ELSEVIER

23 June 1997

PHYSICS LETTERS A

Physics Letters A 230 (1997) 295-300

On integrable two-dimensional generalizations of nonlinear Schrödinger type equations

A.V. Mikhailov [a,b], R.I. Yamilov [a,1]

[a] Applied Mathematics Department, University of Leeds, Leeds LS2 9JT, UK

[b] Landau Institute for Theoretical Physics, 2 Kosygina Street, Moscow 117940, Russian Federation

Received 2 January 1997; revised manuscript received 2 April 1997; accepted for publication 9 April 1997

Communicated by A.P. Fordy

Abstract

New Bäcklund transformations of integrable 2 + 1 dimensional generalisations of nonlinear Schrödinger type equations are found. The corresponding Miura transformations and modified equations are constructed. The Bäcklund transformations being treated as two-dimensional chain equations provide examples of new integrable difference-differential equations. © 1997 Elsevier Science B.V.

1. Introduction

In the one-dimensional case there is a comprehensive description and classification of integrable nonlinear Schrödinger type equations [1-3]. In the two-dimensional case the problem is much less studied and such a coherent picture does not exist at present. Apart from the famous Benney-Roskes (Davey-Stewartson) equation [4,5], only a few examples of integrable equations of this type are known (see for instance Refs. [6,7]). The aim of this work is to extend the list of two-dimensional integrable equations in order to collect more "experimental" material for future comprehensive theory. Here we extend the one-dimensional theory of Bäcklund and Miura transformations. Simultaneously, Bäcklund transformations being treated as two-dimensional chain equations give examples of new integrable two-dimensional lattice equations.

2. The Benney-Roskes family of equations

The Benney-Roskes (Davey-Stewartson) (BRDS) equation can be written in the following form,

$$i\partial_T\psi = \psi_{xx} + \psi_{yy} + 2\Phi\psi, \quad \Phi_{xy} = |\psi|^2_{xx} + |\psi|^2_{yy}. \quad (1)$$

There are a few standard forms of writing the above equation (see e.g. Refs. [8,9]). They even have different names (DS-1, DS-2, DS-3) and are indeed different from the analytical point of view. The symmetry structure for all of these forms is exactly the same and we shall treat them as a single equation. Moreover, instead of (1) we shall consider the following two systems of equations,

$$\partial_t u = u_{xx} + 2pu, \quad -\partial_t v = v_{xx} + 2pv,$$
$$p_y = (uv)_x, \quad (2a)$$

[1] On leave from Ufa Institute of Mathematics, Russian Academy of Sciences, Chernyshevsky Street 112, 450000 Ufa, Russian Federation.

0375-9601/97/$17.00
PII S0375-9601(97)00283-1

$$\partial_\tau u = u_{yy} + 2qu, \qquad -\partial_\tau v = v_{yy} + 2qv,$$
$$q_x = (uv)_y. \tag{2b}$$

It is known that Eqs. (2) are integrable [9]. Eqs. (2) are compatible and any linear combination of them is integrable. The BRDS equation (1) is nothing but the sum of these flows [2]. They have the same algebra of higher symmetries, Bäcklund and Miura transformations. Moreover, both systems (2) have the same L operator

$$L = \begin{pmatrix} \partial_x & -v \\ u & \partial_y \end{pmatrix} \tag{3}$$

in the Lax representation [8].

Following Ref. [10] (see also Refs. [11,12]) we shall consider a sequence u_n, v_n, p_n, q_n generated by an auto-Bäcklund transformation (adjacent sites with index n and $n+1$ are related by the Bäcklund transformation). Variables $u_n, v_n, p_n, q_n, n \in \mathbb{Z}$ and their derivatives in x and y we shall call *the extended set of dynamical variables*. Variables p, q we shall call *nonlocal*, because they cannot be locally expressed (without taking a quadrature) in terms of variables u, v which satisfy evolutionary equations. Presence of nonlocal variables is the important feature of integrable multidimensional equations. The corresponding L operator will also be indexed by n.

In the operator form the auto-Bäcklund transformation is defined by (cf. Ref. [13])

$$W_n L_n = L_{n+1} W_n. \tag{4}$$

One of the Bäcklund transformations [7,14] of (2) is nothing but the Darboux equation [3] (see for example Ref. [16])

$$\chi_{nxy} = \exp(\chi_{n+1} - \chi_n) - \exp(\chi_n - \chi_{n-1}), \tag{5}$$

where $u_n = \exp \chi_n$, $v_n = -\exp(-\chi_{n-1})$. The corresponding operator W_n is of the form

$$W_n = \begin{pmatrix} 0 & -\exp(-\chi_n) \\ \exp \chi_n & \partial_y - \partial_x + \chi_{nx} \end{pmatrix}. \tag{6}$$

One can consider (4), (6) as the Lax representation for (5) (cf. Ref. [15] where another representation has been given). The transformation (5) is explicit, because one can explicitly express $u_{n+1}, v_{n+1}, u_{n+2}, v_{n+2}, \ldots$ in terms of u_n, v_n and their derivatives.

An implicit auto-Bäcklund transformation of (2) corresponds to (cf. Ref. [17])

$$W_n = \begin{pmatrix} 1 & -v_{n+1} \\ u_n & \partial_y - \partial_x - w_n \end{pmatrix}. \tag{7}$$

It follows from (3), (4), (7) that

$$-u_{nx} = u_{n+1} + u_n w_n, \tag{8a}$$

$$v_{nx} = v_{n-1} + v_n w_{n-1}, \tag{8b}$$

$$w_{ny} = (u_n v_{n+1})_x = u_n v_n - u_{n+1} v_{n+1}. \tag{8c}$$

We would like to emphasize that a new nonlocal dynamical variable w_n has been introduced in (7), (8). The prolongation of transformation (8) on nonlocal variables p_n, q_n is of the form

$$p_n - p_{n+1} = w_{nx}, \qquad q_n - q_{n+1} = (u_n v_{n+1})_y. \tag{9}$$

The Bäcklund transformation (8) is a two-dimensional analog of

$$-u_{nx} = u_{n+1} + \lambda_n u_n + u_n^2 v_{n+1},$$
$$v_{nx} = v_{n-1} + \lambda_{n-1} v_n + v_n^2 u_{n-1}, \tag{10}$$

which is the Bäcklund transformation of the nonlinear Schrödinger equation (cf. Ref. [11]). Indeed, assuming $x = y$ one can integrate Eq. (8c) and express $w_n = u_n v_{n+1} + \lambda_n$, where λ_n is a constant of the integration. In the one-dimensional case w_n is the first nontrivial canonical conserved density (see Ref. [18]) of Eqs. (10). The well known Bäcklund transformation of NLS (see for example Refs. [19,20]),

$$(u_{n+1} + u_n)_x = (u_{n+1} - u_n) \times [\lambda_n^2 - (u_{n+1} + u_n)(v_{n+1} + v_n)]^{1/2},$$
$$(v_{n+1} + v_n)_x = (v_{n+1} - v_n) \times [\lambda_n^2 - (u_{n+1} + u_n)(v_{n+1} + v_n)]^{1/2}, \tag{11}$$

is nothing but a composition of two transformations. The first one is (10) and the second one can be obtained from (10) by an exchange $u_n \leftrightarrow v_n$. In (8) we

[2] In order to make the notations consistent we have to change $\partial_t + \partial_T \to \mathrm{i}\partial_T$, and assume that $\psi = u, \psi^* = v, \Phi = p + q$.

[3] After Ref. [15], where the integrability of Eq. (5) was shown, this equation is often called the two-dimensional Toda lattice. We consider that it would be more natural to call it after Darboux, which has discovered and used this equation at the beginning of the century.

do not have the "spectral" parameter λ_n. In the two-dimensional case it can be absorbed in the definition of w_n.

To our knowledge (8) is a new system of integrable two-dimensional lattice equations. It has infinitely many symmetries. Indeed, any member of the BRDS hierarchy is a symmetry of (8). The chain (8) defines the auto-Bäcklund transformation for both Eqs. (2) and any of their linear combinations. Eq. (8) is a generalization of the Darboux lattice (5). Indeed, let us make the following change of variables: $u_n = \epsilon^n \exp(\psi_n), v_n = \epsilon^{-n}\exp(-\varphi_{n-1})$. In the new variables Eqs. (8) are

$$-\psi_{nx} = \epsilon \exp(\psi_{n+1} - \psi_n) + w_n,$$
$$-\varphi_{nx} = \epsilon \exp(\varphi_n - \varphi_{n-1}) + w_n,$$
$$w_{ny} = \exp(\psi_n - \varphi_{n-1}) - \exp(\psi_{n+1} - \varphi_n). \qquad (12)$$

In the limit $\epsilon \to 0$ the above equations turn into (5).

We shall follow a modern interpretation of Miura transformations [12], namely, we shall treat them as invertible transformations in the extended set of dynamical variables generated by a sequence of auto-Bäcklund transformations. Let us consider the following transformation,

$$U_n = u_n, \quad V_n = v_{n+1}, \qquad (13)$$

of the extended set. This transformation is obviously invertible. In the new variables chain equations (8) take the form

$$-U_{nx} = U_{n+1} + U_n W_n, \quad V_{nx} = V_{n-1} + V_n W_n,$$
$$W_{ny} = (U_n V_n)_x = U_n V_{n-1} - U_{n+1} V_n. \qquad (14)$$

It follows from the above system of equations that variables u_n, v_n can be expressed in terms of U_n, V_n,

$$u_n = U_n, \quad v_n = V_{n-1} = V_{nx} - V_n W_n, \qquad (15)$$

where

$$W_{ny} = (U_n V_n)_x.$$

This defines a Miura transformation

$$u = U, \quad v = V_x - VW, \quad W_y = (UV)_x. \qquad (16)$$

To find the transformed equations let us rewrite the system of equations (2a) in the new variables U, V. It follows from (13) that

$$U_{nt} = U_{nxx} + 2p_n U_n, \qquad -V_{nt} = V_{nxx} + 2p_{n+1} V_n,$$

where

$$p_{ny} = (U_n V_{n-1})_x, \quad p_{n+1,y} = (U_{n+1} V_n)_x.$$

The variable V_{n-1} is already expressed in terms of U_n, V_n (15). It follows from (14) that

$$U_{n+1} = -(U_{nx} + U_n W_n). \qquad (17)$$

Thus the Miura transformation (16) of the system of equations (2a) is of the form

$$U_t = U_{xx} + 2AU, \qquad -V_t = V_{xx} + 2BV,$$
$$A_y = (UV_x - UVW)_x, \quad B_y = -(VU_x + UVW)_x,$$
$$W_y = (UV)_x. \qquad (18)$$

There are three "nonlocal" terms A, B, W in the above equation, but they are linearly dependent: $A_y - B_y = W_{xy}$. Therefore introducing $P = A + B$ we can rewrite Eqs. (18) in the form

$$U_t = U_{xx} + U(P + W_x),$$
$$-V_t = V_{xx} + V(P - W_x),$$
$$P_y = (UV_x - VU_x - 2UVW)_x,$$
$$W_y = (UV)_x. \qquad (19)$$

The Miura transformation (16) of the system of equations (2b) can be accomplished in a very similar way,

$$U_\tau = U_{yy} + U(Q + (UV)_y),$$
$$-V_\tau = V_{yy} + V(Q - (UV)_y),$$
$$Q_x = (UV_x - VU_x - 2UVW)_y,$$
$$W_y = (UV)_x. \qquad (20)$$

The BRDS equation is a two-dimensional integrable generalisation of the nonlinear Schrödinger equation (NLS), but Eqs. (19), (20) and any of their linear combinations are corresponding generalizations of one of three well known derivative nonlinear Schrödinger equations DNLS1 [21]. The Miura transformation

(16) is a two-dimensional generalization of the corresponding one-dimensional transformation [2]. It is known [12] that there are two Miura transformations that link NLS and DNLS1. To construct the second Miura transformation we express variables $u = u_{n+1}, v = v_{n+1}$ in terms of $U = U_n, V = V_n$. It follows from (13) $u_{n+1} = U_{n+1}, v_{n+1} = V_n$ and from (17) that

$$u = -(U_x + UW), \quad v = V. \tag{21}$$

This is the second Miura transformation linking Eqs. (2) and (19), (20).

The action of the Miura transformation can be prolonged on the nonlocal variables (cf. Ref. [7]). The first Miura transformation (16) gives

$$2p = P + W_x, \quad 2q = Q + (UV)_y; \tag{22}$$

the prolongation of the second transformation (21) is of the form

$$2p = P - W_x, \quad 2q = Q - (UV)_y. \tag{23}$$

We would like to emphasize that together with the Miura transformations (16), (21) we have the Bäcklund transformations of the modified equations (19), (20). The Bäcklund transformations are given in the form of the chain equations (14).

3. Two other examples

It was shown in Ref. [7] that another derivative nonlinear Schrödinger equation DNLS2 [22],

$$u_t = u_{xx} + 2uvu_x, \quad -v_t = -v_{xx} + 2uvv_x, \tag{24}$$

and the Kaup equation [23]

$$\begin{aligned} &u_t = u_{xx} + 2(u+v)u_x, \\ &-v_t = -v_{xx} + 2(u+v)v_x, \end{aligned} \tag{25}$$

also have integrable two-dimensional generalizations [7]. In this section we shall give implicit auto-Bäcklund transformation of these two-dimensional equations. The Bäcklund transformations themselves are new integrable two-dimensional systems of chain equations. As above, we shall construct Miura transformations and the corresponding modified equations.

Let us start with the DNLS2 equation. It has two different integrable two-dimensional generalizations [7],

$$\begin{aligned} &u_t = u_{xx} + 2pu_x, \quad -v_t = v_{xx} - 2pv_x, \\ &p_y = (uv)_x, \end{aligned} \tag{26a}$$

$$\begin{aligned} &u_\tau = u_{yy} + u((uv)_y + q), \quad -v_\tau = v_{yy} - v((uv)_y - q), \\ &q_x = (vu_x - uv_x)_y. \end{aligned} \tag{26b}$$

We have checked that they are compatible and that any linear combination of these flows is integrable. An explicit auto-Bäcklund transformation for Eqs. (26) has been found in Ref. [7]. The implicit auto-Bäcklund transformation of the DNLS2 Eq. (24) is of the form [11,24]

$$\begin{aligned} &u_{nx} = w_n(u_{n+1} - u_n), \quad v_{nx} = w_{n-1}(v_n - v_{n-1}), \\ &w_n = u_n v_{n+1} + \lambda_n. \end{aligned} \tag{27}$$

The first canonical density of (27) is $w_n = u_n v_{n+1} + \lambda_n$. Now, comparing (27) with (10) and (8), it is easy to guess that the two-dimensional generalisation of (27) is of the form

$$\begin{aligned} &u_{nx} = w_n(u_{n+1} - u_n), \quad v_{nx} = w_{n-1}(v_n - v_{n-1}), \\ &w_{ny} = (u_n v_{n+1})_x. \end{aligned} \tag{28}$$

A routine check that (28) represents a true Bäcklund transformation of (26) was quite tedious and we used a computer with a symbolic software (Mathematica) to accomplish this verification. System (28) is a new integrable two-dimensional chain equation.

Like the case of the BRDS equation, the invertible change of variables (13) leads to a Miura transformation

$$u = U, \quad v = V - V_x/W, \tag{29}$$

where $W_y = (UV)_x$. This Miura transformation links (26) and the corresponding modified equations,

$$\begin{aligned} &U_t = U_{xx} + U_x(2W - (\log W)_x + P), \\ &-V_t = V_{xx} - V_x(2W + (\log W)_x + P), \\ &P_y = [(VU_x - UV_x)/W]_x, \quad W_y = (UV)_x, \\ &U_\tau = U_{yy} + U((UV)_y + Q), \end{aligned} \tag{30}$$

$$-V_\tau = V_{yy} - V((UV)_y - Q),$$
$$Q_x = (VU_x - UV_x - 2U_xV_x/W)_y,$$
$$W_y = (UV)_x. \tag{31}$$

The second Miura transformation which links (26) and (30), (31) is of the form

$$u = U + U_x/W, \quad v = V. \tag{32}$$

Another example is a two-dimensional generalization of the Kaup equation. There are two integrable generalizations of this equation (see for example Ref. [7], in a slightly different form the first equation has been previously found in Ref. [6], the second one in Ref. [25])

$$u_t = u_{xx} + 2pu_x, \quad -v_t = v_{xx} - 2pv_x,$$
$$p_y = (u+v)_x, \tag{33a}$$

$$u_\tau = u_{yy} + (u^2 + uv)_y + q,$$
$$-v_\tau = v_{yy} - (v^2 + uv)_y + q,$$
$$q_x = (vu_x - uv_x)_y. \tag{33b}$$

We found the following implicit auto-Bäcklund transformation of (33),

$$u_{nx} = w_n(u_{n+1} - u_n), \quad v_{nx} = w_{n-1}(v_n - v_{n-1}),$$
$$w_{ny} = (u_n + v_{n+1})_x. \tag{34}$$

Again, it is a simple generalisation of the known one-dimensional auto-Bäcklund transformation of the Kaup equation (25) [11,24] (in the one-dimensional case $w_n = u_n + v_{n+1}$ is the first canonical density). By direct computation we have verified the fact that (34) defines a Bäcklund transformation of (33).

The invertible transformation (13) gives the following Miura transformation,

$$u = U, \quad v = V - V_x/W, \tag{35}$$

which links Eqs. (33) with

$$U_t = U_{xx} + U_x(2W - (\log W)_x + P),$$
$$-V_t = V_{xx} - V_x(2W + (\log W)_x + P),$$
$$P_y = [(U_x - V_x)/W]_x, \quad W_y = (U+V)_x, \tag{36}$$

$$U_\tau = U_{yy} + (U^2 + UV)_y + Q,$$
$$-V_\tau = V_{yy} - (V^2 + UV)_y + Q,$$
$$Q_x = (VU_x - UV_x - 2U_xV_x/W)_y,$$
$$W_y = (U+V)_x. \tag{37}$$

The second Miura transformation that links (33) with (36), (37) is of the form

$$u = U + U_x/W, \quad v = V. \tag{38}$$

4. Conclusions

We have found implicit auto-Bäcklund transformations (8), (28), (34) of the Benney–Roskes equation (2), two-dimensional generalizations of the DNLS2 (26) and Kaup (33) equations respectively. The explicit auto-Bäcklund transformations of these equations which were recently found in Ref. [7] can be obtained as degenerations of our implicit ones. Eqs. (8), (28), (34) themselves are new integrable two-dimensional lattice equations. Using these implicit auto-Bäcklund transformations we have constructed Miura transformations (16), (21), (29), (32), (35), (38) and the corresponding modified Eqs. (19), (20), (30), (31), (36), (37). The latter equations are new integrable examples of two-dimensional generalizations of NLS-type equations. In the one-dimensional case there are three well known forms of DNLS, namely the equation found by Kaup and Newell [26], the equation of Chen, Lee and Liu [22] and the equation of Ablowitz and Segur [21]. Up to so-called symmetrical transformations all of these equations are equivalent [2]. A two-dimensional generalization (26) of the Chen–Lee–Liu equation (24) (DNLS2) has been recently found in Ref. [7], generalizations of the Ablowitz–Segur equation (DNLS1) are (19), (20). We have found a two-dimensional generalization of the third DNLS equation (Kaup–Newell equation) in a way a little bit different which we will publish in our next paper. Existence of a two-dimensional analog of symmetrical transformations is still an open problem.

Acknowledgement

One of the authors (R.Y.) is pleased to acknowledge the NATO–Royal Society Fellowship Programme for their financial support and the University of Leeds for the hospitality. We would like to thank A. Degasperis and A.C. Newell for useful discussions and providing us with Refs. [4,19,20].

References

[1] A.V. Mikhailov, A.B. Shabat and R.I. Yamilov, Usp. Mat. Nauk 42 (1987) 3 [Russ. Math. Surv. 42 (1987) 1].

[2] A.V. Mikhailov, A.B. Shabat and R.I. Yamilov, Commun. Math. Phys. 115 (1988) 1.

[3] A.V. Mikhailov, A.B. Shabat and V.V. Sokolov, in: What is integrability?, ed. V.E. Zakharov (Springer, Berlin, 1991) pp. 115-184.

[4] D.J. Benney and G.J. Roskes, Stud. Appl. Math. 48 (1969) 337.

[5] A. Davey and K. Stewartson, Proc. R. Soc. A 338 (1974) 101.

[6] B.G. Konopelchenko, Phys. Lett. A 156 (1991) 221.

[7] A.B. Shabat and R.I. Yamilov, Phys. Lett. A 227 (1997) 15.

[8] V.E. Zakharov, S.V. Manakov, S.P. Novokov and L.P. Pitaevski, Theory of solitons. The inverse problem method (Nauka, Moscow, 1980) [in Russian; English transl. (Plenum, New York, 1984)].

[9] V.E. Zakharov, in: Lecture Notes in Physics, Vol. 153. Integrable systems in multi-dimensional spaces (Springer, Berlin, 1983) pp. 190-216.

[10] R. Benguria and D. Levi, Proc. Natl. Acad. Sci. USA 77 (1980) 5025.

[11] A.B. Shabat and R.I. Yamilov, Alg. Anal. 2 (1990) 183 [Leningrad Math. J. 2 (1991) 377].

[12] R.I. Yamilov, Teor. Mat. Fiz. 85 (1990) 368 [Theor. Math. Phys. 85 (1991) 1269].

[13] V.E. Zakharov and A.B. Shabat, Funkt. Anal. Prilozh. 13 (1979) 13 [Funct. Anal. Appl. 13 (1980) 166].

[14] A.N. Leznov, A.B. Shabat and R.I. Yamilov, Phys. Lett. A 174 (1993) 397.

[15] A.V. Mikhailov, Pis'ma Zh. Eksp. Teor. Fiz. 30 (1979) 443 [JETP Lett. 30 (1979) 414].

[16] G. Darboux, Théorie générale des surfaces, II (Chelsea, New York, 1972).

[17] M. Boiti, B.G. Konopelchenko and F. Pempinelli, Inv. Probl. 1 (1985) 33.

[18] D. Levi and R. Yamilov, Conditions for the existence of higher symmetries of evolutionary equations on the lattice, submitted to J. Math. Phys. (1997).

[19] H.H. Chen, Phys. Rev. Lett. 33 (1974) 925.

[20] K. Konno and M. Wadati, Prog. Theor. Phys. 53 (1975) 1652.

[21] M.J. Ablowitz and H. Segur, Solitons and the inverse scattering transform (SIAM, Philadelphia, 1981).

[22] H.H. Chen, Y.C. Lee and C.S. Liu, Phys. Scr. 20 (1979) 490.

[23] D.J. Kaup, Progr. Theor. Phys. 54 (1975) 72.

[24] V.E. Adler and R.I. Yamilov, J. Phys. A 27 (1994) 477.

[25] M. Boiti, J.J.P. Leon and F. Pempinelli, Inv. Probl. 3 (1980) 371.

[26] D.J. Kaup and A.C. Newell, J. Math. Phys. 19 (1978) 898.