\infty \Big(\int_0^t f(x)\,dx\Big)e^{-st}\,dt = -\left(-\frac{1}{s}\right)\int_0^\infty f(t)e^{-st}\,dt = \frac{1}{s}(\mathcal{L}f)(s)$$

Assume that for $n = k$ the statement is true then by letting $n = k+1$ we get

$$\begin{aligned}(\mathcal{L}\{\mathcal{J}^{k+1}(f)\})(s) = (\mathcal{L}\{\mathcal{J}(\mathcal{J}^k(f)\}))(s) &= \frac{1}{s}(\mathcal{L}\{\mathcal{J}^k(f)\})(s)\\ &= \frac{1}{s}\cdot\frac{1}{s^k}(\mathcal{L}f)(s) = \frac{1}{s^{k+1}}(\mathcal{L}f)(s)\end{aligned}$$

□

[1]Note that if the integral holds for all $s \in \mathbb{C}$ with Re$(s) \geqslant \beta$ then the same integral expression holds when s is restricted on the real interval $[\beta,\infty)$.

Definition 3.2. *Let $f(t)$ be a real valued function defined on the interval $[0,\infty)$ such that its Laplace transform $(\mathcal{L}f)(s)$ exists. The inverse Laplace transform of $(\mathcal{L}f)(s)$ is defined as*

$$f(t) = (\mathcal{L}^{-1}(\mathcal{L}f)(s))(t) = \frac{1}{2\pi i}\lim_{T\to\infty}\int_{\gamma-Ti}^{\gamma+Ti}(\mathcal{L}f)(s)e^{st}\,ds \tag{17}$$

where $\gamma = \mathrm{Re}\,(s)$ such that γ is greater than the real part of any singularity of $(\mathcal{L}f)(s)$ in the complex plane.[1]

Theorem 4 (Uniqueness of Transforms)**.** [4] *For any two real valued functions $f(t)$ and $g(t)$ such that their Laplace transforms exist. If $(\mathcal{L}f)(s) = (\mathcal{L}g)(s)$, then $f(t) = g(t)$ for all $t \geqslant 0$.*

3.2. A Uniqueness Result. It has been already noted by Khabibullin that his conjecture holds with identity for a certain function $f(x)$ on the non-negative real line. In what follows we provide a proof using the Laplace transform to reach his conclusion. We present formally this result.

Proposition 3.2. *Let $\alpha > 0$ and $f(t)$ be a real valued function. For any $n \geqslant 1$ and $n \in \mathbb{N}$ if*

$$(\mathcal{J}^n f)(t) = \frac{t^{\alpha+n-1}}{(n-1)!}, \quad 0 \leqslant t < \infty, \tag{18}$$

then

$$\int_0^\infty f(t)\frac{dt}{1+t^{2\alpha}} = \frac{\pi}{2}\prod_{k=1}^{n-1}\Big(1+\frac{\alpha}{k}\Big). \tag{19}$$

Moreover, there is a unique function $f(t)$ that satisfies the conjecture with equality and that is

$$f(t) = t^{\alpha-1}\frac{\Gamma(\alpha+n)}{\Gamma(n)\Gamma(\alpha)}. \tag{20}$$

Proof. Let $(\mathcal{J}^n f)(t) = t^{\alpha+n-1}/(n-1)!$ for all $t \in [0,\infty)$. Taking the Laplace transform of both sides yields

$$(\mathcal{L}f)(s) = \frac{1}{s^\alpha}\frac{\Gamma(\alpha+n)}{\Gamma(n)}. \tag{21}$$

By the Theorem 4 (Uniqueness of Transforms) the inverse Laplace transform of both sides of (21) must be equal for all $t \geqslant 0$. The inverse Laplace transform of the left side is simply $f(t)$. The right hand side (RHS) is

$$\mathcal{L}^{-1}(\frac{1}{s^\alpha}\frac{\Gamma(\alpha+n)}{\Gamma(n)})(t) = \frac{\Gamma(\alpha+n)}{\Gamma(n)\Gamma(\alpha)}(\mathcal{L}^{-1}(\frac{\Gamma(\alpha)}{s^\alpha}))(t) = t^{\alpha-1}\frac{\Gamma(\alpha+n)}{\Gamma(n)\Gamma(\alpha)}.$$

Hence, we have

$$f(t) = t^{\alpha-1}\frac{\Gamma(\alpha+n)}{\Gamma(n)\Gamma(\alpha)}.$$

for all $t \geqslant 0$. This proves the uniqueness. Now integrating $f(t)$ over the non-negative real line against the function $g(t) = (1+t^{2\alpha})^{-1}$ yields

$$\int_0^\infty f(t)\frac{1}{1+t^{2\alpha}}\,dt = \frac{\Gamma(\alpha+n)}{\Gamma(n)\Gamma(\alpha)}\int_0^\infty\frac{t^{\alpha-1}}{1+t^{2\alpha}}\,dt = \frac{\Gamma(\alpha+n)}{\alpha\Gamma(n)\Gamma(\alpha)}\int_0^\infty\frac{1}{1+t^{2\alpha}}\,d(t^\alpha).$$

Substituting $t^\alpha \equiv u$, we obtain

$$\frac{\Gamma(\alpha+n)}{\alpha\Gamma(n)\Gamma(\alpha)}\int_0^\infty\frac{1}{1+u^2}\,du = \frac{\Gamma(\alpha+n)}{\alpha\Gamma(n)\Gamma(\alpha)}\Big[\arctan(u)\Big|_0^\infty\Big] = \frac{\pi\Gamma(\alpha+n)}{2\alpha\Gamma(n)\Gamma(\alpha)} = \frac{\pi}{2\alpha}\cdot\frac{1}{B(\alpha,n)},$$

where $B(x,y)$ is the beta function (Euler's integral of the first kind) defined as

[1]This integral is also known as the Bromwich integral.

$$B(x,y)=\int_0^1 u^{x-1}(1-u)^{y-1}\,du.$$

for $\mathrm{Re}\,(x), \mathrm{Re}\,(y) > 0$. One can show that

$$B(\alpha,n)=\Big(\alpha\prod_{k=1}^{n-1}\Big(1+\frac{\alpha}{k}\Big)\Big)^{-1}.$$

Therefore, substituting for beta function one gets,

$$\int_0^\infty f(t)\frac{1}{1+t^{2\alpha}}\,dt=\frac{\pi}{2}\prod_{k=1}^{n-1}\Big(1+\frac{\alpha}{k}\Big).$$

□

Notice that another way to prove the result above is to apply the differential operator $d/\,dt$ n times on both sides of (18) and then to integrate. It implies immediately the result. Nevertheless, the Laplace transform method gives the uniqueness of the solution without having to prove it by contradiction. Indeed from Proposition 3.2. we can get a corollary.

Corollary 3.3. *Let $f(x), g(x)$ and $h(x)$ be three real valued functions defined on a domain $D \subseteq \mathbb{R}$. Assume that $g(x)$ is a smooth function, namely, its derivatives of all orders exists. For all $n \geqslant 1$ if*

$$(\mathcal{J}^n f)(x)=g(x) \tag{22}$$

then

$$\int_D f(x)h(x)\,dx=\int_D g^{(n)}(x)h(x)\,dx \tag{23}$$

provided the integrals exist.

Proof. Proof is a direct application of the differential operator d/dx on both sides of the premise identity n times. Then we integrate function $h(x)$ over domain D having the existence of the integral. It completes the proof. □

4. Khabibullin's Theorem

It has been already proved by Khabibullin and Sharipov [5] that the conjecture holds true for any $0 < \alpha \leqslant 1/2$ and any natural number $n > 1$. This is exactly the result presented in Theorem 2. In what follows using a different approach of contour integration we provide a new proof of this result.

4.1. A Preliminary Analysis. It is clear that we need to find a non-negative function such that when both sides of post-Laplace transformed of (13) are multiplied by this non-negative function and then integrated over the non-negative real line, it would yield the result of the conjecture. Therefore, now the question is the existence of a non-negative function such that its Laplace transform is exactly $(1+t^{2\alpha})^{-1}$. We also need to find it explicitly. To answer this question, first we consider the following complex valued function

$$h(z)=\frac{1}{1+z^{2\alpha}}, \quad z\in\mathbb{C}.$$

Notice that the above function will have different characteristics depending on the values of α. If $2\alpha \in \mathbb{Z}$, then $h(z)$ will be a meromorphic function with all the poles lying on the unit circle. However, if $2\alpha \notin \mathbb{Z}$ then $h(z)$ will be a multivalued function and its analysis should be handled with care by choosing an appropriate branch. First we consider $h(z)$ for $0 < \alpha < \frac{1}{2}$. Writing $z^{2\alpha} = \exp(2\alpha \log(z)) = \exp(2\alpha(\mathrm{Log}\,(z) + 2\pi i m))$ and choosing the principal branch as $m = 0$, we obtain $z^{2\alpha} = \exp(2\alpha\,\mathrm{Log}\,(z)) = \exp(2\alpha(\mathrm{Log}\,(|z|) + i\theta))$, where $-\pi \leqslant \theta < \pi$. Given

$0 < 2\alpha < 1$, we get $|2\alpha\theta| < \pi$ which would make $z^{2\alpha} \neq -1$ for any z in the principle branch. This in turn implies that $h(z)$ is analytic on the principle branch. Now we are interested in the inverse Laplace transform of $h(z)$, namely,

$$(\mathcal{L}^{-1}h(z))(t) = \frac{1}{2\pi i}\int_{\gamma-i\infty}^{\gamma+i\infty} h(z)e^{zt}\,dz \quad ,\gamma > 0$$

To find the value of the above integral, we study the following integral over a keyhole contour:

$$\oint_C h(z)e^{zt}\,dz,$$

where the keyhole contour C is as depicted in the figure 1.

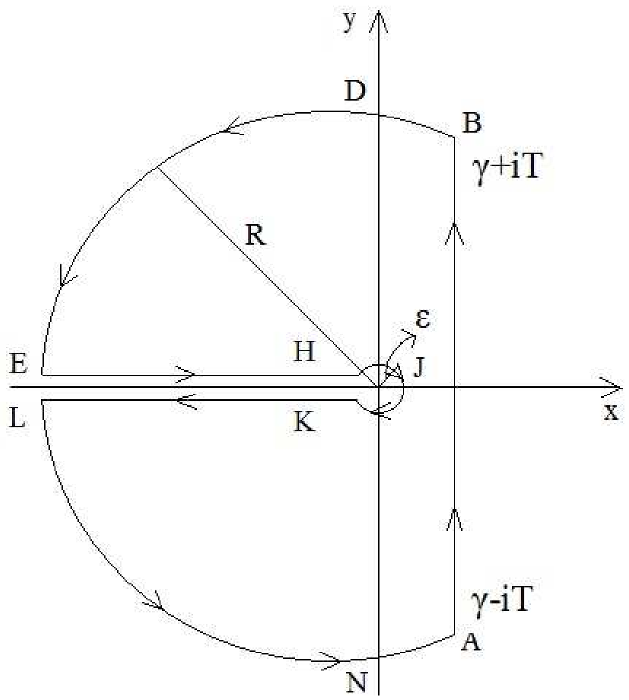

Figure 1. Keyhole contour

Because the integrand is analytic within this contour, by Cauchy Theorem we have

$$\oint_C h(z)e^{zt}\,dz = 0$$

The integral over the contour can be partitioned into several integrals as follows

$$\oint_C h(z)e^{zt}\,dz = \int_{\gamma-iT}^{\gamma+iT} h(z)e^{zt}\,dz + \int_{BDE} h(z)e^{zt}\,dz + \int_{-R}^{-\epsilon} h(x)e^{xt}\,dx + \\ + \int_{\Gamma_\epsilon} h(z)e^{zt}\,dz + \int_{-\epsilon}^{-R} h(xe^{-2\pi i})e^{xt}\,dx + \int_{LNA} h(z)e^{zt}\,dz.$$

Letting $R \to \infty$ and $\epsilon \to\ 0$, one gets

$$\Big|\int_{BDE} h(z)e^{zt}\,dz\Big| \leqslant \int_{BDE}\Big|h(z)e^{zt}\,dz\Big| = \int_{BDE}|h(z)||e^{zt}||\,dz| \\ < \frac{e^{-Rt}}{R^{2\alpha}-1}\cdot\pi R \sim e^{-R}R^{1-2\alpha} \to 0.$$

In the same way

$$\Big|\int_{LNA} h(z)e^{zt}\,dz\Big| \to 0.$$

For the integral over the small circle we get the following estimate

$$\Big|\int_{\Gamma_\epsilon} h(z)e^{zt}\,dz\Big| \leqslant \int_{\Gamma_\epsilon} |h(z)||e^{zt}||\,dz| \leqslant \frac{e^{\epsilon t}}{1-\epsilon^{2\alpha}}\cdot 2\pi\epsilon \sim \epsilon \to 0.$$

The contribution comes only from the integral over the vertical line and the two integrals over the branch cut above and below the non-positive real line. Using the result of Cauchy Theorem and passing to the limit as $R \to \infty$ and $\epsilon \to 0$, one has

$$\int_{\gamma-i\infty}^{\gamma+i\infty} h(z)e^{zt}\,dz + \int_{-\infty}^{0} h(x)e^{xt}\,dx + \int_{0}^{-\infty} h(xe^{-2\pi i})e^{xt}\,dx = 0.$$

The above equation is equivalent to the following one:

$$\int_{\gamma-i\infty}^{\gamma+i\infty} h(z)e^{zt}\,dz = \int_{0}^{\infty} [h(-xe^{-2\pi i}) - h(-x)]e^{-xt}\,dx.$$

Upon substitution by $h(x) = (1+x^{2\alpha})^{-1}$ we get

$$\begin{aligned}\int_{\gamma-i\infty}^{\gamma+i\infty} h(z)e^{zt}\,dz &= \int_{0}^{\infty} \Big[\frac{1}{1+(-xe^{-2\pi i})^{2\alpha}} - \frac{1}{1+(-x)^{2\alpha}}\Big]e^{-xt}\,dx \\ &= \int_{0}^{\infty} \Big[\frac{1}{1+x^{2\alpha}e^{-2\alpha\pi i}} - \frac{1}{1+x^{2\alpha}e^{2\alpha\pi i}}\Big]e^{-xt}\,dx \\ &= \int_{0}^{\infty} \frac{2ix^{2\alpha}\sin(2\alpha\pi)}{1+2x^{2\alpha}\cos(2\alpha\pi)+x^{4\alpha}}e^{-xt}\,dx.\end{aligned}$$

Therefore, dividing both sides by $2\pi i$ yields

$$\frac{1}{2\pi i}\int_{\gamma-i\infty}^{\gamma+i\infty} h(z)e^{zt}\,dz = \frac{1}{\pi}\int_{0}^{\infty} \frac{x^{2\alpha}\sin(2\alpha\pi)}{1+2x^{2\alpha}\cos(2\alpha\pi)+x^{4\alpha}}e^{-xt}\,dx \tag{24}$$

Notice that for $0 < 2\alpha < 1$ we have $\sin(2\alpha\pi) \geqslant 0$. Also

$$1+2x^{2\alpha}\cos(2\alpha\pi)+x^{4\alpha} = (x^{2\alpha}+\cos(2\alpha\pi))^2 + 1 - \cos^2(2\alpha\pi) \geqslant 0$$

as $0 \leqslant \cos^2(2\alpha\pi) \leqslant 1$. Hence, the right hand side of (24) satisfies the criteria of our desired function, it is non-negative and its Laplace transform is exactly $h(z)$. A particular case is when $\alpha = \frac{1}{4}$ for which the integral on right side of (24) is equal to

$$\frac{1}{\pi}\int_{0}^{\infty} \frac{\sqrt{x}}{1+x}e^{-xt}\,dx = \frac{1}{\sqrt{\pi t}} - e^{t}(1-\operatorname{erf}(\sqrt{t})),$$

where $\operatorname{erf}(t)$ is the error function defined as

$$\operatorname{erf}(t) = \frac{2}{\sqrt{\pi}}\int_{0}^{t} e^{-x^2}\,dx.$$

Note that in this case $h(z) = (1+\sqrt{z})^{-1}$. For the special parameter value $\alpha = 1/2$ we have $h(z) = (1+z)^{-1}$ with the corresponding inverse Laplace transform $(\mathcal{L}^{-1}h(z))(t) = e^{-t}$. However, the contour of the integration would be a half disk on the left half-plane, where $h(z)$ has a simple pole at $z = -1$ with residue 1.

4.2. Khabibullin's Theorem. We summarize the results of the previous section in the following lemma.

Lemma 5. *Let $h(z) = (1 + z^{2\alpha})^{-1}$ be a complex-valued function defined via the principal branch $-\pi \leqslant \arg z < \pi$. If $0 < \alpha \leqslant 1/2$, then*

$$(\mathcal{L}^{-1}h(z))(t) \geqslant 0, \quad 0 \leqslant t < \infty. \tag{25}$$

Proof. Let $h(z) = (1 + z^{2\alpha})^{-1}$ be a complex-valued function such that $0 < \alpha \leqslant 1/2$. Using the preliminary analysis of the previous section, we have that

$$(\mathcal{L}^{-1}h(z))(t) = \begin{cases} \dfrac{1}{\pi}\displaystyle\int_0^\infty \frac{x^{2\alpha}\sin(2\alpha\pi)}{1 + 2x^{2\alpha}\cos(2\alpha\pi) + x^{4\alpha}} e^{-xt}\,dx & : 0 < \alpha < 1/2 \\ e^{-t} & : \alpha = 1/2 \end{cases}$$

Clearly, $e^{-t} > 0$ for all $t \in \mathbb{R}$. In the other case the integral expression is convergent for all non-negative values of t and we proved that it was non-negative in the preliminary analysis. □

Theorem 6 (Khabibullin's Theorem)**.** *Let $0 < \alpha \leqslant 1/2$ and $f(x)$ be a real valued function defined on the non-negative real line such that $xf(x)$ is increasing. If for every $n \in \mathbb{N}$*

$$(\mathcal{J}^n f)(t) \leqslant \frac{t^{\alpha+n-1}}{(n-1)!}, \quad 0 \leqslant t < \infty, \tag{26}$$

then

$$\int_0^\infty f(t)\frac{1}{1 + t^{2\alpha}}\,dt \leqslant \frac{\pi}{2}\prod_{k=1}^{n-1}\left(1 + \frac{\alpha}{k}\right). \tag{27}$$

Proof. Let $0 < \alpha \leqslant 1/2$ and $f(x)$ be a real valued function defined on the non-negative real line such that $xf(x)$ is increasing. Assume that for all $n > 1$ we have that the premise inequality holds true. Multiplying both sides of this inequality by e^{-st} keeps the inequality since for any $s, t \in \mathbb{R}$ the quantity e^{-st} is positive. Integrating with respect to t over the non-negative real line we get the Laplace transform of both sides, i.e.

$$\int_0^\infty e^{-st}(\mathcal{J}^n f)(t)\,dt \leqslant \int_0^\infty e^{-st}\frac{t^{\alpha+n-1}}{(n-1)!}\,dt \Leftrightarrow (\mathcal{L}f)(s) \leqslant \frac{1}{s^\alpha}\frac{\Gamma(\alpha+n)}{\Gamma(n)}$$

for all $s \in \mathbb{R}^+$. First we consider $0 < \alpha < 1/2$. Multiplying both sides of the last inequality by the function

$$g(s) = \frac{1}{\pi}\int_0^\infty \frac{x^{2\alpha}\sin(2\alpha\pi)}{1 + 2x^{2\alpha}\cos(2\alpha\pi) + x^{4\alpha}} e^{-xs}\,dx.$$

and integrating with respect to s over the non-negative real line yields

$$\int_0^\infty g(s)(\mathcal{L}f)(s)\,ds \leqslant \int_0^\infty g(s)\frac{1}{s^\alpha}\frac{\Gamma(\alpha+n)}{\Gamma(n)}\,ds$$
$$\int_0^\infty g(s)\Big(\int_0^\infty f(t)e^{-st}\,dt\Big)\,ds \leqslant \frac{\Gamma(\alpha+n)}{\Gamma(n)\Gamma(\alpha)}\int_0^\infty g(s)\Big(\int_0^\infty t^{\alpha-1}e^{-st}\,dt\Big)\,ds.$$

By the argument of the absolute convergence we can interchange the order of integration to obtain

$$\int_0^\infty f(t)\Big(\int_0^\infty g(s)e^{-st}\,ds\Big)\,dt \leqslant \frac{\Gamma(\alpha+n)}{\Gamma(n)\Gamma(\alpha)}\int_0^\infty t^{\alpha-1}\Big(\int_0^\infty g(s)e^{-st}\,ds\Big)\,dt$$
$$\int_0^\infty f(t)(\mathcal{L}g)(t)\,dt \leqslant \frac{\Gamma(\alpha+n)}{\Gamma(n)\Gamma(\alpha)}\int_0^\infty t^{\alpha-1}(\mathcal{L}g)(t)\,dt.$$

Using Lemma 5, we get that $(\mathcal{L}g)(t) = h(t)$, where $h(t) = (1 + t^{2\alpha})^{-1}$. Therefore,

$$\int_0^\infty f(t)(1 + t^{2\alpha})^{-1}\,dt \leqslant \frac{\Gamma(\alpha+n)}{\Gamma(n)\Gamma(\alpha)} \int_0^\infty t^{\alpha-1}(1+t^{2\alpha})^{-1}\,dt = \frac{\pi}{2\alpha}\cdot\frac{1}{B(\alpha,n)}$$

For $\alpha = 1/2$ use $g(s) = e^{-s}$ and take the same approach. It completes the proof. □

Other proofs based on different techniques exist in the mathematical literature. For instance, one could refer to Sharipov [5], where a method of kernels is applied. This method of kernels is borrowed from an earlier work of Sharipov [6], where a conversion formula relating objective functions in the pair of integral inequalities in terms of each other was obtained.

4.3. Khabibullin's Conjecture. It is tempting to employ Laplace transform in studying the conjecture as in the last theorem. However, this technique is shown to be inapplicable. The reason is that when $\alpha > 1/2$, the function $h(z) = (1 + z^{2\alpha})^{-1}$ is not guaranteed to have an inverse Laplace transform that is non-negative on $[0, \infty)$. The simplest example is when $\alpha = 1$ for which the function $h(z) = (1 + z^2)^{-1}$. The inverse Laplace transform of $h(z)$ can be shown to be the sine function, i.e., $(\mathcal{L}^{-1}h(z))(s) = \sin(s)$ for $s \geqslant 0$. This is a periodic function that takes negative values as often as it takes positive ones. Therefore one can not simply multiply both sides of the post Laplace-transformed of the premise inequality (13) by the sine function as the inequality sign is not assured to hold for all values of s over the non-negative real line. Even if one tries to use the non-negative part of the inverse Laplace-transformed of $h(z)$, i.e. $(\mathcal{L}^{-1}h(z))^+(s) = \max\{(\mathcal{L}^{-1}h(z))(s), 0\}$ or its absolute value $|(\mathcal{L}^{-1}h(z))(s)|$, the procedure will fail. This is because in these examples the absolute convergence of obtained integrals is not guaranteed and thus interchanging the order of integration might not be allowed. This in turn hinders us from getting some concluding inequality. To see this closely, we consider again the case $\alpha = 1$ so that complex valued function $h(z) = (1+z^2)^{-1}$. Then $(\mathcal{L}^{-1}h(z))^+(s) = \sin^+(s) = \max\{\sin(s), 0\}$. Multiplying both sides of the post Laplace-transformed premise inequality (13) by $\sin^+(s)$ and integrating over the non-negative real line yields

$$\int_0^\infty \sin^+(s)(\mathcal{L}(f))(s)\,ds \leqslant \int_0^\infty \sin^+(s)\frac{1}{s^\alpha}\frac{\Gamma(\alpha+n)}{\Gamma(n)}\,ds.$$

The RHS can be written as

$$\int_0^\infty \sin^+(s)\frac{1}{s^\alpha}\frac{\Gamma(\alpha+n)}{\Gamma(n)}\,ds = \frac{\Gamma(\alpha+n)}{\Gamma(n)\Gamma(\alpha)}\int_0^\infty \sin^+(s)\Big(\int_0^\infty t^{\alpha-1}e^{-st}\,dt\Big)\,ds.$$

If one assumes that the integrals above are absolutely convergent and thus the order of integration is not relevant then we have,

$$\frac{\Gamma(\alpha+n)}{\Gamma(n)\Gamma(\alpha)}\int_0^\infty t^{\alpha-1}\Big(\int_0^\infty \sin^+(s)e^{-st}\,ds\Big)\,dt. \tag{28}$$

The inner integral in (28) can be computed as follows:

$$\int_0^\infty \sin^+(s)e^{-st}\,ds = \int_0^\infty \max\{\sin(s),0\}e^{-st}\,ds = \sum_{n=0}^{\infty}\int_{\pi n}^{\pi(n+1)} \max\{\sin(s),0\}e^{-st}\,ds.$$

Clearly, for n odd the integrand in the last integral is zero

$$\begin{aligned}\therefore \sum_{n=0}^{\infty}\int_{\pi n}^{\pi(n+1)} \max\{\sin(s),0\}e^{-st}\,ds &= \sum_{n-even}\int_{\pi n}^{\pi(n+1)} \sin(s)e^{-st}\,ds\\ &= \sum_{n-even} e^{-n\pi t}\int_0^\pi (-1)^n\sin(s)e^{-st}\,ds.\end{aligned}$$

Given an even n, one can express the sum of the infinite terms above as

$$\sum_{n-even} e^{-n\pi t}\int_0^{\pi}(-1)^n\sin(s)e^{-st}\,ds = \sum_{n=0}^{\infty} e^{-2n\pi t}\int_0^{\pi}\sin(s)e^{-st}\,ds = \frac{1}{1-e^{-2\pi t}}\int_0^{\pi}\sin(s)e^{-st}\,ds.$$

The integral can be shown to be

$$\int_0^{\pi}\sin(s)e^{-st}\,ds = \operatorname{Im}\left\{\int_0^{\pi}e^{is-st}\,ds\right\} = \operatorname{Im}\left\{\frac{1}{i-t}(e^{i\pi-\pi t}-1)\right\} = \frac{1+e^{-\pi t}}{1+t^2},$$

where $\operatorname{Im}(z)$ is the imaginary part of a complex number z. Therefore, the RHS in (28) is

$$\frac{\Gamma(\alpha+n)}{\Gamma(n)\Gamma(\alpha)}\int_0^{\infty}t^{\alpha-1}\frac{1+e^{-\pi t}}{1-e^{-2\pi t}}(1+t^2)^{-1}\,dt = (n+1)\int_0^{\infty}\frac{1+e^{-\pi t}}{1-e^{-2\pi t}}(1+t^2)^{-1}\,dt \tag{29}$$

as $\alpha = 1$. However, the last integral does not converge to a finite value and so it contradicts our assumption of absolute convergence and the possibility of interchanging the order of integration. In [7] Sharipov constructed a counterexample showing that Khabibullin's conjecture is not valid for $n = 2$ and $\alpha = 2$. Then he introduced the concept of Khabibullin's constants $C[\mathrm{Kh}](n,\alpha)$ and suggested a slightly modified version of Khabibullin's conjecture which is valid for all $\alpha > 0$ (see Theorem 5.1 in [7]). Therefore, since the conjecture does not hold for all n and α, an interesting issue would be to investigate whether there exists some $\alpha > 1/2$ such that the conjecture is true. To show this, it is sufficient to find such α that the function $h(z) = (1+z^{2\alpha})^{-1}$ defined over the principal branch $-\pi \leqslant \arg z < \pi$ has an inverse Laplace transform that is non-negative on $[0,\infty)$. Also we require that the double integrals obtained in the proof of Theorem 6 are absolutely convergent. A good starting point would be to consider positive integer values for 2α. The above example showed that when $\alpha = 1$ the Laplace method is not applicable and therefore one might assume values of α greater than one. In general situation, for positive integer values of 2α the function $h(z) = (1+z^{2\alpha})^{-1}$ has an inverse Laplace transform given by the following expression

$$(\mathcal{L}^{-1}h(z))(t) = \frac{1}{2\pi i}\int_{\gamma-i\infty}^{\gamma+i\infty}h(z)e^{zt}\,dz = \sum_{k\in\mathbb{Z}_{2\alpha}}\frac{\exp(tz_k)}{\prod_{k\neq m}(z_k-z_m)}, \tag{30}$$

where $z_k = \exp(\pi i(2k+1)/2\alpha)$ and $\mathbb{Z}_{2\alpha} = \{0,1,2,\ldots,2\alpha-1\}$. It is not immediately obvious whether the sum in (30) is non-negative for any certain integer value of 2α. This is because the summation has many oscillatory terms. But if one is able to find a way to show that for some value $\alpha > 1$ the summation in (30) is non-negative, then this would imply that the Laplace technique becomes applicable which in turn would prove the conjecture for a particular situation.

5. Concluding remarks

This work presents the Khabibullin's conjecture about a pair of integral inequalities in its different forms. Use of the integral operator leads us to a reformulation of one of the equivalent forms of the conjecture. This new form makes possible a direct application of the Laplace transform twice on both sides of the premise inequality. We were able to restate through this method two well known results: the existence of a unique real valued function that satisfied the conjecture with identity and that the conjecture holds true for any parameter of positive value but not larger than a half. Techniques of contour integration are employed to prove the theorem. However, Laplace transform failed to prove the conjecture for parameter values strictly greater than a half and for this a brief reasoning was laid out.

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