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ON THE ASYMPTOTIC BEHAVIOR OF CAUCHY-STIELTJES INTEGRAL IN THE POLYDISC

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Abstract. In the paper the asymptotic behavior of Cauchy-Stieltjes integral of a complex-valued Borel measure on the skeleton in the polydisc is described. The main result holds outside a set of zero ω -capacity. It generalizes the theorem for the one-dimensional case.

Keywords: modulus of continuity, Cauchy-Stieltjes integral, polydisc, set of zero ω -capacity, non-tangential limit.

For $z = (z_1, \dots, z_n) \in \mathbb{C}^n$, $n \in \mathbb{N}$, let $|z| = \max\{|z_j| : 1 \leq j \leq n\}$ be the polydisc norm. Denote by $U^n = \{z \in \mathbb{C}^n : |z| < 1\}$ the unit polydisc with the distinguished boundary $T^n = \{z \in \mathbb{C}^n : |z_j| = 1, 1 \leq j \leq n\}$, and $\tau = [-\pi; \pi]$. For $z \in U^n$, $z_j = r_j e^{i\varphi_j}$, $w = (w_1, \dots, w_n) \in T^n$, $w_j = e^{i\theta_j}$, $1 \leq j \leq n$ we write $C_\alpha(z, w) = \prod_{j=1}^n \frac{1}{(1 - z_j \bar{w}_j)^{\alpha_j}}$, where $\alpha = (\alpha_1, \dots, \alpha_n)$, $\alpha_j > 0$, $1 \leq j \leq n$, $C_{\alpha_j}(z_j, w_j) = \frac{1}{(1 - z_j \bar{w}_j)^{\alpha_j}}$ is the generalized Cauchy kernel for the unit disc, $C_{\alpha_j}(0, w_j) = 1$. The symbol K will denote a constant not necessary the same in each occurrence.

The function in U^n defined by the equality

$$f(z_1, \dots, z_n) = \int_{T^n} C_\alpha(z, w) d\mu(w), \quad z \in U^n \tag{1}$$

with $|\mu|(T^n) < +\infty$, where $|\mu|$ is the total variation of μ , is called the Cauchy-Stieltjes integral of a complex-valued Borel measure μ . The function $f(z_1, \dots, z_n)$ is analytic in U^n .

For $\psi = (\psi_1, \dots, \psi_n) \in \tau^n$, $\gamma = (\gamma_1, \dots, \gamma_n) \in [0; \pi]^n$ we define the Stolz angle $S(\psi, \gamma) = S(\psi_1, \gamma_1) \times \dots \times S(\psi_n, \gamma_n)$ in the polydisc, where $S(\psi_j, \gamma_j)$ is the Stolz angle for the unit disc with the vertex $e^{i\psi_j}$,

$$S(\psi_j, \gamma_j) = \{|z_j - e^{i\psi_j}| \leq A(\gamma_j)(1 - r_j)\}, \quad 1 \leq j \leq n,$$

$$A(\gamma_j) = \sqrt{1 + 4\text{tg}^2 \frac{\gamma_j}{2}}.$$

In the case of the unit disc ($n = 1$), there is a strong dependence between local smoothness of the measure μ and the growth of f in the direction of $e^{i\psi}$ (see [1]–[3]). In particular, differentiability of μ implies boundedness of the Poisson-Stieltjes integral [4]. The idea of such results goes back to P. Fatou [4], and G. Hardy and J. Littlewood [5].

However, in the case $n > 1$ local differentiability of μ need not imply boundedness of Poisson-Stieltjes integral (see [6, Section 2.3]). In [7], [8] an interplay between smoothness and the growth of the Poisson-Stieltjes integral was considered. In particular, the growth of such integrals was characterized in terms of smoothness of the corresponding (positive) measure μ .

Let $\omega: \mathbb{R}_+^n \rightarrow \mathbb{R}_+$ be a semi-additive continuous increasing function in each variable vanishing if at least one of the arguments equals zero. We call ω a modulus of continuity.

A Borel set $E \subset T^n$ is called a set of positive ω -capacity if there exists a nonnegative measure ν on T^n such that

$$\int_E d\nu = \int_{T^n} d\nu = 1 \tag{2}$$

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and

$$\sup_{x \in \mathbb{R}^n} \int_{\tau^n} \frac{d\nu(e^{it_1}, \dots, e^{it_n})}{\omega(|t_1 - x_1|, \dots, |t_n - x_n|)} < +\infty. \tag{3}$$

Otherwise, E is called a *set of zero ω -capacity*.

The following properties of sets of zero ω -capacity are easy to check:

1) If E_1 and E_2 are Borel subsets of T^n , $E_1 \subset E_2$, and E_2 has ω -capacity zero, then E_1 has ω -capacity zero.

2) If Borel sets $E_i, i = 1, 2, \dots$ have ω -capacity zero, then the set $E = \bigcup_{i=1}^{\infty} E_i$ has ω -capacity zero too.

3) If ω_1 and ω_2 are moduli of continuity, $\omega_1(t) \leq \omega_2(t), t \in \mathbb{R}_+^n$, and a Borel subset E of T^n has positive ω_1 -capacity, then E has positive ω_2 -capacity.

4) If $\int_0^1 \dots \int_0^1 \frac{dt_1 \dots dt_n}{\omega(t_1, \dots, t_n)} < \infty$ and a set $E \subset T^n$ has zero ω -capacity, then E has zero n -dimensional Lebesgue measure.

Let us prove the last property. Indeed, if m is the Lebesgue on T^n and $m(E) > 0$, let

$$d\nu(e^{ix_1}, \dots, e^{ix_n}) = (\mathcal{X}_E(e^{ix_1}, \dots, e^{ix_n}) / m(E)) dm(e^{ix_1}, \dots, e^{ix_n}),$$

where $\mathcal{X}_E(e^{ix_1}, \dots, e^{ix_n})$ is the characteristic function of E . Thus,

$$\begin{aligned} \int_{\tau^n} \frac{d\nu(e^{ix_1}, \dots, e^{ix_n})}{\omega(|x_1 - t_1|, \dots, |x_n - t_n|)} &= \frac{1}{m(E)} \int_{\tau^n} \frac{\mathcal{X}_E(e^{it_1}, \dots, e^{it_n}) dm(e^{it_1}, \dots, e^{it_n})}{\omega(|x_1 - t_1|, \dots, |x_n - t_n|)} \leq \\ &\leq \frac{1}{m(E)} \int_{\tau^n} \frac{dt_1 \dots dt_n}{\omega(|x_1 - t_1|, \dots, |x_n - t_n|)} \leq \frac{2}{m(E)} \int_0^\pi \dots \int_0^\pi \frac{dt_1 \dots dt_n}{\omega(t_1, \dots, t_n)} < \infty. \end{aligned}$$

Therefore, ω -capacity of E is positive.

The notion of zero ω -capacity for Borel subsets of T provides a useful measure of finiteness of exceptional sets for the radial (non-tangential) growth of functions of the form (1) (see Theorem B ([3]) below, [1], and [2]). In the case $n = 1, \omega(t) = t^\beta, \beta \in (0, 1)$, the definition and properties of ω -capacity are given in [9, Chapter 3].

Theorem A ([3]). *Let $\alpha > 0, \psi \in [-\pi; \pi], g$ be a function of bounded variation on $[-\pi; \pi]$, and a modulus of continuity ω satisfies the condition*

$$\int_0^1 t^{-\alpha-1} \omega(t) dt = \infty. \tag{4}$$

If

$$|g(t) - g(\psi)| = o(\omega(|t - \psi|)), \quad t \rightarrow \psi$$

then

$$\left| \int_{-\pi}^\pi C_\alpha(z, e^{-it}) dg(t) \right| / \int_{|1 - ze^{-i\psi}|}^1 t^{-\alpha-1} \omega(t) dt, \quad z \in U \tag{5}$$

has the non-tangential limit zero at $e^{i\psi}$.

Theorem B ([3]). *Let $\alpha > 0, g$ be a function of bounded variation on $[-\pi; \pi]$, and a modulus of continuity ω satisfies condition (4). Then (5) has the non-tangential limit zero at all ψ in $[-\pi; \pi]$, except, possibly, a set of zero ω -capacity.*

We give an example of a set of zero ω -capacity that will be used later. For $a \in \mathbb{R}$, $1 \leq j \leq n$ we denote

$$T_a^{(j)} = \{(e^{i\theta_1}, \dots, e^{i\theta_n}) \in T^n : \theta_j = a\},$$

$$\tau_a^{(j)} = \{(\theta_1, \dots, \theta_n) \in \tau^n : \theta_j = a \pmod{2\pi}\}.$$

Let $e_j = \{a_j^1, \dots, a_j^s, \dots\}$, $s \in \mathbb{N}$. We are to prove that the set $E_j = \bigcup_{a \in e_j} T_a^{(j)}$ is of positive ω -capacity. Suppose the contrary. Then the conditions (2) and (3) hold. It follows from (2) that

$$\exists a_j^s : \nu(T_{a_j^s}^{(j)}) > 0.$$

Consequently, using (3) and the definition of ω , we get

$$\begin{aligned} \infty &> \sup_{x \in \mathbb{R}^n} \int_{\tau^n} \frac{d\nu(e^{i\theta_1}, \dots, e^{i\theta_n})}{\omega(|\theta_1 - x_1|, \dots, |\theta_n - x_n|)} > \\ &> \sup_{x_j = a_j^s} \int_{\tau^n} \frac{d\nu(e^{i\theta_1}, \dots, e^{i\theta_n})}{\omega(|\theta_1 - x_1|, \dots, |\theta_j - x_j|, \dots, |\theta_n - x_n|)} \geq \\ &\geq \int_{\tau_a^{(j)}} \frac{d\nu(e^{i\theta_1}, \dots, e^{i\theta_n})}{\omega(|\theta_1 - x_1|, \dots, 0, \dots, |\theta_n - x_n|)} = \infty. \end{aligned}$$

Hence, the set E_j has ω -capacity zero. In particular, $T_a^{(j)}$ has ω -capacity zero.

Theorem 1. Let $\alpha_j > 0, \beta_j > 0, 1 \leq j \leq n, n \in \mathbb{N}$, ω be a modulus of continuity satisfying

$$\int_0^1 \dots \int_0^1 \frac{\omega(t_1, \dots, t_n)}{t_1^{\alpha_1+1} \dots t_n^{\alpha_n+1}} dt_1 \dots dt_n = +\infty.$$

Let μ be a complex-valued Borel measure on T^n with $|\mu|(T^n) < +\infty$, and $e^{i\psi} \in T^n$.

If

$$|\mu|(\{e^{i\theta} \in T^n : |\theta_j - \psi_j| \leq t_j, 1 \leq j \leq n\}) = o(\omega(t_1, \dots, t_n)), \min_j t_j \rightarrow 0+,$$

then

$$\left| \int_{T^n} C_\alpha(z, w) d\mu(w) \right| = o \left(\log^n \frac{1}{\delta} \cdot \int_{|z_1 - e^{i\psi_1}|}^1 \dots \int_{|z_n - e^{i\psi_n}|}^1 \frac{\omega(t_1, \dots, t_n) dt_1 \dots dt_n}{t_1^{\alpha_1+1} \dots t_n^{\alpha_n+1}} \right),$$

where $\delta \rightarrow 0, |z_j| = 1 - \delta^{\frac{1}{\beta_j}}, z \in S(\psi, \gamma)$.

Theorem 2. Let $\alpha_j > 0, \beta_j > 0, 1 \leq j \leq n, n \in \mathbb{N}$, ω be a modulus of continuity satisfying

$$\int_0^1 \dots \int_0^1 \frac{\omega(t_1, \dots, t_n)}{t_1^{\alpha_1+1} \dots t_n^{\alpha_n+1}} dt_1 \dots dt_n = +\infty,$$

and μ be a complex-valued Borel measure on T^n with $|\mu|(T^n) < +\infty$. Then

$$\left| \int_{T^n} C_\alpha(z, w) d\mu(w) \right| = o \left(\log^n \frac{1}{\delta} \cdot \int_{|z_1 - e^{i\psi_1}|}^1 \dots \int_{|z_n - e^{i\psi_n}|}^1 \frac{\omega(t_1, \dots, t_n) dt_1 \dots dt_n}{t_1^{\alpha_1+1} \dots t_n^{\alpha_n+1}} \right),$$

where $\delta \rightarrow 0, |z_j| = 1 - \delta^{\frac{1}{\beta_j}}, z \in S(\psi, \gamma)$, for $(e^{i\psi_1}, \dots, e^{i\psi_n}) \in T^n$ except, possibly, a set of zero ω -capacity.

The proof of Theorem 1, as a matter of fact, is contained in that of Theorem 2, which generalizes Theorem B.

Corollary 1. *Let $\omega(t_1, \dots, t_n) = t_1^{\varkappa_1} \cdot \dots \cdot t_n^{\varkappa_n}$ be a modulus of continuity, $\varkappa_j > 0$, $\alpha_j > 0, \beta_j > 0$, $1 \leq j \leq n, n \in \mathbb{N}$, and $\exists j_0 : \alpha_{j_0} \geq \varkappa_{j_0}$, μ be a complex-valued Borel measure on T^n with $|\mu|(T^n) < +\infty$. Then*

$$\left| \int_{T^n} C_\alpha(z, w) d\mu(w) \right| = o \left(\log^n \frac{1}{\delta} \cdot \int_{|z_1 - e^{i\psi_1}|}^1 \dots \int_{|z_n - e^{i\psi_n}|}^1 \frac{dt_1 \dots dt_n}{t_1^{\alpha_1+1-\varkappa_1} \cdot \dots \cdot t_n^{\alpha_n+1-\varkappa_n}} \right),$$

as $\delta \rightarrow 0$, where $|z_j| = 1 - \delta^{\frac{1}{\beta_j}}$, $z \in S(\psi, \gamma)$, for $(e^{i\psi_1}, \dots, e^{i\psi_n}) \in T^n$ except, possibly, a set of zero ω -capacity.

Corollary 2. *Let $\varkappa_j \in (0; 1), \alpha_j > 0, \beta_j > 0$, $1 \leq j \leq n, n \in \mathbb{N}$, and $\exists j_0 : \alpha_{j_0} \geq \varkappa_{j_0}$, $\omega(t_1, \dots, t_n) = t_1^{\varkappa_1} \cdot \dots \cdot t_n^{\varkappa_n}$. Let μ be a complex-valued Borel measure on T^n with $|\mu|(T^n) < +\infty$. Then*

$$\left| \int_{T^n} C_\alpha(z, w) d\mu(w) \right| = o \left(\log^n \frac{1}{\delta} \cdot \int_{|z_1 - e^{i\psi_1}|}^1 \dots \int_{|z_n - e^{i\psi_n}|}^1 \frac{dt_1 \dots dt_n}{t_1^{\alpha_1 - \varkappa_1 + 1} \cdot \dots \cdot t_n^{\alpha_n - \varkappa_n + 1}} \right),$$

as $\delta \rightarrow 0$, where $|z_j| = 1 - \delta^{\frac{1}{\beta_j}}$, $z \in S(\psi, \gamma)$, except, possibly, a set of zero Lebesgue measure.

Corollary 2 follows from property 4 of the sets of zero ω -capacity and Corollary 1.

Corollary 3. *Let $\beta_j > 0$, $\omega(t_1, \dots, t_n) = t_1^{\alpha_1} \cdot \dots \cdot t_n^{\alpha_n} \cdot \log^{l_1} \frac{1}{t_1} \cdot \dots \cdot \log^{l_n} \frac{1}{t_n}$, $\alpha_j > 0$, $1 \leq j \leq n$, $n \in \mathbb{N}$, $l_j \in \mathbb{R}$, $\exists j_0 : l_{j_0} \geq -1$. Let μ be a complex-valued Borel measure on T^n with $|\mu|(T^n) < +\infty$. Then*

$$\left| \int_{T^n} C_\alpha(z, w) d\mu(w) \right| = o \left(\log^n \frac{1}{\delta} \int_{|z_1 - e^{i\psi_1}|}^1 \dots \int_{|z_n - e^{i\psi_n}|}^1 \frac{\log^{l_1} \frac{1}{t_1} \cdot \dots \cdot \log^{l_n} \frac{1}{t_n} dt_1 \dots dt_n}{t_1 \cdot \dots \cdot t_n} \right),$$

as $\delta \rightarrow 0$, where $|z_j| = 1 - \delta^{\frac{1}{\beta_j}}$, $z \in S(\psi, \gamma)$, for $(e^{i\psi_1}, \dots, e^{i\psi_n}) \in T^n$ except, possibly, a set of zero ω -capacity.

Unfortunately, the author does not know whether it is possible to omit the factor $\log^n \frac{1}{\delta}$ in the assertion of the theorem.

To prove Theorem 1 we need the following lemma due to Prof. I.Chyzykhov who has kindly allowed to use it.

Lemma 1. *Let $\omega(t_1, \dots, t_n)$ be a modulus of continuity and μ be a complex-valued Borel measure, $|\mu|(T^n) < +\infty$. Then*

$$|\mu|(\{e^{i\theta} \in T^n : |\theta_j - \varphi_j| \leq t_j, 1 \leq j \leq n\}) = o(\omega(t_1, \dots, t_n)), \quad (6)$$

as $\min_j t_j \rightarrow 0+$, except, possibly, a set of zero ω -capacity of values $(e^{i\varphi_1}, \dots, e^{i\varphi_n})$.

Proof of the lemma. Let $e_j = \left\{ a \in \mathbb{R} : |\mu|(T_a^{(j)}) > 0 \right\}$. For arbitrary j the set e_j is at most countable. Denote $E_j = \bigcup_{a \in e_j} T_a^{(j)}$. As it was proved above, every set E_j has zero ω -capacity.

Thus $E = \bigcup_{j=1}^n E_j$ has ω -capacity zero as well.

Let now $e^{i\varphi} \in T^n \setminus E$. Then $|\mu| \left(T_{\varphi_j}^{(j)} \right) = 0$, $1 \leq j \leq n$.

Let $\min_j t_j \rightarrow 0+$. Passing, if necessary, to a subsequence, we may assume that $t_1 \rightarrow 0+$.

Fixing φ_1 , we consider the set

$$G_{\varphi_1}^k = \left\{ (e^{i\theta_1}, \dots, e^{i\theta_n}) \in T^n : |\varphi_1 - \theta_1| < \frac{1}{k} \right\}, \quad k \in \mathbb{N}.$$

Then $G_{\varphi_1}^{k+1} \subset G_{\varphi_1}^k$ and $\bigcap_{k \in \mathbb{N}} G_{\varphi_1}^k = T_{\varphi_1}^{(1)}$. By countable additivity,

$$\lim_{k \rightarrow \infty} |\mu| (G_{\varphi_1}^k) = |\mu| (T_{\varphi_1}^{(1)}) = 0.$$

Assume that

$$\int_{T^n} \frac{|d\mu| (e^{i\theta_1}, \dots, e^{i\theta_n})}{\omega (|\varphi_1 - \theta_1|, \dots, |\varphi_n - \theta_n|)} < +\infty.$$

Denote $g_{\varphi_1}^k = \{(\theta_1, \dots, \theta_n) : -\pi \leq \theta_j < \pi, 2 \leq j \leq n, e^{i\theta} \in G_{\varphi_1}^k, |\varphi_j - \theta_j| < \frac{1}{k}\}$. Then, using countable additivity of the Lebesgue integral, we obtain

$$\lim_{k \rightarrow \infty} \int_{g_{\varphi_1}^k} \frac{|d\mu| (e^{i\theta_1}, \dots, e^{i\theta_n})}{\omega (|\varphi_1 - \theta_1|, \dots, |\varphi_n - \theta_n|)} = 0.$$

Let $t_2, \dots, t_n \in [0; \pi]$. Then $\forall \varepsilon > 0 \quad \exists k_0 \in \mathbb{N} \quad \forall k > k_0 \quad |t_1| < \frac{1}{k}$ we have

$$\begin{aligned} \varepsilon &> \int_{g_{\varphi_1}^k} \frac{|d\mu| (e^{i\theta_1}, \dots, e^{i\theta_n})}{\omega (|\varphi_1 - \theta_1|, \dots, |\varphi_n - \theta_n|)} \geq \\ &\geq \int_{|\varphi_1 - \theta_1| \leq t_1} \dots \int_{|\varphi_n - \theta_n| \leq t_n} \frac{|d\mu| (e^{i\theta_1}, \dots, e^{i\theta_n})}{\omega (|\varphi_1 - \theta_1|, \dots, |\varphi_n - \theta_n|)} \geq \\ &\geq \frac{|\mu| (\{e^{i\theta} : |\varphi_1 - \theta_1| \leq t_1, \dots, |\varphi_n - \theta_n| \leq t_n\})}{\omega (t_1, \dots, t_n)}. \end{aligned}$$

Hence,

$$|\mu| (\{e^{i\theta_1}, \dots, e^{i\theta_n} \in T^n : |\theta_j - \varphi_j| \leq t_j, 1 \leq j \leq n\}) = o(\omega (t_1, \dots, t_n)), \quad t_1 \rightarrow 0+.$$

The lemma is proved.

Now we can prove Theorem 1.

Proof of Theorem 1. By Lemma 1, there exists a set E of zero ω -capacity with the following property. Given $e^{i\psi} \in T^n \setminus E$, for arbitrary $\varepsilon > 0$ there exists η_ε such that

$$|\mu| (\{e^{i\theta} : |\theta_j - \psi_j| < t_j, 1 \leq j \leq n\}) \leq \varepsilon \omega (t_1, \dots, t_n), \quad 0 \leq \min_{1 \leq j \leq n} t_j < \eta_\varepsilon. \quad (7)$$

For fixed δ , $0 < \delta < \frac{1}{2}$, $m = (m_1, \dots, m_n) \in \mathbb{Z}_+^n$ we denote

$$R_m = \left\{ e^{i\theta} \in T^n : 2^{-\frac{1}{\beta_j}} \leq \frac{|\theta_j - \psi_j|}{2A(\gamma_j) \cdot (2^{m_j} \delta)^{\frac{1}{\beta_j}}} \leq 1, \quad \text{if } m_j > 0; \right. \\ \left. \frac{|\theta_j - \psi_j|}{2A(\gamma_j) \cdot (2^{m_j} \delta)^{\frac{1}{\beta_j}}} \leq 1, \quad \text{if } m_j = 0, 1 \leq j \leq n \right\},$$

where $\beta_j > 0$, $\gamma_j \in [0; \pi)$.

For $z_j \in S(\psi_j, \gamma_j)$, $e^{i\theta} \in R_m$, $m_j > 0$, we obtain

$$\begin{aligned} |z_j - e^{i\theta_j}| &\geq |e^{i\psi_j} - e^{i\theta_j}| - |e^{i\psi_j} - z_j| \geq \frac{2}{\pi} |\psi_j - \theta_j| - A(\gamma_j)(1 - r_j) \geq \\ &\geq \frac{4}{\pi} A(\gamma_j) 2^{\frac{m_j-1}{\beta_j}} \delta^{\frac{1}{\beta_j}} - A(\gamma_j) \delta^{\frac{1}{\beta_j}} \geq \frac{1}{4} A(\gamma_j) 2^{\frac{m_j-1}{\beta_j}} \delta^{\frac{1}{\beta_j}}. \end{aligned}$$

If $z_j \in S(\psi_j, \gamma_j)$, $e^{i\theta} \in R_m$, $m_j = 0$, we have $|z_j - e^{i\theta_j}| \geq 1 - r_j = \delta^{\frac{1}{\beta_j}}$. Therefore, we can write both inequality in the form

$$|z_j - e^{i\theta_j}| \geq K 2^{\frac{m_j}{\beta_j}} \delta^{\frac{1}{\beta_j}}, \quad z_j \in S(\psi_j, \gamma_j), e^{i\theta} \in R_m, \quad (8)$$

where $K = K(\beta_j, \gamma_j)$ is a constant depending on β_j and γ_j only.

For $m = (m_1, \dots, m_n) \in \mathbb{Z}_+^n$ we consider two cases:

$$1) \forall j: 2A(\gamma_j) \cdot (2^{m_j} \delta)^{\frac{1}{\beta_j}} \geq \eta_\varepsilon;$$

$$2) \exists j: 2A(\gamma_j) \cdot (2^{m_j} \delta)^{\frac{1}{\beta_j}} < \eta_\varepsilon.$$

In the first case we have $|C_{\alpha_j}(z_j, w_j)| \leq \frac{K(\beta_j, \alpha_j)}{\eta_\varepsilon^{\alpha_j}}$.

We denote $F = F(\beta, \gamma, \delta, \varepsilon) = \bigcup_{m:1} R_m$. Then using the last estimate of the Cauchy kernel we deduce

$$\left| \int_F C_\alpha(z, w) d\mu(w) \right| \leq |\mu|(T^n) \cdot \frac{K}{\eta_\varepsilon^{\alpha_1 + \dots + \alpha_n}}. \quad (9)$$

We now consider the second case. By (7) we have

$$|\mu|(R_{m_1 \dots m_n}) \leq \varepsilon \cdot \omega \left(2A(\gamma_1) \cdot (2^{m_1} \delta)^{\frac{1}{\beta_1}}, \dots, 2A(\gamma_n) \cdot (2^{m_n} \delta)^{\frac{1}{\beta_n}} \right). \quad (10)$$

Then, using (8) we deduce

$$\begin{aligned} &\left| \sum_m \int_{R_m} C_\alpha(z, w) d\mu(w) \right| \leq \\ &\leq \varepsilon \sum_m \omega \left(2A(\gamma_1) (2^{m_1} \delta)^{\frac{1}{\beta_1}}, \dots, 2A(\gamma_n) (2^{m_n} \delta)^{\frac{1}{\beta_n}} \right) \prod_{j=1}^n \frac{1}{K (2^{m_j} \delta)^{\frac{\alpha_j}{\beta_j}}}, \end{aligned} \quad (11)$$

where the sum is taken over $m = (m_1, \dots, m_n)$ satisfying the condition from the second case.

On the other hand

$$\begin{aligned} &\int_{2A(\gamma_1) \cdot (2^{m_1} \delta)^{\frac{1}{\beta_1}}}^1 \dots \int_{2A(\gamma_n) \cdot (2^{m_n} \delta)^{\frac{1}{\beta_n}}}^1 \frac{\omega(t_1, \dots, t_n) dt_1 \dots dt_n}{t_1^{\alpha_1+1} \dots t_n^{\alpha_n+1}} \geq \\ &\geq \frac{K\omega \left(2A(\gamma_1) \cdot (2^{m_1} \delta)^{\frac{1}{\beta_1}}, \dots, 2A(\gamma_n) \cdot (2^{m_n} \delta)^{\frac{1}{\beta_n}} \right)}{\prod_{j=1}^n (2^{m_j} \delta)^{\frac{\alpha_j}{\beta_j}}}, \end{aligned} \quad (12)$$

where $K = K(\alpha, \beta, \gamma)$. From (11) and (12), we obtain

$$\left| \sum_m \int_{R_{m_1 \dots m_n}} C_\alpha(z, w) d\mu(w) \right| \leq \varepsilon \cdot N^* \cdot \int_{A(\gamma_1) \cdot \delta^{\frac{1}{\beta_1}}}^1 \dots \int_{A(\gamma_n) \cdot \delta^{\frac{1}{\beta_n}}}^1 \frac{\omega(t_1, \dots, t_n) dt_1 \dots dt_n}{t_1^{\alpha_1+1} \dots t_n^{\alpha_n+1}},$$

where $|z_j| = 1 - \delta^{\frac{1}{\beta_j}}$, $1 \leq j \leq n$, N^* is the number of m satisfying the condition from the second case. It is easy to see that

$$N^* \leq \left(\max_j \log_2 \frac{\pi^{\beta_j}}{(2A(\gamma_j))^{\beta_j} \delta} \right)^n \leq K \cdot \log^n \frac{1}{\delta}.$$

Then

$$\left| \sum_m \int_{R_m} C_\alpha(z, w) d\mu(w) \right| \leq K \varepsilon \log^n \frac{1}{\delta} \cdot \int_{A(\gamma_1) \delta^{\frac{1}{\beta_1}}}^1 \dots \int_{A(\gamma_n) \delta^{\frac{1}{\beta_n}}}^1 \frac{\omega(t_1, \dots, t_n) dt_1 \dots dt_n}{t_1^{\alpha_1+1} \dots t_n^{\alpha_n+1}}.$$

Finally, using the definition of the Stolz angle, we get

$$\begin{aligned} & \left| \sum_m \int_{R_{m_1 \dots m_n}} C_\alpha(z, w) d\mu(w) \right| \leq \\ & \leq 2K \varepsilon \log^n \frac{1}{\delta} \cdot \int_{|z_1 - e^{i\psi_1}|}^1 \dots \int_{|z_n - e^{i\psi_n}|}^1 \frac{\omega(t_1, \dots, t_n) dt_1 \dots dt_n}{t_1^{\alpha_1+1} \dots t_n^{\alpha_n+1}}. \end{aligned} \quad (13)$$

The assertion of Theorem 2 follows from (9) and (13).

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