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DESCRIPTION OF ZERO SEQUENCES FOR HOLOMORPHIC AND MEROMORPHIC FUNCTIONS OF FINITE λ-TYPE IN A CLOSED HALF-STRIP

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Abstract. We describe the zero sets of holomorphic and meromorphic functions f of finite λ-type in a closed half-strip satisfying $f(\sigma) = f(\sigma + 2\pi i)$ on the boundary.

Keywords: holomorphic function, meromorphic function, function of finite λ - type, sequence of finite λ-density, λ-admissible sequence

Mathematics Subject Classification: 30D35

1. Introduction

Let f be a meromorphic function in the closure of the half-strip

$$S = \{s = \sigma + it : \sigma \ > \ 0, \quad 0 < t < 2\pi\}.$$

Suppose f has neither zeros nor poles on ∂S, and $f(\sigma) = f(\sigma + 2\pi i)$, $\sigma \geq \ 0$. Denote by $\{s_j\}$ the zero sequence of function f in S, $s_j = \sigma_j + it_j$, by $\{p_j\}$ the sequence of its poles in S.

Let S^* be the strip S with the straight slits $\{\tau\sigma_j + it_j\}$, $\{\tau \mathrm{Re}\, p_j + i\,\mathrm{Im}\, p_j\}$, $\quad 1 \leqslant \tau < \infty$. Given $s_0 \in S^*$, suppose $\log f(s_0)$ is well-defined and let

$$\log f(s) = \log f(s_0) + \int\limits_{s_0}^{s} \frac{f'(\zeta)}{f(\zeta)} d\zeta, \tag{1}$$

where integral is taken along a piecewise-smooth path in $S^* \cup \partial S$, which connects s_0 and s.

By $n(\eta, f)$ we denote the counting function of poles of f in the rectangle $R_\eta = \{\sigma + it : 0 < \sigma \ \leqslant \ \eta, \quad 0 \leqslant \ t < 2\pi\}$. We let

$$N(\sigma, f) = \int\limits_0^\sigma n(\eta, f) d\eta. \tag{2}$$

and

$$c_0(\sigma, f) = \frac{1}{2\pi} \int\limits_0^{2\pi} \log |f(\sigma + it)| dt. \tag{3}$$

The following Lemma is a counterpart of Jensen-Littlewood Theorem ([1]).

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Lemma 1. [2] *Let f be a meromorphic function in the closure of half-strip S, $f(\sigma) = f(\sigma + 2\pi i)$, $\sigma \geq 0$. Then*

$$N(\sigma, \frac{1}{f}) - N(\sigma, f) = c_0(\sigma, f) - \frac{\sigma}{\sigma_0} c_0(\sigma_0, f) + (\frac{\sigma}{\sigma_0} - 1) c_0(0, f),$$
$$\sigma \geq \sigma_0 > 0. \tag{4}$$

The Nevanlinna characteristic of such functions was defined in [2] as

$$T(\sigma, f) = m_0(\sigma, f) - \frac{\sigma}{\sigma_0} m_0(\sigma_0, f) + \left(\frac{\sigma}{\sigma_0} - 1\right) m_0(0, f) + N(\sigma, f), \quad \sigma \geq \sigma_0 > 0,$$

where

$$m_0(\sigma, f) = \frac{1}{2\pi} \int_0^{2\pi} \log^+ |f(\sigma + it)| dt.$$

Definition 1. *A positive non-decreasing continuous unbounded function $\lambda(\sigma)$ defined for all $\sigma \geq \sigma_0 > 0$ is said to be a growth function.*

Definition 2. *Let $\lambda(\sigma)$ be a growth function and f be a meromorphic function in $\overline{S}$, such that $f(\sigma + 2\pi i) = f(\sigma), \sigma \geq \sigma_0 > 0$. We say that f is of finite λ-type if $T(\sigma, f) \leqslant A\lambda(\sigma + B)$, $\sigma \geq \sigma_0$ for some constants $A > 0, B > 0$ and all σ, $\sigma \geq \sigma_0 > 0$.*

We denote by Λ the class of meromorphic functions of finite λ-type in $\overline{S}$ and Λ_H the class of holomorphic functions of finite λ-type in $\overline{S}$.

In this paper we describe the zero sequences of holomorphic functions in Λ_H, as well as zero and pole sequences of meromorphic functions in Λ.

For entire and meromorphic in $\mathbb{C}$ functions similar problems were solved by L. Rubel and B. Taylor ([3]), for holomorphic and meromorphic functions in a punctured plane the same was done by A. Kondratyuk and I. Laine ([4]).

2. Description of zero sequences of holomorphic and meromorphic functions of finite λ-type in a half-strip

Let $Q = \{s_j\}$ be a sequence of complex numbers in $\overline{S}$. By $n(\eta, Q)$ we indicate the counting function of Q in the rectangle R_η and we let

$$N(\sigma, Q) = \int_0^{\sigma} n(\eta, Q) d\eta.$$

Definition 3. *A sequence $Q = \{s_j\}$ from $\overline{S}$ has a finite λ- density if*

$$N(\sigma, Q) \leqslant A\lambda(\sigma + B)$$

for some positive constants A, B and all σ, $\sigma \geq \sigma_0 > 0$.

Definition 4. *A sequence $Q = \{s_j\}$ from $\overline{S}$ is said to be λ-admissible if it has finite λ-density and there are positive constants A, B such that*

$$\frac{1}{k} \left| \sum_{\sigma_1 < Re\, s_j \leqslant \sigma_2} \left(\frac{1}{e^{s_j}}\right)^k \right| \leqslant \frac{A\lambda(\sigma_1 + B)}{e^{k\sigma_1}} + \frac{A\lambda(\sigma_2 + B)}{e^{k\sigma_2}},$$

for all σ_1, σ_2, $\sigma_0 \leqslant \sigma_1 < \sigma_2$ and each $k \in \mathbb{N}$.

Denote

$$c_k(\sigma, f) = \frac{1}{2\pi}\int_0^{2\pi} e^{-ikt}\log|f(\sigma+it)|dt, \quad k\in\mathbb{Z}. \tag{5}$$

For a meromorphic in $\overline{R}_\sigma$ function f such that $f(\sigma) = f(\sigma+2\pi i)$ the following relations hold true (see [2]):

$$\begin{aligned} c_k(\sigma, f) =& \frac{e^{k\sigma}}{2k}\alpha_k(f) - \frac{e^{-k\sigma}}{2k}\overline{\alpha_{-k}}(f)\\ &+\frac{1}{2k}\sum_{s_j\in R_\sigma}\left[\left(\frac{e^\sigma}{e^{s_j}}\right)^k - \left(\frac{e^{\overline{s_j}}}{e^\sigma}\right)^k\right] - \frac{1}{2k}\sum_{p_j\in R_\sigma}\left[\left(\frac{e^\sigma}{e^{p_j}}\right)^k - \left(\frac{e^{\overline{p_j}}}{e^\sigma}\right)^k\right],\\ c_{-k}(\sigma, f) =& \overline{c}_k(\sigma, f) \quad k\in\mathbb{N}, \end{aligned} \tag{6}$$

where s_j, p_j are its zeroes and poles in $\overline{R}_\sigma$ respectively, and

$$\alpha(f) = \frac{1}{2\pi}\int_0^{2\pi} e^{-ikt}\frac{f'(it)}{f(it)}dt, \quad k\in\mathbb{N}.$$

Theorem 1. *A sequence Q in $\overline{S}$ is the zero sequence of the function in Λ_H if and only if it is λ-admissible.*

Proof. Let $Q = \{s_j\}$ be the zero sequence of a function f from Λ_H. Then by (6)

$$\begin{aligned} \frac{c_k(\sigma_2, f)}{e^{k\sigma_2}} - \frac{c_k(\sigma_1, f)}{e^{k\sigma_1}} =& \frac{\alpha_k e^{k\sigma_2} - \overline{\alpha_{-k}}e^{-k\sigma_2}}{2ke^{k\sigma_2}} + \frac{1}{2ke^{k\sigma_2}}\left[\sum_{s_j\in R_{\sigma_2}}\left(\frac{e^{\sigma_2}}{e^{s_j}}\right)^k - \sum_{s_j\in R_{\sigma_2}}\left(\frac{e^{\overline{s_j}}}{e^{\sigma_2}}\right)^k\right]\\ &-\frac{\alpha_k e^{k\sigma_1} - \overline{\alpha_{-k}}e^{-k\sigma_1}}{2ke^{k\sigma_1}} - \frac{1}{2ke^{k\sigma_1}}\left[\sum_{s_j\in R_{\sigma_1}}\left(\frac{e^{\sigma_1}}{e^{s_j}}\right)^k - \sum_{s_j\in R_{\sigma_1}}\left(\frac{e^{\overline{s_j}}}{e^{\sigma_1}}\right)^k\right]\\ =& \frac{\overline{\alpha_{-k}}}{2k}\left[\frac{1}{e^{2k\sigma_1}} - \frac{1}{e^{2k\sigma_2}}\right] + \frac{1}{2k}\left[\sum_{s_j\in R_{\sigma_2}}\frac{1}{(e^{s_j})^k} - \sum_{s_j\in R_{\sigma_1}}\frac{1}{(e^{s_j})^k}\right]\\ &+\frac{1}{2ke^{k\sigma_1}}\sum_{s_j\in R_{\sigma_1}}\left(\frac{e^{\overline{s_j}}}{e^{\sigma_1}}\right)^k - \frac{1}{2ke^{k\sigma_2}}\sum_{s_j\in R_{\sigma_2}}\left(\frac{e^{\overline{s_j}}}{e^{\sigma_2}}\right)^k, \end{aligned}$$

where $0 \leqslant \sigma_1 < \sigma_2$.

Then we obtain

$$\begin{aligned} \frac{1}{k}\sum_{\sigma_1<\operatorname{Re}s_j\leqslant\sigma_2}\frac{1}{(e^{s_j})^k} =& \frac{2c_k(\sigma_2, f)}{e^{k\sigma_2}} - \frac{2c_k(\sigma_1, f)}{e^{k\sigma_1}} + \frac{\overline{\alpha_{-k}}}{k}\left[\frac{1}{e^{2k\sigma_2}} - \frac{1}{e^{2k\sigma_1}}\right] +\\ &+\frac{1}{ke^{k\sigma_2}}\sum_{s_j\in R_{\sigma_2}}\left(\frac{e^{\overline{s_j}}}{e^{\sigma_2}}\right)^k - \frac{1}{ke^{k\sigma_1}}\sum_{s_j\in R_{\sigma_1}}\left(\frac{e^{\overline{s_j}}}{e^{\sigma_1}}\right)^k. \end{aligned} \tag{7}$$

We have

$$\sum_{s_j\in R_{\sigma_i}}\left|\frac{e^{\overline{s_j}}}{e^{\sigma_i}}\right|^k \leqslant \sum_{s_j\in R_{\sigma_i}} 1 \leqslant n(\sigma_i+1, \frac{1}{f}) \leqslant N(\sigma_i+1, \frac{1}{f}) \leqslant A_1\lambda(\sigma_i+1+B_1), \quad \sigma_i\in R_{\sigma_i}, i=1,2,$$

for some constants $A_1, B_1 > 0$.

We also get the estimate for the left-hand side of identity (7):

$$
\begin{aligned}
\frac{1}{k}\left|\sum_{\sigma_1<\operatorname{Re}s_j\leqslant\sigma_2}\frac{1}{e^{ks_j}}\right| \leqslant & \frac{A_2\lambda(\sigma_2+B_2)}{e^{k\sigma_2}}+\frac{A_2\lambda(\sigma_1+B_2)}{e^{k\sigma_1}}+\frac{|\overline{\alpha_{-k}}|}{k}\left[\frac{1}{e^{2k\sigma_2}}+\frac{1}{e^{2k\sigma_1}}\right] \\
& +\frac{1}{ke^{k\sigma_2}}\sum_{s_j\in R_{\sigma_2}}\left|\frac{e^{\overline{s_j}}}{e^{\sigma_2}}\right|^k+\frac{1}{ke^{k\sigma_1}}\sum_{s_j\in R_{\sigma_1}}\left|\frac{e^{\overline{s_j}}}{e^{\sigma_1}}\right|^k \\
\leqslant & \frac{A_2\lambda(\sigma_2+B_2)}{e^{k\sigma_2}}+\frac{A_2\lambda(\sigma_1+B_2)}{e^{k\sigma_1}}+C\left[\frac{1}{e^{2k\sigma_2}}+\frac{1}{e^{2k\sigma_1}}\right] \\
& +\frac{1}{ke^{k\sigma_2}}N\left(\sigma_2+1,\frac{1}{f}\right)+\frac{1}{ke^{k\sigma_1}}N\left(\sigma_1+1,\frac{1}{f}\right) \\
\leqslant & \frac{A\lambda(\sigma_2+B)}{e^{k\sigma_2}}+\frac{A\lambda(\sigma_1+B)}{e^{k\sigma_1}}, \qquad k\in\mathbb{N}, \quad \sigma_2>\sigma_1\geqslant\sigma_0,
\end{aligned}
\tag{8}
$$

where $A=\max\{A_1,A_2,C\}, B=\max\{B_1+1,B_2\}$.

Theorem 2 in [2] implies that the sequence Q has a finite λ-density. Hence, it is λ-admissible.

Let now $Q=\{s_j\}$ be λ-admissible. Then the sequence $Z=\{z_j\}$, $z_j=e^{s_j}\in\mathbb{C}$, is λ_1-admissible in $\mathbb{C}$, where $\lambda_1(r)=\lambda(\log r)$. By the Rubel-Taylor Theorem [3, p. 84], (see also [5, p. 29]), there exists an entire function $F(z)$ of finite λ_1-type with zero sequence $Z=\{z_j\}$. Therefore, the function $f(s)=F(e^s)$ is holomorphic of finite λ-type in $\overline{S}$ with the zero sequence $\{s_j\}$. □

Theorem 2. *A sequence Q in $\overline{S}$ is the zero sequence of a function in Λ if and only if it has finite λ-density.*

Proof. If $Q=\{s_j\}$ is the zero sequence of a function f, $f\in\Lambda$, then from [2], we have

$$N(\sigma,Q)=N(\sigma,\frac{1}{f})\leqslant T(\sigma,f)\leqslant B\lambda(\sigma+C),$$

for all $\sigma\geqslant\sigma_0>0$ and some $B,C>0$.

Let now $Q=\{s_j\}$ be a sequence of finite λ-density. Then the sequence $Z=\{z_j\}$, $\quad z_j=e^{s_j}$, has the finite λ_1-density if $\lambda_1(r)=\lambda(\log r)$. By the Rubel-Taylor Theorem [3, p. 88] (see also [5, p. 35]) there exist a meromorphic function F of finite λ_1-type with zero sequence Z. The function $f(s)=F(e^s)$ is the meromorphic of finite λ-type in $\overline{S}$ with zero sequence $\{s_j\}$. □

Corollary 1. *A sequence $P=\{p_j\}$ is the pole sequence of a function f from Λ if and only if it has finite λ-density.*

Proof. Apply Theorem 2 to the function $\frac{1}{f}$. □

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