INVERTIBLE CHANGES OF VARIABLES GENERATED BY BÄCKLUND TRANSFORMATIONS

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In the classification of partial differential equations, one cannot avoid the use of invertible changes of variables, which include not only the long-known point and contact transformations but also, for example, so-called symmetric and generalized contact transformations (reviewed by Mikhailov, Shabat, and Yamilov [1]). The present paper considers a further class of invertible changes of variables.

We consider vector evolution partial differential equations $\mathbf{u}_t=\mathbf{f}(\mathbf{u}, \mathbf{u}_x, \mathbf{u}_{xx}, \ldots)$. If we extend in some manner the set of dynamical variables $\mathbf{u}, \mathbf{u}_x, \mathbf{u}_{xx}, \ldots$, then many Bäcklund transformations of such equations in the extended set $\mathbf{u}, \mathbf{u}_x, \mathbf{u}_{xx}, \ldots, \mathbf{u}_{\pm 1}, \mathbf{u}_{\pm 2}, \ldots$ can be described as chains of ordinary differential equations $(\mathbf{u}_{n+1})_x=\mathbf{g}((\mathbf{u}_n)_x, \mathbf{u}_n, \mathbf{u}_{n+1})$, $n\in\mathbb{Z}$, that are compatible with these equations. In the case when there is a compatible pair — a partial differential equation and such a chain — we give a method for specifying changes of variables that are invertible in the extended set of variables (Theorem 1). The Korteweg–de Vries equation and the decoupled nonlinear Schödinger equation, under the conditions of validity of Theorem 1, for example.

1. We consider a vector chain of equations of the form

$$(\mathbf{u}_{n+1})_x=\mathbf{f}((\mathbf{u}_n)_x, \mathbf{u}_n, \mathbf{u}_{n+1}), \tag{1.1}$$

where n takes all integer values, the index x denotes the derivative with respect to x, and the symbols $\mathbf{u}_i$, $\mathbf{f}$ denote the vector columns $\mathbf{u}_i=(u_i^1, u_i^2, \ldots, u_i^m)^t$ and vector function $\mathbf{f}=(f^1, f^2, \ldots, f^m)^t$. The chain (1.1) is completely determined by any of its equations, and to avoid cumbersome expressions we shall in such chains omit the index n, giving only the relation with n = 0:

$$\mathbf{u}_{1,x}=\mathbf{f}(\mathbf{u}_x, \mathbf{u}, \mathbf{u}_1), \tag{1.2}$$

where $\mathbf{u}\equiv\mathbf{u}_0$. The chain $\mathbf{u}_t=\mathbf{H}$, where the vector function $\mathbf{H}$ depends on a finite number of variables of the set

$$\mathbf{u}, \mathbf{u}_{\pm 1}, \mathbf{u}_{\pm 2}, \ldots, \mathbf{u}_x, \mathbf{u}_{xx}, \ldots \tag{1.3}$$

is compatible with (1.2) if the equation

$$D_x D(\mathbf{H})=(\partial\mathbf{f}/\partial\mathbf{u}_x)D_x(\mathbf{H})+(\partial\mathbf{f}/\partial\mathbf{u})\mathbf{H}+(\partial\mathbf{f}/\partial\mathbf{u}_1)D(\mathbf{H}) \tag{1.4}$$

is a consequence of (1.2). Here, $\partial\mathbf{f}/\partial\mathbf{u}_x$, $\partial\mathbf{f}/\partial\mathbf{u}$, $\partial\mathbf{f}/\partial\mathbf{u}_1$ are Jacobi matrices (for example, $\partial\mathbf{f}/\partial\mathbf{u}\equiv(\partial f^i/\partial u^j)$), and D_X and D are operators that act on vector functions of a finite number of the variables (1.3): D_X is a differentiation ($D_x(\mathbf{u})=\mathbf{u}_x$), D is a displacement operator ($D(\mathbf{u})=\mathbf{u}_1$; if $\mathbf{h}$ is a vector function of several variables, then $D(\mathbf{h}(\mathbf{a}, \mathbf{b}, \mathbf{c}, \ldots))=\mathbf{h}(D(\mathbf{a}), D(\mathbf{b}), \ldots)$).

The compatibility means in particular that the operators D_X, D and the differentiation D_t ($D_t(\mathbf{u})=\mathbf{H}$) commute. We shall consider chains (1.2) with nondegenerate matrix $\partial\mathbf{f}/\partial\mathbf{u}_x$. In the case of such chains, we can regard the variables (1.3) as independent. By means of (1.2), all the variables $\partial^j\mathbf{u}_i/\partial x^j$ in Eq. (1.4) can be expressed in terms of the independent variables, after which Eq. (1.4) must hold identically. A chain of the form (1.2) can be compatible both with purely "continuous" equations

$$\mathbf{u}_t=\mathbf{F}(\mathbf{u}, \mathbf{u}_x, \mathbf{u}_{xx}, \ldots), \tag{1.5}$$

and purely "discrete" equations

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$$\mathbf{u}_\tau=\Phi(\mathbf{u},\mathbf{u}_{\pm1},\mathbf{u}_{\pm2},\ldots) \tag{1.6}$$

(see the examples below; an example of compatible chains of the form (1.2) and (1.6) appeared in [2]). The continuous Korteweg–de Vries equation

$$u_t=u_{xxx}-12uu_x \tag{1.7}$$

and the discrete Volterra chain

$$u_\tau=u(u_1-u_{-1}) \tag{1.8}$$

come under the ambit of the following theorem.

THEOREM 1. Suppose the chain (1.2) can be expressed in the form

$$D_x(\boldsymbol{\varphi}(\mathbf{u},\mathbf{u}_1))=\boldsymbol{\psi}(\mathbf{u},\mathbf{u}_1), \tag{1.9}$$

where the matrices $\partial\boldsymbol{\varphi}/\partial\mathbf{u}$, $\partial\boldsymbol{\varphi}/\partial\mathbf{u}_1$ are nondegenerate, and the coordinates of the vector functions $\boldsymbol{\varphi}$, $\boldsymbol{\psi}$ of the variables u^i and u^i_1 are functionally independent. Then the substitution

$$\mathbf{v}=\boldsymbol{\varphi}(\mathbf{u},\mathbf{u}_1), \tag{1.10}$$

which relates the variables (1.3) and

$$\mathbf{v},\mathbf{v}_{\pm1},\mathbf{v}_{\pm2},\ldots,\mathbf{v}_x,\mathbf{v}_{xx},\ldots \tag{1.11}$$

is invertible. As a result of this substitution, Eqs. (1.5) and (1.6) which are compatible with (1.9) go over, respectively, into equations of the form

$$\mathbf{v}_t=\mathbf{G}(\mathbf{v},\mathbf{v}_x,\mathbf{v}_{xx},\ldots), \tag{1.12}$$

$$\mathbf{v}_\tau=\boldsymbol{\Psi}(\mathbf{v},\mathbf{v}_{\pm1},\mathbf{v}_{\pm2},\ldots), \tag{1.13}$$

which are compatible with a chain of the form

$$D(\mathbf{p}(\mathbf{v},\mathbf{v}_x))=\mathbf{q}(\mathbf{v},\mathbf{v}_x), \tag{1.14}$$

where the matrices $\partial\mathbf{p}/\partial\mathbf{v}_x$, $\partial\mathbf{q}/\partial\mathbf{v}_x$ are nondegenerate, and the coordinates of the vector functions $\mathbf{p}$, $\mathbf{q}$ of the variables v^i and v^i_x are functionally independent. Conversely, if the chain (1.2) can be expressed in the form (1.14), then the change of variables

$$\mathbf{u}=\mathbf{p}(\mathbf{v},\mathbf{v}_x) \tag{1.15}$$

is invertible, and as a result of the change of variables Eqs. (1.12) and (1.13) which are compatible with (1.14) go over, respectively, into equations of the form (1.5) and (1.6), which are compatible with a chain of the form (1.9).

Proof. It follows from the relations (1.9) and (1.10) that

$$\mathbf{v}_x=\boldsymbol{\psi}(\mathbf{u},\mathbf{u}_1), \tag{1.16}$$

and therefore the invertibility of the change of variables is obvious. We shall assume that $\mathbf{u}$, $\mathbf{u}_1$ can be expressed in terms of $\mathbf{v}$, $\mathbf{v}_x$ in accordance with the formulas (1.15) and

$$\mathbf{u}_1=\mathbf{q}(\mathbf{v},\mathbf{v}_x). \tag{1.17}$$

From these formulas, we readily obtain the relation (1.14). The equations

$$(\partial\mathbf{p}/\partial\mathbf{v}_x)A=\partial\boldsymbol{\varphi}/\partial\mathbf{u}_1,\quad(\partial\mathbf{q}/\partial\mathbf{v}_x)A=-\partial\boldsymbol{\varphi}/\partial\mathbf{u},\quad A=(\partial\boldsymbol{\psi}/\partial\mathbf{u})(\partial\boldsymbol{\varphi}/\partial\mathbf{u}_1)-(\partial\boldsymbol{\psi}/\partial\mathbf{u}_1)(\partial\boldsymbol{\varphi}/\partial\mathbf{u}),$$

which are differential consequences of the relations

$$\mathbf{p}(\boldsymbol{\varphi}(\mathbf{u},\mathbf{u}_1),\ \boldsymbol{\psi}(\mathbf{u},\mathbf{u}_1))=\mathbf{u},\quad \mathbf{q}(\boldsymbol{\varphi}(\mathbf{u},\mathbf{u}_1),\ \boldsymbol{\psi}(\mathbf{u},\mathbf{u}_1))=\mathbf{u}_1, \tag{1.18}$$

explain why the matrices $\partial\mathbf{p}/\partial\mathbf{v}_x$, $\partial\mathbf{q}/\partial\mathbf{v}_x$ are degenerate.

Suppose the equation $\mathbf{u}_t=\mathbf{H}$, where $\mathbf{H}$ is a function of the variables (1.3), is compatible with (1.9). The substitution (1.10) carries it into the equation $\mathbf{v}_t=\mathbf{R}$ ($\mathbf{R}$ depends on the variables (1.11)), and

$$\mathbf{R}=D_t\boldsymbol{\varphi}(\mathbf{u},\mathbf{u}_1)=(\partial\boldsymbol{\varphi}/\partial\mathbf{u})\mathbf{H}+(\partial\boldsymbol{\varphi}/\partial\mathbf{u}_1)D(\mathbf{H}), \tag{1.19}$$

$$D_x\mathbf{R}=D_t\boldsymbol{\psi}(\mathbf{u},\mathbf{u}_1)=(\partial\boldsymbol{\psi}/\partial\mathbf{u})\mathbf{H}+(\partial\boldsymbol{\psi}/\partial\mathbf{u}_1)D(\mathbf{H}) \tag{1.20}$$

(see (1.10), (1.9)). The condition of compatibility (1.4) in the case of the equation $\mathbf{v}_t=\mathbf{R}$

and the chain (1.14) has the form

$$D\left(\frac{\partial \mathbf{p}}{\partial \mathbf{v}}\mathbf{R}+\frac{\partial \mathbf{p}}{\partial \mathbf{v}_x}D_x\mathbf{R}\right)=\frac{\partial \mathbf{q}}{\partial \mathbf{v}}\mathbf{R}+\frac{\partial \mathbf{q}}{\partial \mathbf{v}_x}D_x\mathbf{R}. \tag{1.21}$$

We express $\mathbf{R}$, $D_x(\mathbf{R})$ in terms of $\mathbf{H}$, $D(\mathbf{H})$ by means of (1.19) and (1.20). By virtue of the consequences of (1.18), which are obtained by differentiating Eqs. (1.18) with respect to $\mathbf{u}$, $\mathbf{u}_1$, the expression on the right-hand side of (1.21) is equal to $D(\mathbf{H})$, and the expression to which D is applied on the left-hand side of (1.21) is equal to $\mathbf{H}$, i.e., (1.21) is transformed into an identity. Thus, an equation compatible with (1.9) goes over as a result of the substitution (1.10) into an equation compatible with (1.14).

We obtain Eqs. (1.12) and (1.13) from Eqs. (1.5) and (1.6). By virtue of (1.10) and (1.5),

$$\mathbf{v}_t=(\partial\boldsymbol{\varphi}/\partial\mathbf{u})\mathbf{F}+(\partial\boldsymbol{\varphi}/\partial\mathbf{u}_1)D(\mathbf{F}),$$

and the right-hand side of the equation depends only on $\mathbf{u}$, $\mathbf{u}_1$ and their derivatives with respect to x. Therefore (see (1.15), (1.17)) $\mathbf{v}_t$ depends only on $\mathbf{v}$, $\mathbf{v}_x$, $\mathbf{v}_{xx}, \ldots$, i.e., an equation of the form (1.12) holds. From (1.10) and (1.6), we obtain

$$\mathbf{v}_\tau=(\partial\boldsymbol{\varphi}/\partial\mathbf{u})\boldsymbol{\Phi}+(\partial\boldsymbol{\varphi}/\partial\mathbf{u}_1)D(\boldsymbol{\Phi}),$$

and in the expression on the right there are only the variables $\mathbf{u}_i$. It can be seen from (1.14) that it is only on $\mathbf{v}_x$ and the variables $\mathbf{v}_i$ that the variables $(\mathbf{v}_i)_x$ depend and hence (see (1.15)) the same is true of $\mathbf{u}_i$, and therefore

$$\mathbf{v}_\tau=\boldsymbol{\Psi}(\mathbf{v}_x, \mathbf{v}, \mathbf{v}_{\pm1}, \mathbf{v}_{\pm2}, \ldots). \tag{1.22}$$

Similarly, from (1.16) we find that $\mathbf{v}_{x\tau}$ depends on the same variables as the function $\boldsymbol{\Psi}$ in (1.22). Comparing $\mathbf{v}_{x\tau}$ with $\mathbf{v}_{\tau x}$ obtained by differentiating (1.22), we see that $\boldsymbol{\Psi}$ in (1.22) does not depend on $\mathbf{v}_x$, i.e., an equation of the form (1.13) holds.

The converse of the theorem is proved similarly. From the relations (1.14) and (1.15) we obtain (1.17). Expressing $\mathbf{v}$, $\mathbf{v}_x$ in terms of $\mathbf{u}$, $\mathbf{u}_1$ in accordance with (1.10) and (1.16) and eliminating $\mathbf{v}$, we obtain Eq. (1.9). The condition of compatibility of the chain (1.9) and the equation obtained from the equation $\mathbf{v}_t=\mathbf{R}$ compatible with (1.14) is transformed on this occasion into an identity: $D_x(\mathbf{R})=D_x(\mathbf{R})$. In order to express (1.13) in the form (1.6), we find from (1.15) and (1.13) that $\mathbf{u}_\tau$ depends only on $\mathbf{v}_i$, $(\mathbf{v}_i)_x$, and therefore (see (1.10), (1.16)) (1.16) holds. Finally, by means of (1.15), (1.17), and (1.12) we establish that $\mathbf{u}_t$, $(\mathbf{u}_1)_t$ depends only on $\mathbf{u}_1$, $\mathbf{u}$, $\mathbf{u}_x$, $\mathbf{u}_{xx}, \ldots$. Comparing $D(\mathbf{u}_t)$, $D_t(\mathbf{u}_1)$, we arrive at an equation of the form (1.5).

Remark 1. We shall say that Eqs. (1.5) and (1.12) are related by the substitution $\mathbf{u}=\mathbf{r}(\mathbf{v}, \mathbf{v}_x)$ if

$$(\partial\mathbf{r}/\partial\mathbf{v})\mathbf{G}+(\partial\mathbf{r}/\partial\mathbf{v}_x)D_x(\mathbf{G})=\mathbf{F}(\mathbf{r}(\mathbf{v}, \mathbf{v}_x),\ D_x(\mathbf{r}(\mathbf{v}, \mathbf{v}_x)), \ldots).$$

As the theorem shows, Eqs. (1.5) and (1.12) are related by the two different substitutions (1.15) and (1.17) (the coordinates of the vector functions $\mathbf{p}$, $\mathbf{q}$ are functionally independent). Conversely, if there are two such substitutions, then, eliminating the letter $\mathbf{u}$, we obtain a chain (1.14) compatible with (1.12). The compatibility occurs because differentiation of the relation (1.14) with respect to t leads by virtue of (1.12) to the equation

$$\mathbf{F}(\mathbf{p}(\mathbf{v}_1, (\mathbf{v}_1)_x, (\mathbf{p}(\mathbf{v}_1, (\mathbf{v}_1)_x))_x, \ldots)=\mathbf{F}(\mathbf{q}(\mathbf{v}, \mathbf{v}_x), (\mathbf{q}(\mathbf{v}, \mathbf{v}_x))_x, \ldots),$$

which is a consequence of (1.14).

Remark 2. From the local conservation laws $a_t = b_x$ (or $c_\tau = (D-1)d$) of Eq. (1.5) (or (1.13)), where a, b are scalar functions of a finite number of the variables $\mathbf{u}$, $\mathbf{u}_x$, $\mathbf{u}_{xx}, \ldots$ (c, d are scalar functions of the variables $\mathbf{v}$, $\mathbf{v}_{\pm1}$, $\mathbf{v}_{\pm2}, \ldots$), we can readily construct local conservation laws of Eq. (1.12) (respectively, (1.6)) using the substitution (1.15) (respectively, (1.10)) (see, for example, [3]).

In the scalar case, the simplest example of a chain of the form (1.2) that comes within the ambit of Theorem 1 is the chain

$$(u_1+u)_x=u_1^2-u^2. \tag{1.23}$$

with which the equations

$$u_t = u_{xxx} - 6u^2 u_x, \tag{1.24}$$

$$u_\tau = -(u_1+u)^{-1} + (u+u_{-1})^{-1} \tag{1.25}$$

are compatible. Equation (1.24) is known as the modified Korteweg–de Vries equation, and (1.23) is its Bäcklund transformation. Equations (1.24) and (1.25) are compatible with each other by virtue of (1.23). The example of the compatible triplet of equations (1.23)–(1.25) appeared in [4].

In the vector case, we take the decoupled nonlinear Schrödinger equation

$$u_t = u_{xx} + 2u^2 v, \quad -v_t = v_{xx} + 2v^2 u \tag{1.26}$$

and write its Bäcklund transformation in the form a system of two chains, one of which is

$$(u_1+\varepsilon u)_x + \alpha(u_1+\varepsilon u) = (u_1 - \varepsilon u)\,[\beta - (u_1+\varepsilon u)(v_1+\varepsilon^{-1}v)]^{1/2},$$

while the other is obtained by the substitution $u \leftrightarrow v$, $\varepsilon \leftrightarrow \varepsilon^{-1}$, $\alpha \to -\alpha$. A substitution of the form (1.10) $\hat{u} = u_1 + \varepsilon u$, $\hat{v} = v_1 + \varepsilon^{-1} v$ leads to a system invariant with respect to the transformation

$$u \leftrightarrow v, \quad x \to -x, \quad t \to -t, \tag{1.27}$$

one of the equations of which has the form

$$u_t = u_{xx} + \gamma u^2 v + {}^1/_2(u_x + \alpha u)(\beta u^{-1} v^{-1} - 1)^{-1}[(\ln uv^2)_x - \alpha], \tag{1.28}$$

where $\gamma = \frac{1}{2}$. For any γ, the so-called symmetric transformation relates the system (1.28) to degeneracies of the Landau–Lifshitz model (see [1]).

2. Apart from some comparatively simple transformations, the list of equations of the form $u_t = u_{xxx} + f(u, u_x, u_{xx})$ with a rich set of local conservation laws (defined in Remark 2) consists of the Krichever–Novikov equation, Eqs. (1.7) and (1.24), and also

$$u_t = u_{xxx} - {}^1/_8 u_x^3 + z(u)\, u_x, \tag{2.1}$$

where $z(u) = \alpha \exp(u) + \beta \exp(-u) + \gamma$,

$$u_t = u_{xxx} - \frac{3}{2}\frac{u_x u_{xx}^2}{u_x^2+1} + y(u)\, u_x(u_x^2+1), \tag{2.2}$$

where y satisfies the differential equation

$$(y')^2 = P(y) = -{}^8/_3(y+2\gamma)\,[(y-\gamma)^2 - 4\alpha\beta]$$

(see [5,3]). Equations (1.7), (1.24), (2.2), and (2.1) are related by double differential substitutions (see [3]), and hence (see Remark 1 on Theorem 1) are equivalent from the point of view of the theory discussed here. As is shown by the example of Eq. (1.24), there can be a situation in which a scalar chain of the form (1.2) is compatible with a pair of equations: one of Eqs. (1.7), (1.24), (2.1), and (2.2) and a representative of the complete list of discrete equations of the form $u_\tau = g(u_1, u, u_{-1})$ with rich set of local conservation laws in [6]. We shall give several such pairs on the basis of Theorem 1. We shall say that chains of the form (1.2) are nonlocal, and of the form (1.6) local.

We can write (1.23) in the form (1.14) and introduce the new variable $2\hat{u} = u^2 - u_x$. Equation (1.24) goes over precisely into (1.7), and the chain (1.25) into the chain

$$u_\tau = \varepsilon[h(u_1+u) - h(u+u_{-1})]/[h(u_1+u) + h(u+u_{-1})], \tag{2.3}$$

where $\varepsilon = 1$, $h = (u_1 + u)^{1/2}$. On the other hand, (1.23) enables us to introduce the variable $\hat{u} = -2\ln(u_1 + u)$. Thus, it becomes clear that Eq. (2.1) with $\alpha = \gamma = \beta + 3/2 = 0$ corresponds to the local chain

$$u_\tau = \exp((u_1+u)/2) - \exp((u+u_{-1})/2). \tag{2.4}$$

Omitting the nonlocal chain, we merely mention that it can readily be expressed in the form (1.9) and one can obtain the substitution $\hat{u} = \exp((u_1 + u)/4)$, which leads to Eq. (2.2) with $y(u) = -3/2u^{-3}$. From (2.4) we obtain a chain related to the Volterra equation (1.8) by an obvious point transformation ($\hat{u} = \sigma(u)$).

A further group of examples can be obtained by using the double substitution in [3], which relates the solutions of Eqs. (2.1) and (2.2) in the generic situation (at least one of the numbers α and β nonzero). The substitution has the form $\hat{u}=\pm 2\,\mathrm{arsh}\, u_x+\varphi(u)$, where $z(\varphi(u))=y(u)$, $\hat{u}$ is a solution of (2.1), and u a solution of (2.2). In accordance with Remark 1 on Theorem 1, this double substitution enables us to construct a nonlocal chain of the form (1.14) for Eq. (2.2), and, hence, a chain of the form (1.9) for (2.1):

$$\left(z\left(\frac{u_1+u}{2}\right)\right)_x=\left(P\left(z\left(\frac{u_1+u}{2}\right)\right)\right)^{1/2}\mathrm{sh}\,\frac{u_1-u}{2}, \tag{2.5}$$

where P is the polynomial that determines the function y in (2.2).

In the special case 2z(u) = cosh u + 1/3, the following local chain is compatible with (2.5):

$$u_\tau=\mathrm{th}((u_1+u)/4)-\mathrm{th}((u+u_{-1})/4). \tag{2.6}$$

From it, we obtain for (2.2) a chain related by a point transformation to

$$4v_\tau=(1-v^2)(v_1-v_{-1}) \tag{2.7}$$

(note that $v = \tanh((u_1 + u)/4)$ is the connection between (2.6) and (2.7)). Finally, in our special case (2.5) can be written in the form

$$(u_1+u)_x=\alpha\,\mathrm{sh}(u_1/2)-\alpha\,\mathrm{sh}(u/2),$$

where $\alpha = 4i/\sqrt{6}$, and this enables us to introduce the new variable $4\hat{u} = -u_x + \alpha \sinh(u/2)$. We arrive at the equation

$$u_t=u_{xxx}+(2/3-6u^2)u_x \tag{2.8}$$

with the nonlocal chain

$$(\mathrm{arsh}\, 2\alpha^{-1}(u_1+u))_x=u_1-u \tag{2.9}$$

and local chain of the (2.3) with different ε and h. Since Eqs. (2.8) and (1.24) are identical apart from a Galileo transformation ($\hat{u}(t, \hat{x}) = u(t, x)$, $\hat{x} = x + 3t/2$ (u is a solution of (2.8)), we see that to Eq. (1.24) there correspond not only two nonlocal chains but also two local chains. The chain (2.9) appeared in [7,8].

3. There are not a few vector equations of the form

$$\mathbf{V}_t=\mathbf{F}(\mathbf{V}, \mathbf{V}_x, \mathbf{V}_{xx},\ldots), \tag{3.1}$$

where $\mathbf{V}=(u, v)^t$, compatible with chains of the form

$$\mathbf{V}_x=\mathbf{G}(\mathbf{V}, \mathbf{V}_{\pm1}, \mathbf{V}_{\pm2},\ldots) \tag{3.2}$$

(see [4]). It is here natural to take the independent variables to be

$$u,\ v,\ u_{\pm1},\ v_{\pm1},\ u_{\pm2},\ v_{\pm2},\ldots, \tag{3.3}$$

and the compatibility condition takes the form

$$(\partial\mathbf{G}/\partial\mathbf{V})\mathbf{F}+(\partial\mathbf{G}/\partial\mathbf{V}_1)D(\mathbf{F})+(\partial\mathbf{G}/\partial\mathbf{V}_{-1})D^{-1}(\mathbf{F})+\ldots=$$

$$(\partial\mathbf{F}/\partial\mathbf{V})\mathbf{G}+(\partial\mathbf{F}/\partial\mathbf{V}_x)D_x(\mathbf{G})+(\partial\mathbf{F}/\partial\mathbf{V}_{xx})D_x^2(\mathbf{G})+\ldots.$$

For such equations, one can often introduce substitutions that are invertible in the set of variables (3.3) (see Theorem 2). The ambit of Theorem 2 includes the Schrödinger equation (1.26), and also the Heisenberg and Landau-Lifshitz models written in the form (3.1) (see [4]).

THEOREM 2. Suppose a chain (3.2) compatible with (3.1) has the form

$$u_x=\varphi(u, v, u_1, v_1),\quad v_x=\psi(u_{-1}, v_{-1}, u, v), \tag{3.4}$$

where $\varphi(a, b, c, d)$, $\psi(a, b, c, d)$ are functionally independent as functions of the variables b and c. Then a change of variables $\hat{u} = u$, $\hat{v} = v_1$ that is invertible in the set (3.3) carries (3.1) to an equation of the form (3.1) again, and the old and new equations are related by differential substitutions of the form

$$u=\hat{u},\quad v=A(\hat{u}, \hat{v}, \hat{u}_x, \hat{v}_x), \tag{3.5}$$

$$u_1 = B(\hat{u}, \hat{v}, \hat{u}_x, \hat{v}_x), \quad v_1 = \hat{v}. \tag{3.6}$$

Proof. Writing (3.1) in the new variables, we see that $\hat{u}_t$ depends only on the variables $\hat{u}$, $\hat{v}_{-1}$ and their derivatives with respect to x, and $\hat{v}_t$ only on $\hat{v}$, $\hat{u}_1$ and their derivatives with respect to x. But in the new variables, the relations (3.4) are such that we can obtain expressions of the form

$$\hat{u}_1 = B(\hat{u}, \hat{v}, \hat{u}_x, \hat{v}_x), \quad \hat{v}_{-1} = A(\hat{u}, \hat{v}, \hat{u}_x, \hat{v}_x). \tag{3.7}$$

Therefore, $\hat{u}_t$ and $\hat{v}_t$ can be expressed in terms of $\hat{u}$, $\hat{v}$, $\hat{u}_x$, $\hat{v}_x$, ... The expressions (3.6) are a consequence of (3.7).

Remark 3. When necessary, it is proposed to verify by direct calculations that Eq. (3.1) and the chain (3.2) written in the new variables are again compatible. This was done in the example given below.

The substitutions (3.5) and (3.6) (in connection with chains of the form (3.4) moreover) appeared in [4]. On the basis of Theorem 2, we show how from the known system (1.26) we can obtain the quasilinear system

$$\begin{cases} u_t = u_{xx} - 2u_x(\ln a)_x + {}^1/_2(1-a^{-2})u^{-1}u_x^2, \\ -v_t = v_{xx} - 2v_x(\ln b)_x + {}^1/_2(1-b^{-2})v^{-1}v_x^2, \end{cases} \tag{3.8}$$

where the functions a, b are given by the implicit relations

$$2a(b+1) = vu_x, \quad 2b(a+1) = -uv_x.$$

This system of equations is invariant with respect to the substitution (1.27) and can be represented in the form

$$\mathbf{V}_t = M(\mathbf{V}, \mathbf{V}_x)\mathbf{V}_{xx} + \mathbf{N}(\mathbf{V}, \mathbf{V}_x),$$

where $\mathbf{V} = (u, v)^t$ is a vector column, and the matrix M has vanishing trace and determinant equal to minus unity.

The system (1.26) is compatible with a system of the form (3.4),

$$u_x = u_1 + u^2 v_1, \quad -v_x = v_{-1} + v^2 u_{-1}$$

(see [4]). Going over to new variables in accordance with Theorem 2, we obtain the compatible pair

$$\begin{cases} u_t = u_{xx} - 2(u^2 v_x + u^3 v^2), \\ -v_t = v_{xx} + 2(v^2 u_x - v^3 u^2), \end{cases} \tag{3.9}$$

$$u_x = u_1 + u^2 v, \quad -v_x = v_{-1} + v^2 u. \tag{3.10}$$

The system of chains (3.10) again has the form (3.4). By introducing standard new variables, we obtain from (3.9) and (3.10) a consistent pair in which the discrete system does not belong to the systems (3.4). However, we can make additionally a point transformation ($\hat{u} = \hat{u}(u, v)$, $\hat{v} = \hat{v}(u, v)$) that does not change the form of the vector equations (3.1) and (3.2). The composition of the two invertible transformations

$$\hat{u} = 2u/(uv_1 - 1), \quad \hat{v} = 2v_1/(uv_1 - 1)$$

leads to the continuous system (1.28) with $\alpha = \gamma = \beta + 1 = 0$ and a discrete system of the form (3.4),

$$v_1 u_x = 2A(D(A)+1), \quad -u_{-1}v_x = 2A(D^{-1}(A)+1),$$

where $A = (uv + 1)^{1/2}$. In the next step, the invertible change of variables in Theorem 2 gives the system of equations (3.8).

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LOWER KORTEWEG–DE VRIES EQUATIONS AND SUPERSYMMETRIC STRUCTURE OF THE SINE–GORDON AND LIOUVILLE EQUATIONS

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A continuation of the hierarchy of the Korteweg–de Vries equation in the direction corresponding to negative powers of the spectral parameter is constructed. Among the members of this hierarchy there are equations related by a Miura transformation to the sine–Gordon and Liouville equations. The supersymmetric structure of this connection is clarified.

1. Introduction

Integrable nonlinear differential equations possess an extremely rich internal structure. Indeed, new connections and constructions are still being found even in such well-studied equations as the Korteweg–de Vries (KdV) and sine–Gordon equations. Recently, structures analogous to the structures of supersymmetric quantum mechanics have been found in the framework of the inverse scattering method [1,2]. This work is based on a Miura transformation – on the one hand, it carries solutions of the modified Korteweg–de Vries equation (mKdV) into solutions of the KdV equation, while on the other hand it establishes a connection between the potentials of the Hamiltonian and the potentials of its supercharge. From this point of view, the Zakharov–Shabat operator $\tilde{L}$ that occurs in the Lax pair for the mKdV equation is a supercharge, and its square plays the part of a supersymmetric Hamiltonian. Such a Hamiltonian has the form of a diagonal matrix whose elements are Schrödinger operators L in the Lax pair for the KdV equation. This correspondence also generates a connection between the integrals of the motion of the hierarchies of the KdV and mKdV equations. The mKdV hierarchy also includes the sine–Gordon equation, for the example of which the supersymmetry properties are most clearly manifested. First of all, it is necessary to find its analog from the KdV hierarchy. Each equation of this hierarchy is characterized by the degree of the corresponding operator A (deg A) of the Lax pair. For the higher KdV equations, deg A > 0. In the case of the sine–Gordon equation, deg A = −1, and the operator A corresponding to its analog in the KdV hierarchy must have the same order. In the paper, we construct equations that continue the KdV hierarchy to the case of negative powers of the operators A. We shall refer to these equations as the lower KdV equations. They include the equations related by a Miura transformation to the sine–Gordon and Liouville equations. We establish the supersymmetric structure of this connection and give an invariant definition of the Miura transformation.

We assume that the potential u in the Schrödinger operator L decreases sufficiently rapidly at $\pm\infty$ for all the integrals considered below to converge.

2. Lower Korteweg–de Vries Equations

For study of the KdV equation and its higher analogs, one considers the time-independent Schrödinger operator

$$L=\frac{d}{dx^2}+u \tag{1}$$

and seeks differential operators A_N such that the commutator $[L, A_N]$ is an operator of zeroth degree (a function). To find the explicit form of the operators A_N, one expresses them as sums of powers of the operator L:

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