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RELATIVISTIC TODA CHAINS AND SCHLESINGER TRANSFORMATIONS

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We construct the auto-Schlesinger transformations for all equations in the known list of integrable relativistic Toda chains. Our construction is essentially based on the equations being Lagrangian and on a standard transition to their Hamiltonian form; in this case, the transition is described by the changes of variables that are invertible but not pointwise. We discuss two examples of another type that has similar properties; these are also integrable Lagrangian equations allowing the Schlesinger transformation.

Keywords: integrability, Toda chain, Lagrangian equations, Hamiltonian equations, autotransformations

1. Introduction

As is known, the Ablowitz–Kaup–Newell–Segur (AKNS) system of equations

$$u_t = u_{xx} + 2u^2 v, \qquad -v_t = v_{xx} + 2v^2 u \tag{1}$$

allows the autotransformation

$$\tilde{u} = u_{xx} - \frac{u_x^2}{u} + u^2 v, \qquad \tilde{v} = \frac{1}{u} \tag{2}$$

(see [1]–[3]). Multiple application of this transformation,

$$(u, v) = (u_n, v_n) \quad \to \quad (\tilde{u}, \tilde{v}) = (u_{n+1}, v_{n+1}),$$

results in the chain of relations

$$u_{n+1} = u_{n,xx} - \frac{u_{n,x}^2}{u_n} + \frac{u_n^2}{u_{n-1}}$$

(v_n is excluded in accordance with $v_n = 1/u_{n-1}$), which is written as the known integrable Toda model

$$q_{n,xx} = e^{q_{n+1}-q_n} - e^{q_n-q_{n-1}} \tag{3}$$

in terms of $q_n = \log u_n$. Toda chain (3) thus determines auto-Schlesinger transformation (2) (we follow [3] in using this term) for AKNS system (1). Other autotransformations of the same type for systems of equations similar to (1) can be found in [4]–[6].

The auto-Schlesinger transformation for an integrable nonlinear equation is a special (degenerate) case of the auto-Bäcklund transformation. Applying such a transformation, we do not obtain additional constant parameters, but we can nevertheless sometimes construct a multisoliton solution. In the case where the initial solution (u_0, v_0) of system (1) is such that $v_0 = 0$, the function u_0 satisfies the heat conduction

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equation: $u_{0,t} = u_{0,xx}$. We can easily find solutions of the heat conduction equation with a great number of arbitrary constants such that after a certain number of steps, we obtain a multisoliton solution for both system (1) and the nonlinear Schrödinger equation

$$i\psi_t = \psi_{xx} + 2|\psi|^2\psi$$

(see, e.g., [6]).

It turns out that the relativistic Toda chain

$$\ddot{u}_n = \frac{\dot{u}_{n+1}\dot{u}_n}{1+e^{u_n-u_{n+1}}} - \frac{\dot{u}_n\dot{u}_{n-1}}{1+e^{u_{n-1}-u_n}}, \qquad \dot{u}_n = \frac{du_n}{dt} \tag{4}$$

allows an autotransformation of the same type, namely,

$$\tilde{u}_n = \log\frac{e^{u_{n+1}} + e^{u_n}}{\dot{u}_n} \tag{5}$$

(transformation (5) seems new, as do the other autotransformations presented below). This autotransformation is invertible on the solutions of chain (4), and the inverse transformation is

$$u_n = \log\frac{\tilde{u}_{n,t}}{e^{-\tilde{u}_n} + e^{-\tilde{u}_{n-1}}}. \tag{6}$$

It is interesting that when passing from (4) to the system for u_n and v_n ($v_n = \dot{u}_n$), we can easily write a transformation $(u_n, v_n) \to (\tilde{u}_n, \tilde{v}_n)$ that is invertible in the usual sense (as is (2)). Introducing the corresponding chain of transformations in terms of u_n^k and v_n^k, we exclude v_n^k and obtain the pure difference equation

$$e^{u_n^{k-1}-u_n^k} - e^{u_n^k-u_n^{k+1}} = e^{u_{n+1}^k-u_n^{k+1}} - e^{u_n^{k-1}-u_{n-1}^k}. \tag{7}$$

This is one of the known integrable equations approximating Toda chain (3) (see, e.g., [7]).

In this paper, we consider integrable equations similar to the relativistic Toda chain:

$$\ddot{u}_n = F(\dot{u}_{n+1}, \dot{u}_n, \dot{u}_{n-1}, u_{n+1}, u_n, u_{n-1}). \tag{8}$$

The list of such equations was obtained in [8] by using the method of higher symmetries (also see [9]). This list comprises six equations, and we construct the Schlesinger transformation for each of them. These transformations are similar to (5) and have the form

$$\tilde{u}_n = U(\dot{u}_n, u_{n+1}, u_n). \tag{9}$$

The pure difference equations similar to (7) and related to transformations (9) are known (they can be found, e.g., in [7], [10], [11]), and we do not give them here.

Transformations (9) not only are given by an explicit formula and are invertible (although not pointwise) on the solutions of the corresponding equation but also have yet another property. Theoretically, transformation (9) can be a nonpointwise involution: $\sigma^2[u_n] = \sigma\big[\sigma[u_n]\big] = u_n$ (we use the notation $\tilde{u}_n = \sigma[u_n]$). It can be a nonpointwise group transformation: if $\tilde{u}_n = \sigma[\alpha, u_n]$, where α is the group parameter, then

$$\sigma[0, u_n] = u_n, \qquad \sigma\big[\alpha, \sigma[\beta, u_n]\big] = \sigma[\alpha+\beta, u_n].$$

We can offer examples of each type of the transformations. In our case, transformations (9) are such that

$$\sigma^2[u_n] = V(\dot u_{n+1}, \dot u_n, \dot u_{n-1}, u_{n+2}, u_{n+1}, u_n, u_{n-1}),$$

and the number of variables $\dot u_{n+i}$ and u_{n+i} in $\sigma^3[u_n], \sigma^4[u_n], \dots$ increases. In some sense, this property shows that the transformations presented in this paper allow constructing a set of different solutions in contrast to the involution or group transformation.

All integrable equations in the list obtained in [8] are Lagrangian [10]. In the construction of the Schlesinger transformation, we essentially use the standard transition from the Lagrangian form of equations to the Hamiltonian form. But in contrast to the classical case, the transition here is given by a nonpointwise invertible transformation [9]. We also note that the standard (nondegenerate) auto-Bäcklund transformations are known for most of the equations in the list under discussion [10].

In the last section, we present two examples of another type that has similar properties. These are also Lagrangian equations allowing the Schlesinger transformation. One equation (an equation on a lattice) is intimately related to the difference nonlinear Schrödinger equation, and the other (a partial differential equation) is related to the Landau–Lifshitz equation.

2. A scheme for constructing the Schlesinger transformations

We here explain how to construct the Schlesinger transformations for the relativistic Toda chains. We first discuss the transition from the Lagrangian form of equations to the Hamiltonian form, i.e., the relation between the known integrable equations of type (8) and the also known integrable systems of the class[1]

$$\dot u_n = f(u_{n+1}, u_n, v_n), \qquad \dot v_n = g(v_{n-1}, v_n, u_n). \tag{10}$$

Lists of both Eqs. (8) and systems (10) are presented in the next section. For brevity, we omit the index n in formulas; for example, relativistic Toda chain (4) is written as

$$\ddot u = \frac{\dot u_1 \dot u}{1 + e^{u-u_1}} - \frac{\dot u \dot u_{-1}}{1 + e^{u_{-1}-u}}. \tag{11}$$

The equations in (8) are Lagrangian; in this case, the Lagrangian L and the Euler–Lagrange equation are

$$L = L(\dot u, u_1, u), \qquad \frac{\partial^2 L}{\partial \dot u^2} \neq 0, \qquad \frac{d}{dt}\frac{\partial L}{\partial \dot u} = \frac{\partial}{\partial u}(1 + T^{-1})L \tag{12}$$

(here, T is the shift operator: $n \to n+1$, in particular, $T^{-1}L = L(\dot u_{-1}, u, u_{-1})$). The local conservation laws $\dot p = (T-1)q$, namely,

$$\frac{d}{dt}(\dot u L_{\dot u} - L) = (T^{-1} - 1)(\dot u_1 L_{u_1}) \tag{13}$$

(for brevity, the partial derivatives are here denoted by subscripts) correspond to the standard conservation laws for such equations. If the Lagrangian can be written as $L = L(\dot u, u_1 - u)$, then there is also the local conservation law

$$\frac{d}{dt} L_{\dot u} = (1 - T^{-1}) L_u. \tag{14}$$

[1] We obtained the list of integrable Hamiltonian systems (10) using a symmetry test [12]. The main part of the list was first published in [5]. The L-A pairs were constructed for some systems in [13]. The total list was published in [9], where, in particular, it was shown that all the systems have higher symmetries and conservation laws.

Relativistic Toda chain (11) is assigned the Lagrangian

$$L = \dot{u} \log \frac{\dot{u}}{e^{u_1 - u} + 1}. \tag{15}$$

In this case, the Legendre transformation $H = v\dot{u} - L$, $v = L_{\dot{u}}$ results in a invertible change of variables between $u, \dot{u}$ and u, v that is given by a formula of the type $v = \theta(\dot{u}, u_1, u)$. More exactly, if we pass from Eq. (8) to the system for $y = u$ and $z = \dot{u}$, then the systems for y, z and u, v are related by the nonpointwise, but invertible, change of variables given by $u = y$, $v = \theta(z, y_1, y)$ (the pointwise change of variables would have the form $u = U(y, z)$, $v = V(y, z)$). In this case, the Hamilton equations corresponding to the Hamiltonian H are

$$\dot{u} = \frac{\delta H}{\delta v}, \qquad \dot{v} = -\frac{\delta H}{\delta u}, \qquad H = H(v, u_1, u), \tag{16}$$

where

$$\begin{aligned} \frac{\delta H}{\delta v} &= \frac{\partial}{\partial v} \sum_i T^i(H) = \frac{\partial H}{\partial v}, \\ \frac{\delta H}{\delta u} &= \frac{\partial}{\partial u} \sum_i T^i(H) = \frac{\partial}{\partial u}(1 + T^{-1})H. \end{aligned} \tag{17}$$

For comparison, we recall the classical formulas in the same notation,

$$L = L(u, \dot{u}), \qquad \frac{d}{dt}\frac{\partial L}{\partial \dot{u}} = \frac{\partial L}{\partial u}, \qquad H = H(u, v), \qquad \dot{u} = \frac{\partial H}{\partial v}, \qquad \dot{v} = -\frac{\partial H}{\partial u},$$

and the change of variables is defined by $v = \theta(u, \dot{u})$.

It turns out that the known equations of class (8) are such that we can transform system (16) to form (10) using an additional pointwise change of variables of the form $\hat{v} = \hat{\theta}(u, v)$ (u is unchanged). In the case of relativistic Toda chain (11), the additional change of variables $\hat{v} = u - v + 1$ results in the system of equations

$$\dot{u} = e^{u_1 - v} + e^{u - v}, \qquad \dot{v} = e^{u - v_{-1}} + e^{u - v}, \tag{18}$$

and the direct relation between (11) and (18) has the form

$$u = u, \qquad v = \log \frac{e^{u_1} + e^u}{\dot{u}}. \tag{19}$$

The additional change of variables does not change the form of the Hamiltonian in (16) but generally changes the Hamiltonian structure:

$$\dot{u} = \varphi(u, v)\frac{\delta H}{\delta v}, \qquad \dot{v} = -\varphi(u, v)\frac{\delta H}{\delta u}, \qquad H = \Phi(u, v) + \Psi(u_1, v). \tag{20}$$

For system (18), we have $\varphi = -1$ and $H = e^{u_1 - v} + e^{u - v}$. It is evident that the inverse transition from Hamiltonian systems (20) to Lagrangian equations is also possible. If we start with the system of equations of form (10), (20) (where $f_v \neq 0$), then the invertible transformation leading to (8), (12) is given by the first equation in the system (compare (18) and (19)). The formula for constructing the Lagrangian is slightly modified in comparison with the classical formula:

$$L = \psi(u, v)\dot{u} - H, \qquad \psi_v = \frac{1}{\varphi}. \tag{21}$$

The arbitrariness in the choice of the function ψ is inessential because Lagrangian equation (12) is unchanged under changing the Lagrangian to

$$\hat{L} = \alpha L + \beta + \sigma(u)\dot{u} + (T-1)\omega(u), \tag{22}$$

where $\alpha \neq 0$ and β are constants and σ and ω are arbitrary functions.

Such a relation between integrable equations (8) and systems (10) was discussed in [9]. Two lists were obtained independently (the systems of form (10) were the first; cf., e.g., [5] and [8]); their total equivalence up to nonpointwise, invertible transformations similar to (19) was discovered only more recently.

We discuss the proof of the statement that the indicated change of variables and formula (21) transform system (10), (20) (with $f_v \neq 0$) to Eqs. (8), (12). We use just this statement in what follows. Introducing the dependent variables y_i and z_i such that $y = u$ and $z = \dot{u}$ and using (10), we have the invertible transformation

$$y = u, \qquad z = f(u_1, u, v) \tag{23}$$

connecting the sets of variables y_i, z_i and u_i, v_i together with the relations for the partial derivatives

$$\frac{\partial}{\partial v} = f_v \frac{\partial}{\partial z}, \qquad \frac{\partial}{\partial u} = \frac{\partial}{\partial y} + f_u \frac{\partial}{\partial z} + T^{-1}(f_{u_1}) \frac{\partial}{\partial z_{-1}}. \tag{24}$$

Formulas (21), more exactly,

$$L(z, y_1, y) = \psi(u, v) f(u_1, u, v) - H(v, u_1, u),$$

and (24) allow evaluating

$$L_z = \psi, \qquad L_y = \psi_u f - H_u, \qquad L_{y_1} = -H_{u_1}$$

(we also use Hamiltonian structure (20)). It is now easy to verify that (12) is a consequence of (10), (20):

$$\frac{d}{dt}\frac{\partial L}{\partial z} = \frac{d\psi}{dt} = \psi_u f + \psi_v g = \psi_u f - \big(H_u + T^{-1}(H_{u_1})\big) = L_y + T^{-1}(L_{y_1}).$$

The coefficient of $\dot{z} = \ddot{u}$ in the Euler–Lagrange equation differs from zero, $L_{zz} = (\varphi f_v)^{-1}$, and this equation can therefore also be written in form (8).

We can now explain the source of auto-Schlesinger transformation (5). Relativistic Toda chain (4) allows the involution

$$\mathrm{Inv}_u: \quad \tilde{u}_n = -u_{-n}, \qquad \tilde{t} = -t, \tag{25}$$

and its Hamiltonian form (18) is invariant under the involution

$$\mathrm{Inv}_{u,v}: \quad \hat{u}_n = -v_{-n}, \qquad \hat{v}_n = -u_{-n}, \qquad \hat{t} = -t. \tag{26}$$

Introducing the notation $\mathrm{Lg}\colon (u_n, \dot{u}_n) \to (u_n, v_n)$ for map (19) generated by the Legendre transformation, we can consider the composition

$$\mathrm{AT} = \mathrm{Inv}_u \circ \mathrm{Lg}^{-1} \circ \mathrm{Inv}_{u,v} \circ \mathrm{Lg}\colon \quad \tilde{u}_n = v_n = \log \frac{e^{u_{n+1}} + e^{u_n}}{\dot{u}_n}, \qquad \tilde{t} = t. \tag{27}$$

Therefore, if u_n is a solution of relativistic Toda chain (4), then the function v_n obtained from (18) is also a solution of (4). Formula (5) with v_n in place of $\tilde{u}_n$ (coinciding with the second formula in (19)) is just another form of the first equation in system (18). It is natural that inverse transformation (6) proves to be the rewritten second equation in the same system.

In the general case, formulas (10) and (20) indicate the equal status of u and v: the function v also must satisfy an equation of form (8), (12) (this can be easily proved). In the case of system (18), the equations for u and v proved to be identical. This is proved by the transition to a special form of each integrable system (10), (20); we provide this form in the next section. For example, system (18) was presented in [5] and [9] in the polynomial form

$$\dot{u} = uv(u_1 + u), \qquad \dot{v} = -uv(v + v_{-1}).$$

We slightly change it using the pointwise transformation $\tilde{u} = \log u$, $\tilde{v} = -\log v$.

In the next section, all integrable systems (10), (20) are represented in a form allowing an involution similar to (26). In addition, the corresponding Lagrangian equation is also invariant under some involution, and the two involutions taken together guarantee that u and v satisfy the same Lagrangian equation. As a result, we can obtain the Schlesinger transformation expressing $\tilde{u} = v$ in terms of u_1, u, and $\dot{u}$ using the first equation in (10).

Concluding this theoretical section, we note that for Lagrangian equations (12), there is an intimate connection, as usual, between conservation laws and higher symmetries. Given a higher symmetry, we can easily construct a conservation law as shown in [10]. On the other hand, using a change of variables of form (23) between equation (12) and its Hamiltonian form (20), we can obtain a rather simple formula for constructing a higher symmetry in accordance with a conservation law.

In fact, if $\dot{p} = (T-1)q$ is the local conservation law for system (20) (the function q and the density p of the conservation law depend on u_i and v_i), then the system

$$u_\tau = \varphi \frac{\delta p}{\delta v}, \qquad v_\tau = -\varphi \frac{\delta p}{\delta u}$$

is a symmetry, i.e., a system compatible with (20). We can easily rewrite all the formulas in terms of u_i and $\dot{u}_i$. It follows that if the function

$$p = p(\dot{u}_{i_1}, \dot{u}_{i_1 - 1}, \ldots, \dot{u}_{i_2}, u_{j_1}, u_{j_1 - 1}, \ldots, u_{j_2})$$

is the density of the conservation law for Lagrangian equation (12), then the equation

$$u_\tau = \frac{1}{L_{\dot{u}\dot{u}}} \frac{\delta p}{\delta \dot{u}}, \qquad \frac{\delta p}{\delta \dot{u}} = \frac{\partial}{\partial \dot{u}} \sum_{k=-i_1}^{-i_2} T^k(p) \tag{28}$$

is its symmetry.

For example, the functions

$$p^{(1)} = \dot{u}_1 \dot{u} S(u_1 - u) + \frac{1}{2}\dot{u}^2, \qquad p^{(2)} = \frac{1}{\dot{u}}(1 + e^{u_1 - u})(1 + e^{u - u_{-1}}),$$

where $S(z) = 1/(1 + e^{-z})$, are the densities of the conservation laws for relativistic Toda chain (11), and formula (28) gives the respective symmetries

$$u_{\tau_1} = \dot{u}_1 \dot{u} S(u_1 - u) + \dot{u}\dot{u}_{-1} S(u - u_{-1}) + \dot{u}^2, \qquad u_{\tau_2} = -p^{(2)}$$

for this chain because $L_{\dot{u}\dot{u}} = 1/\dot{u}$ (see (15)).

It is easy to see that in the case of standard conservation laws (13) and (14), formulas (28) result in trivial pointwise Lie symmetries that have the respective forms $u_\tau = \dot{u}$ and $u_\tau = 1$.

3. The list of the Schlesinger transformations

As previously stated, the lists of the integrable Hamiltonian systems of form (10) and Lagrangian equations of form (8) comprise six objects. We first consider five cases with a similar structure.

In all five cases, systems (10), (20) can be written as

$$\dot u = r(u_1 - u, u - v), \qquad \dot v = r(v - v_{-1}, u - v) \tag{29}$$

(i.e., the system is defined by one function $r(x, y)$ of two variables), which provides the invariance under involution (26). For example, system (18) is assigned the function $r(x, y) = (e^x + 1)e^y$. The transition to the Lagrangian equations is given as described in the preceding section: the relation between v and $\dot u$ is defined by the first equation in the system, and the Lagrangian is constructed in accordance with formula (21). In all these cases, we can easily verify that Lagrangian (21) can be written as (for simplicity, we sometimes use formula (22))

$$L = R(\dot u) - \dot u A(w_1) - B(w_1), \qquad w = u - u_{-1}, \tag{30}$$

where $R'' \neq 0$. Such a Lagrangian is assigned the equation

$$\ddot u = Q(\dot u)\big(\dot u_1 a(w_1) - \dot u_{-1} a(w) + b(w_1) - b(w)\big), \tag{31}$$

$$Q(z) = \frac{1}{R''(z)}, \qquad a(z) = A'(z), \qquad b(z) = B'(z), \tag{32}$$

which is obvioustly invariant under involution (25). Therefore, as in the case of relativistic Toda chain (11), the formula for $\tilde u = v$ obtained from the first equation in system (29) yields the auto-Schlesinger transformation for Eq. (31). The inverse transformation can be found from the second equation in system (29).

In what follows, we present the main formulas for each of the five cases. We indicate the function $r = r(x, y)$ defining system of equations (29) and the functions φ and H that define Hamiltonian structure (20). We then present the functions $Q(z)$, $a(z)$, and $b(z)$ defining Lagrangian equation (31). We do not write the Lagrangian because it is easily reconstructed from formulas (30) and (32). Finally, we present the Schlesinger transformation for Eq. (31) (a formula for a new solution $\tilde u = v$). In some cases, there is a dependence on the arbitrary constants μ and ν. Relativistic Toda chain (11) belongs to case III with $\mu = 1$ and $\nu = 0$.

The list of equations and transformations (five of six cases)

I. $\quad r = e^x + e^y, \qquad \varphi = -e^{v-u}, \qquad H = e^{u_1 - v} + \frac{1}{2}e^{2(u-v)},$

$\quad Q = 1, \qquad a = e^z, \qquad b = -e^{2z},$

$\quad \tilde u = u - \log(\dot u - e^{u_1 - u}).$

II. $\quad r = xy, \qquad \varphi = v - u, \qquad H = v(u - u_1),$

$\quad Q = z, \qquad a = \frac{1}{z}, \qquad b = z,$

$\quad \tilde u = u + \frac{\dot u}{u - u_1}.$

III. $\quad r=(e^x+\mu)(e^y+\nu), \qquad \varphi=-(1+\nu e^{v-u}), \qquad H=e^{u_1-v}+\mu e^{u-v},$

$$Q=z, \qquad a=\frac{1}{1+\mu e^{-z}}, \qquad b=-\nu e^z,$$

$$\tilde{u}=u-\log\biggl(\frac{\dot{u}}{e^{u_1-u}+\mu}-\nu\biggr).$$

IV. $\quad r=\frac{x}{x+y}, \qquad \varphi=v-u, \qquad H=\log\frac{u-v}{u_1-v},$

$$Q=z(1-z), \qquad a=\frac{1}{z}, \qquad b=0,$$

$$\tilde{u}=\frac{u+u_1(\dot{u}-1)}{\dot{u}}.$$

V. $\quad r=\frac{e^x+\mu}{e^{x+y}+1}, \qquad \varphi=\mu-e^{v-u}, \qquad H=\log\frac{e^{u_1-v}+1}{1-\mu e^{u-v}},$

$$Q=z(z-\mu), \qquad a=\frac{1}{e^z+\mu}, \qquad b=0,$$

$$\tilde{u}=u_1-\log\biggl(\frac{e^{u_1-u}+\mu}{\dot{u}}-1\biggr).$$

We consider the sixth case separately and in more detail. This case differs significantly from the others. System (10) has the form

$$\begin{gathered}\dot{u}=\frac{2r}{u_1-v}+r_v, \qquad \dot{v}=\frac{2r}{u-v_{-1}}-r_u,\\ r=r(u,v)=r(v,u), \qquad r_{uuu}=0,\end{gathered} \tag{33}$$

i.e., r is a symmetric polynomial with six arbitrary constant coefficients, namely,

$$r(u,v)=\alpha u^2v^2+\beta uv(u+v)+\gamma(u^2+v^2)+\delta uv+\varepsilon(u+v)+\mu. \tag{34}$$

System (33) is intimately related to the Landau–Lifshitz equation (this system defines the auto-Bäcklund transformation for the latter; see [5] and [13]). Hamiltonian structure (20) is defined by the functions

$$\varphi=r, \qquad H=\log\frac{r}{(u_1-v)^2}. \tag{35}$$

The involution allowed by the system of equations is now

$$\hat{u}_n=v_{-n}, \qquad \hat{v}_n=u_{-n}, \qquad \hat{t}=-t. \tag{36}$$

We have some technical problems here related to deriving the explicit formula for v from (33) and representing Lagrangian (21) in terms of $\dot{u}$, u_1, and u such that the obtained expression is not unduly cumbersome. In parallel with the notation $r=r(u,v)$, we introduce the notation $s=r(u,u_1)$ for a polynomial of the same form (34) but in other variables. For any polynomial (34), the identities

$$\frac{2r}{u_1-v}+r_v=\frac{2s}{u_1-v}-s_{u_1}, \qquad r_v=s_{u_1}+(v-u_1)s_{u_1u_1} \tag{37}$$

hold, and, in particular, we can rewrite the first equation in (33) in terms of s,

$$\dot{u} = \frac{2s}{u_1 - v} - s_{u_1}. \tag{38}$$

Now using (37), (38), and the first equation in (33), we can easily express the functions v, r_v, and r and consequently the Hamiltonian H in (35) in explicit terms of $\dot{u}$, u_1, and u. To obtain the formulas for the Lagrangian L in (21), it then remains to represent the function ψ in terms of the new variables.

We failed to derive the explicit formula for ψ, but it suffices to obtain the relations for the partial derivatives of the function ψ in terms of the new variables by rewriting the defining equation $\psi_v = 1/\varphi = 1/r$. Using formulas (24) (we recall that $f(u_1, u, v)$ here is the right-hand side of the first equation in the initial Hamiltonian system (33)), we obtain the relations

$$\psi_v = f_v \hat{\psi}_{\dot{u}} = \frac{1}{r}, \qquad \psi_{u_1} = \hat{\psi}_{u_1} + f_{u_1} \hat{\psi}_{\dot{u}} = 0 \tag{39}$$

for the function $\hat{\psi}(u_1, u, \dot{u}) = \psi(u, v)$. On the other hand, it follows from the two different representations for the function f (see (33) and (38)) that

$$f_{u_1} = -\frac{2r}{(u_1 - v)^2}, \qquad f_v = \frac{2s}{(u_1 - v)^2}. \tag{40}$$

Relations (39) and (40) taken together allow representing the partial derivatives $\hat{\psi}_{u_1}$ and $\hat{\psi}_{\dot{u}}$ in terms of u_1, u, and $\dot{u}$. There is no need for a formula for $\hat{\psi}_u$. In particular, we find that $\hat{\psi}_{u_1 \dot{u}} = \hat{\psi}_{\dot{u} u_1} = 0$, and we have the representation $\hat{\psi} = A(u_1, u) + B(u, \dot{u})$ with the known derivatives A_{u_1} and $B_{\dot{u}}$ for A and B.

The presented scheme explains how to write the next defining equations for the Lagrangian L:

$$L = \log \frac{s}{\dot{u}^2 - R(u)} + \dot{u}\bigl(A(u_1, u) + B(u, \dot{u})\bigr), \tag{41}$$

$$s = r(u, u_1), \qquad R(u) = s_{u_1}^2 - 2s s_{u_1 u_1}, \tag{42}$$

$$A_{u_1} = \frac{1}{s}, \qquad B_{\dot{u}} = \frac{2}{\dot{u}^2 - R(u)}. \tag{43}$$

We recall that r is polynomial (34) defining Hamiltonian system (33). If we consider s in (42) a square polynomial in u_1, then R is the usual discriminant of this polynomial and therefore depends on only u and is a polynomial of degree not higher than four. The functions A and B are not uniquely defined by Eqs. (43), but this has no effect on the corresponding Lagrangian equation by virtue of (22).

The equation corresponding to Lagrangian (41)–(43) is

$$2\ddot{u} = \bigl(\dot{u}^2 - R(u)\bigr)\left(\frac{s_u - \dot{u}_1}{s} + \frac{\check{s}_u + \dot{u}_{-1}}{\check{s}}\right) + R'(u), \tag{44}$$

$$\check{s} = T^{-1}s = r(u_{-1}, u). \tag{45}$$

This equation is invariant under the involution $\tilde{u}_n = u_{-n}$, $\tilde{t} = -t$, which taken together with involution (36) explains why the function $\tilde{u} = v$ found from (33) is a new solution of Eq. (44) with (45). To define v, it is convenient to use relation (38), which gives the formula

$$\tilde{u} = u_1 - \frac{2s}{\dot{u} + s_{u_1}} \tag{46}$$

for the autotransformation of Eq. (44) with (45).

4. Similar examples

We show that there are equations of other classes that have similar properties. These are the integrable Lagrangian equations allowing the auto-Schlesinger transformation. We present two examples: the first is a difference-differential equation, and the second is a partial differential equation. The equation on a lattice is intimately related to the difference nonlinear Schrödinger equation (which coincides with the Ablowitz–Ladik equation). The other example is obtained from the stereographic projection of the Landau–Lifshitz equation. We also briefly discuss the Schlesinger transformation for the corresponding Hamiltonian systems.

The construction scheme here is the same as in the preceding sections. We start with the known Hamiltonian system of equations and obtain an equation equivalent to this system but new in form. In the difference-differential case, we use the system of equations

$$\begin{aligned} \dot u &= (uv+\nu)(u_1+\alpha u_{-1})-\eta u,\\ \dot v &= -(uv+\nu)(v_{-1}+\alpha v_1)+\eta v, \end{aligned} \tag{47}$$

which is integrable for any values of the constant coefficients ν, α, and η (see, e.g., [9]). The Hamiltonian structure (identical to that in (20)) is defined by the functions

$$\varphi = uv+\nu, \qquad H = v(u_1+\alpha u_{-1})-\eta\log\varphi,$$

i.e., the Hamiltonian has a form different from (20).

We note that system (47) generalizes one of the systems in the preceding section because for $\alpha=\eta=0$, we have case III with $\mu=0$ after the pointwise change of variables

$$\tilde u = \log u, \qquad \tilde v = -\log v. \tag{48}$$

In addition, system (47) allows two integrable reductions. In the case of $\alpha=\nu=1$ and $\eta=2$, the complex reduction $\psi=u=\bar v$, $\theta = it$ results in the known difference nonlinear Schrödinger equation (or the Ablowitz–Ladik equation [14])

$$i\psi_\theta = \psi_1-2\psi+\psi_{-1}+|\psi|^2(\psi_1+\psi_{-1}).$$

In the case of $\alpha=-1$ and $\eta=0$, another reduction $v=u$ yields the modified Volterra equation

$$\dot u = (u^2+\nu)(u_1-u_{-1}).$$

The same change of variables (48) transforms system (47) such that the system becomes invariant under involution (26):

$$\begin{aligned} \dot u &= (e^{u-v}+\nu)(e^{u_1-u}+\alpha e^{u_{-1}-u})-\eta,\\ \dot v &= (e^{u-v}+\nu)(e^{v-v_{-1}}+\alpha e^{v-v_1})-\eta. \end{aligned} \tag{49}$$

System (49) is completely similar to the systems in the preceding section. The first equation defines the invertible relation between $\dot u$ and v; the standard transition to the Lagrangian form yields the following results.

If we introduce the notation $w=u-u_{-1}$ as in (30), then the new equation can be written as

$$\begin{aligned} \ddot u &= (T-1)\big((\dot u+\eta)(\dot u_{-1}+\eta)V\big)+\nu(\dot u+\eta)(1-T)(e^w-\alpha e^{-w}),\\ V &= \frac{e^{w_1+w_{-1}}-\alpha^2e^{-2w}}{(e^{w_1}+\alpha e^{-w})(e^{w_{-1}}+\alpha e^{-w})}. \end{aligned} \tag{50}$$

The Lagrangian is defined as

$$L = (\dot u + \eta)\log\frac{\dot u + \eta}{U} + \nu U, \qquad U = e^{w_1} + \alpha e^{-w}.$$

In this case, the Euler–Lagrange equation differs from the equation in (12) only in the expression $\delta L/\delta u = \partial(T + 1 + T^{-1})L/\partial u$ in the right-hand side. We note that the obtained Eq. (50) has the form

$$\ddot u = F(\dot u_1, \dot u, \dot u_{-1}, u_2, u_1, u, u_{-1}, u_{-2}) \tag{51}$$

(compare (8)), and integrable equations of such a form are apparently unknown.

The higher conservation laws and symmetries for Eq. (50) can be obtained from the conservation laws and symmetries of the corresponding system (47) [9]. As an example, we present the formulas for the density of the simplest local conservation law

$$p = \left(\frac{\dot u + \eta}{U} - \nu\right)(c_1 e^{w_1} + c_2 e^{-w}) + c_3 \log\frac{\dot u + \eta}{U}$$

and for the simplest higher symmetry

$$u_\tau = \frac{\dot u + \eta}{U}(c_1 e^{w_1} + c_2 e^{-w})$$

(here, c_1, c_2, and c_3 are arbitrary constants). Finally, we can easily verify that Eq. (50) is invariant under involution (25). As before, this implies that the function $\tilde u = v$ found from system (49) satisfies Eq. (50) along with u. Therefore, the auto-Schlesinger transformation for the new Eq. (50) is

$$\tilde u = u - \log\left(\frac{\dot u + \eta}{U} - \nu\right). \tag{52}$$

In the second example, we rely on the known integrable system of equations that looks like

$$\begin{aligned} \dot u &= u_{xx} + \frac{2}{v-u}\big(u_x^2 + R(u)\big) + \frac12 R'(u), \\ \dot v &= -v_{xx} + \frac{2}{v-u}\big(v_x^2 + R(v)\big) - \frac12 R'(v). \end{aligned} \tag{53}$$

Here, $d^5R(z)/dz^5 = 0$, i.e., there are five arbitrary constant coefficients in the system. The stereographic projection of the Landau–Lifshitz equation on a sphere is represented in such a form (see, e.g., [5], [13]).

Superficially, the Hamiltonian structure has the previous form,

$$\dot u = \varphi(u,v)\frac{\delta H}{\delta v}, \qquad \dot v = -\varphi(u,v)\frac{\delta H}{\delta u},$$

but the formal variational derivative is given by

$$\frac{\delta}{\delta u} = \frac{\partial}{\partial u} - D_x\frac{\partial}{\partial u_x} + D_x^2\frac{\partial}{\partial u_{xx}} - \dots,$$

where D_x is the total derivative with respect to x (the definition for $\delta/\delta v$ is similar). We represent the coefficient φ and the Hamiltonian H for system (53) in terms of the functions ψ, S, and Q. Let

$$\psi = \frac{1}{v-u}, \qquad S = u_x^2 + R(u), \qquad Q = u_{xx} + \frac12 R'(u). \tag{54}$$

Then

$$\varphi = -\frac{1}{\psi^2}, \qquad H = S\psi^2 + Q\psi + \frac{1}{12}R''(u).$$

The transition to the Lagrangian equation is again given by the invertible change of variables found from the first equation in system (53). The equation itself is cumbersome, and we only present the integrable Lagrangian

$$L = \frac{1}{S}(\dot u - Q)^2 - \frac{1}{3}R''(u). \tag{55}$$

The explicit form of the equation can be obtained using the standard formula

$$D_t L_{\dot u} = \frac{\delta L}{\delta u} = L_u - D_x L_{u_x} + D_x^2 L_{u_{xx}}. \tag{56}$$

In this case, the involutions are $u \leftrightarrow v$, $t \to -t$ for system (53) and $t \to -t$ for Eq. (55) with (56). Therefore, the formula for $\tilde u = v$ found from (53) again gives the autotransformation for Lagrangian equation (55) with (56):

$$\tilde u = u + \frac{2S}{\dot u - Q}. \tag{57}$$

We present the two densities $p^{(i)}$ of the local conservation laws $p_t^{(i)} = q_x^{(i)}$ for the obtained equation (the first of these is the standard one):

$$p^{(1)} = \dot u L_{\dot u} - L = \frac{1}{S}(\dot u^2 - Q^2) + \frac{1}{3}R''(u), \qquad p^{(2)} = \frac{1}{S}u_x \dot u.$$

Equation (55) with (56) belongs to the class of equations of the form

$$\ddot u = F(\dot u, \dot u_x, \dot u_{xx}, u, u_x, u_{xx}, u_{xxx}, u_{xxxx}) \tag{58}$$

(as well as the well-known Boussinesq equation) and is seemingly new in form. In the particular case of $R = 0$ corresponding to the Heisenberg model, the formulas are significantly simplified, and we easily can write the explicit form of the equation:

$$D_t\left(\frac{\dot u}{u_x}\right) = D_x\left(\frac{u_{xxx}}{u_x} + \frac{3}{2}\frac{\dot u^2 - u_{xx}^2}{u_x^2}\right),$$

$$L = \frac{(\dot u - u_{xx})^2}{u_x^2}, \qquad \tilde u = u + \frac{2u_x^2}{\dot u - u_{xx}}.$$

The Lagrangian equations and the corresponding Hamiltonian systems indicated in this paper are related by invertible changes of variables; therefore, all the autotransformations can be easily extended to the Hamiltonian systems of equations. For example, using the transformation $\mathrm{AT}\colon (u, \dot u) \to (\tilde u, \tilde u_t)$ given by (27) (we also can express $\tilde u_t$ in terms of u_i and $\dot u_i$ by differentiating the formula in (27) with respect to t and using relativistic Toda chain (11)), we pass to the composition $\mathrm{Lg} \circ \mathrm{AT} \circ \mathrm{Lg}^{-1}\colon (u, v) \to (\tilde u, \tilde v)$ (the map Lg is defined by (19)) and obtain the autotransformation for Hamiltonian system of equations (18):

$$\tilde u = v, \qquad \tilde v = \log\frac{e^{v_1} + e^{v}}{e^{-v} + e^{-v_{-1}}} - u.$$

Schlesinger transformation (52) is similarly extended from Eq. (50) to the corresponding system (49) and consequently also to the initial Hamiltonian system (47):

$$\tilde u = \frac{1}{v}, \qquad \tilde v = (uv + \nu)\frac{v_1 v_{-1}}{v} - \nu v. \tag{59}$$

This transformation depends on only ν but is applicable for any values of the coefficients ν, α, and η in system (47). The autotransformation for system of equations (53) is well known (see, e.g., [5], [6]) and is similar to transformation (2) for system (1).

5. Conclusion

The main results in this paper are the auto-Schlesinger transformations that have form (9) and are written for all integrable Lagrangian equations (8) in the list obtained in [8]. The list comprises six equations, five of which have the same structure (31). The explicit form of these equations and the corresponding auto-transformations are given in the separate list in Sec. 3. The sixth equation together with the transformation are given by (44) and (46) (also see (34), (42), and (45)).

In the construction of the Schlesinger transformations, we essentially use the fact (pointed out in [9]) that the standard Legendre relation between Lagrangian form (12) and Hamiltonian form (20) of the same equation in this case is described by the invertible "triangle" change of variables of form (23), which is explained by the specific character of the Lagrangians and Hamiltonians. As a result, we obtain Schlesinger transformation (9) by simply rewriting one of the equations in the corresponding Hamiltonian system (20), (10) as the formula for defining $\tilde{u}_n = v_n$.

In addition, we demonstrate that proceeding in accordance with the scheme described in this paper, we can construct integrable Lagrangian equations along with the allowed autotransformations in other classes of equations. As an example, we obtained Eq. (50) with transformation (52), which belongs to class (51) and is related to Ablowitz–Ladik system (47). Another example, Eqs. (54), (55), and (56), was obtained from the Landau–Lifshitz equation, more exactly, from its stereographic projection (53). This example belongs to the equations of type (58), and its auto-Schlesinger transformation is given by (57). We can construct other equations of classes (51) and (58) similarly, and we plan to do this in a subsequent paper.

Finally, because the Lagrangian equations and Hamiltonian systems given in this paper are related by invertible changes of variables, we can extend the autotransformations in hand to all Hamiltonian systems of equations. As an example, we presented such transformation (59) for Ablowitz–Ladik system (47).

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