Conditions for the existence of higher symmetries of evolutionary equations on the lattice

D. Levi[a)]
Dipartimento di Fisica—Universitá di Roma 3
and INFN-Sezione di Roma, Via della Vasca Navale 84, 00146 Roma, Italy

R. Yamilov[b)]
Department of Applied Mathematical Studies, University of Leeds,
Leeds LS2 9JT, United Kingdom

(Received 24 March 1997; accepted for publication 10 June 1997)

In this paper we construct a set of five conditions necessary for the existence of generalized symmetries for a class of differential-difference equations depending only on nearest neighboring interaction. These conditions are applied to prove the existence of new integrable equations belonging to this class. © *1997 American Institute of Physics.* [S0022-2488(97)02411-0]

I. INTRODUCTION

Nonlinear differential-difference equations are always more important in applications. They enter as models for many biological chains, are encountered frequently in queuing problems and as discretizations of field theories. So, both as themselves and as approximations of continuous problems, they play a very important role in many fields of mathematics, physics, biology, and engineering.

Not many tools are available to solve such kinds of problems. Apart from a few exceptional cases the solution of nonlinear differential-difference equations can be obtained only by numerical calculations or by going to the continuous limit when the lattice spacing vanishes and the system is approximated by a continuous nonlinear partial differential equation. Exceptional cases are those equations that, in one way or another, are either linearizable or integrable via the solution of an associated spectral problem on the lattice. In such cases we can write down a denumerable set of exact solutions corresponding to symmetries of the nonlinear differential-difference equations. Such symmetries can be either, depending just on the dependent field and independent variable, and are denoted as point symmetries, or can depend on the dependent field in various positions of the lattice, and in this case we speak of generalized symmetries. Any differential-difference equation can have point symmetries, but the existence of generalized symmetries is usually associated only to the integrable ones.

Few classes of integrable nonlinear differential-difference equations are known[1–5] and are important for all kind a of applications, both as themselves and as a starting point for perturbation analysis.[6] However, not all cases of physical interest are covered, and so it would be nice to be able to recognize if a given nonlinear differential-difference equation is integrable or not, so it can be used as a model of nonlinear systems on the lattice or as a starting point of perturbation theory. A way to accomplish such a goal can be obtained using the so-called formal symmetry approach introduced by A. Shabat and collaborators in Ufa (see e.g., review articles[7–9]) by which the authors classified all equations of a certain class that possess few generalized symmetries of a certain kind. Such an approach has been introduced at first to classify partial differential equations, but then the procedure has been extended to the case of differential-difference equations.[2–4] In such an approach, one introduces conditions under which one can prove the existence of at least

[a)]Electronic mail: levi@roma1.infn.it
[b)]On leave from: Ufa Institute of Mathematics, Russian Academy of Sciences, 112 Chernyshevsky Street, Ufa 450000, Russia. Electronic mail: yamilov@imat.rb.ru

0022-2488/97/38(12)/6648/27/$10.00

one (or more) generalized symmetries. These conditions are basic tools to start the procedure of classification, i.e. to look for the form of the nonlinear differential difference equations, which is compatible with these conditions. This process gives rise to classes of equations. These conditions can be used as they are; for examples, they have been used in the program DELIA[10] to discover if an evolutionary scalar equation is integrable.

The class of nonlinear differential-difference equations we will consider in the following is given by

$$u_{n,t}(t)=f_n(u_{n-1}(t),u_n(t),u_{n+1}(t)), \tag{1.1}$$

where $u_n(t)$ is a complex-dependent field expressed in terms of its dependent variables, t varying over the complex numbers while n is varying over the integers. Equation (1.1) is a differential functional relation that correlates the ''time'' evolution of a function calculated at the point n to its values in its nearest neighboring points $(n+1,\ n-1)$. A peculiarity of the choice of Eq. (1.1) is the fact that the right-hand side of it not just a function, i.e. it is not the same for all points in the lattice but for each point of the lattice one has an *a priori* different right-hand side. In fact, we can think of Eq. (1.1) as an infinite system of different differential equations for the infinite number of functions u_n. By proper choices of the functions f_n, Eq. (1.1) can be reduced to a system of k coupled differential difference equations for the k unknown u_m^k or to a system of dynamical equations on the lattice. In fact, for example, by imposing periodicity conditions on the dependent field in the lattice variables one is able to rewrite Eq. (1.1) as a coupled system of nonlinear differential difference equations. Let us assume that f_n and u_n are periodic functions of n of period k, i.e.

$$f_n(u_{n-1}(t),u_n(t),u_{n+1}(t))=\sum_{j=0}^{k-1} P_{n-j}^k\, f^j(u_{m-1}(t),u_m(t),u_{m+1}(t)),$$

$$u_n=\sum_{j=0}^{k-1} P_{n-j}^k u_m^j,$$

where we have defined the projection operator P_n^k such that for any integer m such that $n=km+j$ with $0\leqslant j\leqslant k-1$, we have

$$P_{km}^k=1, \quad P_{km+j}^k=0 \quad (j=1,2,...,k-1), \tag{1.2}$$

then Eq. (1.1) becomes the system:

$$\begin{aligned} u_{m,t}^0&=f^{\,0}(u_{m-1}^{k-1}(t),u_m^0(t),u_m^t(t)),\\ u_{m,t}^1&=f^1(u_m^0(t),u_m^1(t),u_m^2(t)),\\ &\cdots\quad\cdots\\ u_{m,t}^{k-1}&=f^{k-1}(u_m^{k-2}(t),u_m^{k-1}(t),u_{m+1}^0(t)). \end{aligned} \tag{1.3}$$

Of particular interest is the case of periodicity $k=2$, when we have

$$\begin{aligned} u_{m,t}^0&=f^{\,0}(u_{m-1}^1,u_m^0,u_m^1),\\ u_{m,t}^1&=f^1(u_m^0,u_m^1,u_{m+1}^0). \end{aligned} \tag{1.4}$$

A subclass of Eq. (1.4), of particular relevance for its physical applications, is given by dynamical systems on the lattice, i.e. equations of the type

$$\chi_{n,tt}=g(\chi_{n+1}-\chi_n,\chi_n-\chi_{n-1}). \tag{1.5}$$

Equation (1.5) is obtained from Eq. (1.4) for $u_n^0=\chi_{n,t}$, $u_n^1=\chi_{n+1}-\chi_n$ by choosing $f^0=g(u_n^1,u_{n-1}^1)$ and $f^1=u_n^0-u_{n+1}^0$. Then, by choosing

$$g(z,z')=e^z-e^{z'},$$

Eq. (1.1) reduces to the Toda lattice equation,

$$\chi_{n,tt}=e^{\chi_{n+1}-\chi_n}-e^{\chi_n-\chi_{n-1}}. \tag{1.6}$$

In terms of the projection operator (1.2), Eq. (1.6) can obviously also be written in polynomial form as

$$u_{n,t}=(P_{n+1}^2u_n+P_n^2)(u_{n+1}-u_{n-1}), \tag{1.7}$$

the polynomial Toda Lattice.

In the present paper, the general theory of the symmetry approach in the differential-difference case is discussed in detail for the first time and an explicit dependence on n is introduced. In the previous literature, in the framework of the formal symmetry approach, only n-independent differential difference equations were considered; the following classes of equations were completely classified:

$$u_{n,t}=f(u_{n-1},u_n,u_{n+1}) \tag{1.8}$$

(Volterra-type equations, see Ref. 2) and

$$u_{n,tt}=f(u_{n,t},u_{n-1},u_n,u_{n+1}) \tag{1.9}$$

(Toda-type equations; see Ref. 3). Reference 2 is a one page paper in Russian in which only the classification theorem is formulated with a few examples. A detailed version of Ref. 2 can be found only in the unpublished work.[11] It should be remarked that the classification of chains (1.8) is also briefly discussed in Ref. 7. Theoretically, in our class, we can consider chains that can be expressed as systems of 2,3,4,..., n-independent equations, and chains that are systems of an infinite number of different equations. In fact, if in the case of the class of equations (1.8) an equation is defined by a function f, in the case of (1.1) we have an infinite set $\{f_n\}$ of *a priori* quite different functions. So, this paper is a further step in the development of the general theory of the formal symmetry approach (readers can find elements of a previous version of the general theory in Refs. 3 and 7).

Section II is devoted to the construction of a certain number of conditions (the simpler ones) necessary to prove that an equation of the class (1.1) has generalized symmetries and higher-order conservation laws. Section III is devoted to a discussion of the results presented in Sec. II especially in connection with the reductions (1.4) and (1.5). The obtained conditions are applied in Sec. IV to a few examples of interest. In particular, we will study three classes of systems of nonlinear differential equations on the lattice:

(1)
$$u_{n,t}=\beta(u_n)(u_{n+1}-u_{n-1}); \tag{1.10}$$

this class of equations includes the well-known Volterra equation;

(2)
$$u_{n,t}=P_{n+1}^2e^{u_{n+1}}g_n(u_{n+1}-u_{n-1})+P_n^2\lambda_n(u_{n+1}-u_{n-1}). \tag{1.11}$$

This equation describes a class of dynamical equations,

$$v_{ktt}=\exp\left(\frac{v_{k+1}-v_k}{\epsilon_{k+1}}\right)G_k\left(\frac{v_{k+1}-v_k}{\epsilon_{k+1}}-\frac{v_k-v_{k-1}}{\epsilon_k}\right), \tag{1.12}$$

where

$$u_{2k}=\frac{v_{k+1}-v_k}{\epsilon_{k+1}},\quad u_{2k-1}=v_{kt},\quad g_{2k-1}=G_k,\quad \lambda_{2k}=\epsilon_{k+1}^{-1}, \tag{1.13}$$

having a four-dimensional group of point symmetries and including the Toda lattice as one of its members.[12] Here P_n^2 is the projection operator of period 2, as introduced in Eq. (1.2), g_n is an arbitrary analytic function of its argument and λ_n are arbitrary n-dependent constants.

(3)
$$u_{n,t}=\varphi_n(u_{n+1}-u_{n-1}). \tag{1.14}$$

By setting

$$\varphi_{2n}(z)=b_n\,z,\quad b_n\neq 0,\quad \varphi''_{2n-1}(z)=F_n(z)\neq 0,\quad \forall n, \tag{1.15}$$

and rewriting $u_{2n}=w_n$, $u_{2n-1}=v_n$, one gets from (1.14) chains of the form

$$M_n\omega_{n,t}=v_{n+1}-v_n,\quad v_{n,t}=F_n(\omega_n-\omega_{n-1}), \tag{1.16}$$

which correspond to a dynamical system of the following form:

$$M_n\omega_{n,tt}=F_{n+1}(\omega_{n+1}-\omega_n)-F_n(\omega_n-\omega_{n-1}). \tag{1.17}$$

If we set

$$F_n(z)=B_nz^2+C_nz, \tag{1.18}$$

and define $y_n=\omega_{2n}$ and $x_n=\omega_{2n-1}$, Eqs (1.17), (1.18) reduce, by an appropriate choice of the constants B_n, C_n, and M_n to the system

$$M_2y''_n=f(x_{n+1}-y_n)-g(y_n-x_n),\quad M_1x''_n=g(y_n-x_n)-f(x_n-y_{i-n}), \tag{1.19}$$

with

$$f(z)=\epsilon\beta_2z^2+k_2z,\quad g(z)=\epsilon\beta_1z^2+k_1z,$$

which describes the evolution of diatomic chains[13] and explicitly reads as

$$\begin{aligned}M_1x_{n,tt}&=k_1(y_n-x_n)-k_2(x_n-y_{n-1})+\epsilon[\beta_1(y_n-x_n)^2-\beta_2(x_n-y_{n-1})^2],\\ M_2y_{n,tt}&=k_2(x_{n+1}-y_n)-k_1(y_n-x_n)+\epsilon[\beta_2(x_{n+1}-y_n)^2-\beta_1(y_n-x_n)^2].\end{aligned} \tag{20}$$

As a last example, at the end of Sec. IV, we will use the obtained conditions to study the integrability of an n-dependent generalization of a discrete analog of the Krichever–Novikov equation:

$$u_{n,t}=\frac{p(u_n)u_{n+1}u_{n-1}+q(u_n)(u_{n+1}+u_{n-1})+r(u_n)}{u_{n+1}-u_{n-1}}, \tag{1.21}$$

where

$$p(u_n) = \alpha u_n^2 + 2\beta u_n + \gamma, \tag{1.22a}$$

$$r(u_n) = \gamma u_n^2 + 2\delta u_n + \omega, \tag{1.22b}$$

$$q(u_n) = \beta u_n^2 + \lambda u_n + \delta. \tag{1.22c}$$

Equation (1.21) depends on six arbitrary complex constants and is invariant under linear–fractional transformations, as under those transformations only the coefficients of the polynomials p, q, r are changed, but not the polynomials themselves. Equation (1.21) was obtained for the first time in Ref. 2 when classifying discrete evolutionary equations of the form (1.8). It satisfies all the five integrability conditions, has an infinite set of higher local conservation laws and should have an infinite set of generalized symmetries (but nobody has yet proved it). It is the only example of a nonlinear chain of the form (1.8), up to now obtained, which cannot be reduced to the Toda or Volterra equations by Miura transformations. By carrying out the continuous limit, in the same way as one does to obtain the Korteweg–de Vries equation from the Volterra equation, we get the Krichever–Novikov equation:[14]

$$u_t = u_{xxx} - \frac{3}{2}\frac{u_{xx}^2}{u_x} + \frac{R(u)}{u_x}, \tag{1.23}$$

where $R(u)$ is an arbitrary fourth degree polynomial of its argument with constant coefficients.

The complete classification of all the classes of equations of the form (1.1) that satisfy the conditions obtained in Sec. II is left to a future work. Few conclusive remarks are contained in Sec. V.

II. CONSTRUCTION OF THE CLASSIFYING CONDITIONS

If Eq. (1.1) is to represent an evolutionary difference equation, then the function f_n must depend in an essential way from the points $(n \pm 1)$, the nearest neighboring points with respect to the point n in which we compute the ''time'' evolution. This implies that we must add to Eq. (1.1) the condition

$$\frac{\partial f_n}{\partial u_{n+1}} \neq 0, \quad \frac{\partial f_n}{\partial u_{n-1}} \neq 0, \qquad \text{for any } n. \tag{2.1}$$

Before considering in detail the problem of costructing generalized symmetries to Eq. (1.1), we will introduce few definitions necessary for the future calculations.

A function g_n depending on the set of fields u_n, for n varying on the lattice, will be called a rectricted function and will be denoted by the symbol RF if it is defined on a compact support, i.e. if

$$g_n = g_n(u_{n+i}, u_{n+i-1}, \ldots, u_{n+j+1}, u_{n+j}), \quad i \geqslant j, \tag{2.2}$$

and i and j are finite integer numbers. If there exist, in the range of the possible values of n, values k and m such that

$$\frac{\partial g_k}{\partial u_{k+i}} \neq 0, \quad \frac{\partial g_m}{\partial u_{m+j}} \neq 0, \tag{2.3}$$

then we say that the function g_n has a length $i-j+1$. For example, g_n could be given by the function

$$g_n = n u_{n+1} + u_n + [1 + (-1)^n] u_{n-1};$$

then $i=1$, $j=-1$ and the length of g_n is equal to 3 even if only the even functions are depending on u_{n-1}.

Let us define the shift operator D such that

$$Dg_n(u_{n+i},\ldots,u_{n+j})=g_{n+1}(u_{n+i+1},\ldots,u_{n+j+1}).$$

Then we can split the RF into equivalent classes.

Definition: Two RF,

$$a_n(u_{n+i_a},\ldots,u_{n+j_a}) \quad \text{and} \quad b_n(u_{n+i_b},\ldots,u_{n+j_b})$$

are said to be *equivalent*,

$$a_n \sim b_n,$$

iff

$$a_n-b_n=(D-1)c_n, \tag{2.4}$$

where c_n is a RF.

If, for example, we have $a_n=u_n+u_{n+1}$, it is immediate to see that a_n is equivalent to a function $b_n=2u_n$, as $a_n-b_n=u_{n+1}-u_n=(D-1)u_n$.

Let us notice that any function that is equal to a total difference is equivalent to zero, i.e. $a_n=(D-1)c_n\sim 0$. If a RF a_n of length $i-j+1(i>j)$ is equivalent to zero, then there will exist, by necessity, a RF c_n of the length $i-j$ such that $a_n=(D-1)c_n$. As

$$a_n(u_{n+i},\ldots,u_{n+j})=c_{n+1}(u_{n+i},\ldots,u_{n+j+1})-c_n(u_{n+i-1},\ldots,u_{n+j}),$$

one can easily see that

$$\frac{\partial a_n}{\partial u_{n+i}}=\frac{\partial c_{n+1}(u_{n+i},\ldots,u_{n+j+1})}{\partial u_{n+i}},$$

and consequently,

$$\frac{\partial^2 a_n}{\partial u_{n+i}\,\partial u_{n+j}}=0. \tag{2.5}$$

In the case $i=j$,

$$a_n(u_{n+i})=c_{n+1}(u_{n+k_1+1},\ldots,u_{n+k_2+1})-c_n(u_{n+k_1},\ldots,u_{n+k_2}).$$

As for $k_2<i\,\partial a_n/\partial u_{n+k_2}=-\partial c_n/\partial u_{n+k_2}=0$, then $c_n=d_n(u_{n+k_1},\ldots,u_{n+i})$. For $k_1\geqslant i$ also $\partial a_n/\partial u_{n+k_1+1}=0$, then d_n cannot depend on u_{n+k} for any k, and consequently,

$$\frac{da_n}{du_{n+i}}=0, \tag{2.6}$$

i.e., a_n is an *invariant* function, where by it we mean a function that depends only on n.

We can moreover define the *''formal'' variational derivative* of a RF a_n of length $i-j+1$ as

$$\frac{\partial a_n}{\partial u_n}=\sum_{k=n-i}^{n-j}\frac{\partial a_k}{\partial u_n}. \tag{2.7}$$

If a_n is linear in u_n, then $\partial a_n/\partial u_n$ is an invariant function, but if it is nonlinear, then $\partial a_n/\partial u_n = \tilde{g}_n(u_{n+N},\ldots,u_{n-N})$, where for some k $\partial \tilde{g}_k/\partial u_{k+N} \neq 0$ and for some m $\partial \tilde{g}_m/\partial u_{m-N} \neq 0$. Consequently, this quantity is strictly related to the notion of a variational derivative, and this is the reason for its name. It is immediate to prove that if a_n is a RF equivalent to zero, then the formal variational derivative of a_n is zero. The vice versa is also true, i.e., if $\partial a_n/\partial u_n = 0$, then a_n is equivalent to zero. In fact, using Eq. (2.7) we have $\partial^2 a_n/\partial u_{n+i}\, \partial u_{n+j} = 0$, which implies that $a_n = b_n(u_{n+i},\ldots,u_{n+j+1}) + c_n(u_{n+i-1},\ldots,u_{n+j}) \sim d_n(u_{n+i-1},\ldots,u_{n+j})$, i.e., a_n is equivalent to a RF of length $i-j$. Carrying out recursively this reasoning, we arrive at the conclusion that $a_n \sim F_n(u_n)$ with $F_n' = 0$, i.e. a_n must be an invariant function, i.e., equivalent to zero.

Given a nonlinear chain (1.1), we will say that the RF $g_n(u_{n+i},\ldots,u_{n+j})$ is a generalized (or higher) local symmetry of *order* i (more precisely, of left order i) of our equation iff

$$u_{n,\tau} = g_n(u_{n+i},\ldots,u_{n+j}), \tag{2.8}$$

is compatible with (1.1), i.e. iff

$$\partial_t \partial_\tau (u_n) = \partial_\tau \partial_t (u_n). \tag{2.9}$$

Explicitating condition (2.9), we get

$$\partial_t g_n = \partial_\tau f_n = \frac{\partial f_n}{\partial u_{n+1}}\, u_{n+1,\tau} + \frac{\partial f_n}{\partial u_n}\, u_{n,\tau} + \frac{\partial f_n}{\partial u_{n-1}}\, u_{n-1},$$

$$\tau = \left[\frac{\partial f_n}{\partial u_{n+1}}\, D + \frac{\partial f_n}{\partial u_n} + \frac{\partial f_n}{\partial u_{n-1}}\, D^{-1} \right] g_n = f_n^* g_n,$$

i.e.,

$$(\partial_t - f_n^*) g_n = 0, \tag{2.10}$$

where by f_n^* we mean the Frechet derivative of the function f_n, given by

$$f_n^* = \frac{\partial f_n}{\partial u_{n+1}}\, D + \frac{\partial f_n}{\partial u_n} + \frac{\partial f_n}{\partial u_{n-1}}\, D^{-1} = f_n^{(1)} D + f_n^{(0)} + f_n^{(-1)} D^{-1}. \tag{2.11}$$

Equation (2.10) is an equation for g_n once the function f_n is given, an equation for the symmetries. In this work we limit ourselves to local symmetries, i.e. symmetries that are given by RF. A nonlocal extension can be carried out by introducing, for example, a new field v_n: $(D-1)v_n = u_n$, i.e. $v_n = -\Sigma_{j=n}^{\infty} u_j$ or $v_n = \Sigma_{j=-\infty}^{n-1} u_j$ (compare Ref. 7). Extension in such a direction will be carried out in future work.

Given a symmetry we can construct a new symmetry by applying a recursive operator, i.e. an operator that transforms symmetries into symmetries. Given a symmetry g_n of Eq. (1.11), an operator

$$L_n = \sum_{j=-\infty}^{m} l_n^{(j)}(t) D^j, \tag{2.12}$$

will be a recursive operator for Eq. (1.1) if $\tilde{g}_n$, given by

$$\tilde{g}_n = L_n g_n, \tag{2.13}$$

is a new generalized symmetry associated to (1.1). Equation (2.10) and Equation (2.13) imply that

$$A(L_n)=L_{n,t}-[f_n^*,L_n]=0. \tag{2.14}$$

Moreover, from (2.10) it follows that

$$A(g_n^*)=\partial_\tau(f_n^*). \tag{2.15}$$

In fact, from (2.10) we get

$$B_n=\partial_t(g_n)=\sum_k \frac{\partial g_n}{\partial u_{n+k}} f_{n+k},$$

and consequently we have

$$B_n^*=\sum_m \frac{\partial B_n}{\partial u_{n+m}} D^m=\sum_{m,k} \frac{\partial^2 g_n}{\partial u_{n+k}\,\partial u_{n+m}} f_{n+k} D^m+\sum_{m,k} \frac{\partial g_n}{\partial u_{n+k}} \frac{\partial f_{n+k}}{\partial u_{n+m}} D^m=\partial_t(g_n^*)+g_n^* f_n^*. \tag{2.16}$$

Equation (2.15) is then obtained by introducing (2.16) into the Frechet derivative of (2.9). Equation (2.15) implies that, as its right-hand side (rhs) is an operator of the order 1 [see (2.11)], the highest terms on the left-hand side (lhs) must be zero.

We can define as *approximate* symmetry of *order* i and *length* m, the operator

$$G_n=\sum_{k=i-m+1}^{i} g_n^{(k)} D^k,$$

such that the highest m terms of the operator,

$$A(G_n)=\sum_{k=i-m}^{i+1} a_n^{(k)} D^k,$$

are zeros. Taking into account Eq. (2.15), we find that we must have $i-m+2>1$ if the equation

$$A(G_n)=0 \tag{2.17}$$

is to be satisfied.

From these results we can derive the first integrability condition, which can be stated in the following theorem, whose proof is contained in Appendix A.

Theorem 1: If Eq. (1.1) has a local generalized symmetry of order $i\geqslant 2$, then it must have a conservation law given by

$$\partial_n \log f_n^{(1)}=(D-1)q_n^{(1)}, \tag{2.18}$$

where $q_n^{(1)}$ is a RF.

In this way we have shown the existence of the first *canonical* conservation law. The next canonical conservation laws could be obtained in the same way, by assuming the existence of a higher symmetry, so that we are allowed to consider an approximate symmetry of higher length. These canonical conservation laws would, however, be very complicated (they will depend on the order of the generalized symmetry) and very difficult to reduce to simple expressions not depending on its order. So we prefer to follow an alternative approach that requires the existence of two higher symmetries. This procedure can be carried out, as we already know one canonical conservation law.

Let us now assume that there exist two RF g_n and $\widetilde{g}_n$ that generate symmetries of order i and $i+1$, respectively. In correspondence to these symmetries, we can construct two approximate symmetries G_n and $\widetilde{G}_n$ of orders i and $i+1$, respectively, and from (2.1), $g_{\underset{\sim}{n}}^{(i)}$ and $\widetilde{g}_n^{(i+1)}$ will be different from zero for all n [see (A3), Appendix A]. Starting from G_n and $\widetilde{G}_n$, we can construct the operator

$$\hat{G}_n = G_n^{-1} \widetilde{G}_n . \tag{2.19}$$

As from (2.14) we have

$$A(G_n^{-1}) = -G_n^{-1} A(G_n) G_n^{-1}, \quad A(L_n K_n) = A(L_n) K_n + L_n A(K_n),$$

we obtain

$$A(\hat{G}_n) = G_n^{-1} [-A(G_n)\hat{G}_n + A(\widetilde{G}_n)]. \tag{2.20}$$

Let us notice that, as G_n is an approximate symmetry, its inverse will be an operator with an infinite number of terms. Consequently $\hat{G}_n$, though it is an approximate symmetry of order 1 and length i (the lowest of the lengths of G_n and $\widetilde{G}_n$) is represented by an infinite sum. This shows that under the hypothesis that two local higher symmetries exist, we can restrict ourselves to consider approximate symmetries of order 1. In such a way $g_n^{(1)} = f_n^{(1)}$, and for $q_n^{(1)}$ the following simple formula can be obtained: $q_n^{(1)} = g_n^{(0)} - f_n^{(0)}$. We can now state the following theorem, proved in Appendix B.

Theorem 2: If Eq. (1.1) satisfies conditions (2.1) and it has two generalized local symmetries of order i and $i+1$, with $i \geqslant 4$, then the following conservation laws must be true:

$$\partial_t p_n^{(k)} = (D-1) q_n^{(k)} \quad (k=1,\ 2,\ 3),$$

$$p_n^{(1)} = \log \frac{\partial f_n}{\partial u_{n+1}}, \quad p_n^{(2)} = q_n^{(1)} + \frac{\partial f_n}{\partial u_n}, \tag{2.21}$$

$$p_n^{(3)} = q_n^{(2)} + \frac{1}{2} (p_n^{(2)})^2 + \frac{\partial f_n}{\partial u_{n+1}} \frac{\partial f_{n+1}}{\partial u_n},$$

where $q_n^{(k)}$ $(k=1,\ 2,\ 3)$ are some RFs.

So, if Eq. (1.1) has local generalized symmetries of high enough order, we can construct a few conservation laws depending on the function at the rhs of Eq. (1.1).

One can divide the conservation laws into conjugacy classes under an equivalence condition. Two conservation laws,

$$p_{n,t} = (D-1) q_n, \quad r_{n,t} = (D-1) s_n,$$

are *equivalent* if

$$p_n \sim r_n . \tag{2.22}$$

A local conservation law is *trivial* if $p_n \sim 0$. If $p_n \sim r_n(u_n)$, with $r_n' \neq 0$ at least for some n, then we have a conservation law of *zeroth order*, while if

$$p_n \sim r_n(u_{n+N}, \ldots, u_n), \quad N > 0,$$

and

$$\frac{\partial^2 r_n}{\partial u_n\ \partial u_{n+N}} \neq 0,$$

for at least some n, the conservation law is of *order* N.

An alternative way to define equivalence classes of local conservation laws is via the formal variational derivative. Let us denote by $\tilde{p}_n$ the formal variational derivative of the density p_n of a local conservation law, i.e.

$$\tilde{p}_n = \frac{\delta p_n}{\delta u_n}. \tag{2.23}$$

If the local conservation law is trivial, then $\tilde{p}_n = 0$, if it is of zeroth order, then $\tilde{p}_n = \tilde{p}_n(u_n) \neq 0$ for at least some n while if it is of order N, then

$$\tilde{p}_n = \tilde{p}_n(u_{n+N}, \ldots, u_n, \ldots, u_{n-N}),$$

where

$$\frac{\partial \tilde{p}_n}{\partial u_{n+N}} \neq 0, \quad \frac{\partial \tilde{p}_n}{\partial u_{n-N}} \neq 0,$$

for at least some n. Then, for any conserved density p_n, by direct calculation, we derive that the following relation is valid:

$$p_{n,t} \sim \tilde{p}_n\ f_n \sim 0. \tag{2.24}$$

By carrying out the formal variational derivative of Eq. (2.24), taking into account that in a summation the following equality is valid:

$$\frac{\partial \tilde{p}_{n+k}}{\partial u_n} = \frac{\partial \tilde{p}_n}{\partial u_{n+k}},$$

we get that the formal variational derivative $\tilde{p}_n$ of a conserved density p_n satisfies the following equation:

$$(\partial_t + f_n^{*T})\tilde{p}_n = 0, \tag{2.25}$$

where the transposed Frechet derivative of f_n is given by

$$f_n^{*T} = \frac{\partial f_{n+1}}{\partial u_n} D + \frac{\partial f_n}{\partial u_n} + \frac{\partial f_{n-1}}{\partial u_n} D^{-1} = f_{n+1}^{(-1)} D + f_n^{(0)} + f_{n-1}^{(1)} D^{-1}. \tag{2.26}$$

Let us consider the Frechet derivative of $\tilde{p}_n$ for a local conservation law of order N. In such a case, we have

$$\tilde{p}_n{}^* = \sum_{k=-N}^{N} \tilde{p}_n{}^{(k)} D^k, \quad \tilde{p}_n{}^{(k)} = \frac{\partial \tilde{p}_n}{\partial u_{n+k}}. \tag{2.27}$$

We can construct the following operator:

$$B(S_n) = S_{n,t} + S_n\ f_n^* + f_n^{*T} S_n, \tag{2.28}$$

where

$$S_n=\sum_{j=-\infty}^{l} s_n^{(j)}(t)D^j. \tag{2.29}$$

Taking into account (2.10) and (2.25), one can easily prove that

$$\tilde{p}_n=S_n g_n. \tag{2.30}$$

Let us construct

$$B(\tilde{p}_n{}^*)=\sum_k b_n^{(k)}(t)D^k, \tag{2.31}$$

where $b_n^{(k)}(t)$ are some RFs, then from Eq. (2.28) it follows that

$$b_n^{(k)}=\tilde{p}_{n,t}^{(k)}+\sum_j (\tilde{p}_n^{(j)} f_{n+j}^{(k-j)}+f_{n+j}^{(-j)}\tilde{p}_{n+j}^{(k-j)}), \tag{2.32}$$

and then by differentiating Eq. (2.25) with respect to u_{n+k}, we can rewrite Eq. (2.32), after a long but straight forward calculation, in the form

$$b_n^{(k)}=-\sum_j \tilde{p}_{n+j}\frac{\partial^2 f_{n+j}}{\partial u_n \partial u_{n+k}}, \tag{2.33}$$

and thus prove, as f_n depends just on u_n and $u_{n\pm1}$, that $b_n^{(k)}$ are different from zero only for $-2\leqslant k\leqslant 2$.

In such a way, for a sufficiently high-order conserved density p_n, we can require that

$$B(\tilde{p}_n{}^*)=0, \tag{2.34}$$

is approximately solved. If the first $m<N-1$ terms of the Frechet derivative of $\tilde{p}_n$ satisfy Eq. (2.34), then we say that we have an approximate conserved density of *order* N and *length* m.

Let us mention here that sometimes the $\tilde{p}_n$ solution of (2.25), is called a conserved covariant, while $H_n=S_n^{-1}$ and the solutions of (2.34) are called, respectively, a Noether operator and an inverse Noether operator.[15] The Noether operator maps conserved covariants into symmetries while the inverse Noether operator maps symmetries into conserved covariants. This corresponds to the familiar relation between symmetries and conservation laws in Lagrangian or Hamiltonian mechanics (Noether's theorem). In some cases H_n can be the Hamiltonian operator for our equation and the inverse of formula (2.30),

$$g_n=H_n\frac{\delta p_n}{\delta u_n}, \tag{2.35}$$

will be local.

Taking all the results up to now obtained into account, we can state the following theorem, which will be proved in Appendix C.

Theorem 3: If the chain (1.1) satisfies conditions (2.1), it has a conservation law of order $N\geqslant 3$, and condition (2.18) is satisfied, then the following conditions must take place:

$$r_n^{(k)}=(D-1)s_n^{(k)} \quad (k=1,2), \tag{2.36a}$$

with

$$r_n^{(1)}=\log[-f_n^{(1)}/f_n^{(-1)}],\quad r_n^{(2)}=s_{n,t}^{(1)}+2f_n^{(0)},\tag{2.36b}$$

where $s_n^{(k)}$ are RFs.

III. DISCUSSION OF THE CONDITIONS

First of all let us notice that the request that nontrivial local conservation laws exist is more restrictive than that of the existence of symmetries. In fact, there are many instances in which generalized symmetries do exist, but not nontrivial conservation laws. This may be the case for many c-integrable equations, i.e. nonlinear equations that can be transformed into linear ones by an invertible transformation.[7]

If one compares Theorem 1 and Theorem 2 of Sec. II, one can think that among conditions (2.18) and (2.21) with $k=2$, 3 there is a difference of importance, as conditions (2.21) require the existence of two generalized symmetries, while for condition (2.18) only one generalized symmetry is sufficient. However, we could obtain conditions (2.21) with $k=2,3$, assuming that only one symmetry of order $i\geqslant 4$ exists, but calculations in the proof would be more difficult. For example, in the case $k=2$, following the notation of Appendix A, we can define

$$\hat{g}_n=\frac{g_n^{(i-1)}}{\Pi_{k=n}^{n+i-2}f_k^{(1)}}=\sum_{k=n}^{n+i-1}p_k^{(2)}.$$

Then, for $i\geqslant 3$, it follows that

$$\partial_t\hat{g}_n+\hat{g}_n(p_{n+i-1}^{(2)}-p_n^{(2)})=(D-1)\left[\frac{g_n^{(i-2)}}{\Pi_{k=n}^{n+i-3}f_k^{(1)}}-\sum_{k=n}^{n+i-1}f_k^{(-1)}f_{k-1}^{(1)}\right]\sim 0,$$

and hence we get the wanted result:

$$\hat{g}_n(p_{n+i-1}^{(2)}-p_n^{(2)})\sim 0,\quad \partial_t\hat{g}_n\sim i\,\partial_t p_n^{(2)}.$$

Conditions (2.21) required only that $f_n^{(1)}\neq 0$. An analogous set of conditions could be derived if we requested that just $f_n^{(-1)}\neq 0$ for all n. They can be derived in a straightforward way, considering expansions in negative powers of D, instead of positive, as we have done up to now. This derivation is left to the readers as an exercise. This set of conditions also will have the form of canonical conservation laws:

$$\partial_t\hat{p}_n^{(k)}=(D-1)\hat{q}_n^{(k)},\tag{3.1}$$

and conserved densities will be symmetric to the ones of (2.21). For example,

$$\hat{p}_n^{(1)}=\log\left(-\frac{\partial f_n}{\partial u_{n-1}}\right).\tag{3.2}$$

Let us notice, moreover, that if $H_n^{(1)}$ and $H_n^{(2)}$ are two solutions of (2.34) of different order, the operator

$$K_n=(H_n^{(1)})^{-1}H_n^{(2)}\tag{3.3}$$

satisfies (2.14), and thus it is a recursive operator. Consequently, if we start from two approximate solutions of (2.34), i.e. two Frechet derivatives of formal variational derivatives of conserved densities, we can, using (2.34), get an approximate symmetry. So, one can derive all the conditions

(2.21), (2.36), (3.1), assuming the existence of two higher-order local conservation laws. In the case of conditions (3.1), one should use the same formula (3.3), but $(H_n^{(1)})^{-1}$ will be a series in positive powers of the shift operator D:

$$(H_n^{(1)})^{-1}=\sum_{k=-N}^{+\infty} h_n^{(k)}D^k.$$

This proves the statement written at the beginning of this section that conservation laws are ''more fundamental'' than symmetries, as from conservations laws we get symmetries.

If we compare conditions (2.21), (2.36), (3.1), we can see, for example, that

$$r_n^{(1)}=p_n^{(1)}-\hat{p}_n^{(1)}, \tag{3.4}$$

i.e. the first of conditions (2.36) implies that $p_n^{(1)}\sim\hat{p}_n^{(1)}$, i.e. the first canonical conservation laws of (2.21) and (3.1) are equivalent. The same result could be obtained for the second condition of (2.32). In particular, the set of conditions (3.1) can be derived, starting from conditions (2.21) and (2.36). However, these conditions are of great importance in themselves, as there might be equations of interest that satisfy (2.21) and (3.1), but not (2.36).

The solution of the conditions, be those obtained by requesting the existence of the generalized symmetries or those of local conservation laws, provide the highest-order coefficients of the Frechet derivative of a symmetry or of the formal variational derivative of a conserved density. Those coefficients are the building blocks for the reconstruction of the symmetries or of the formal variational derivatives of the conserved densities. In fact the knowledge of $g_n^{(k)}=\partial g_n/\partial u_{n+k}$ with $k=i,\ i-1,...$, for a few values of k, gives a set of partial differential equations for g_n with respect to its variable, whose solution provides the needed symmetry. In the same way we can reconstruct variational derivatives of conserved densities. There is, however, a more direct way to obtain conserved densities. In fact, if we know the highest coefficients of L_n, the solution of Eq. (2.14), we can obtain several conserved densities by the following formula:

$$p_n^{(j)}=\mathrm{res}(L_n^j) \quad (j=1,2,...)$$

(see Appendix B).

Equation (1.1) with the conditions (2.1) can be splitted into two different classes. In fact, Eq. (2.21) with $k=1$ can be written in the form

$$p_{n,t}^{(1)}\sim\frac{\partial p_n^{(1)}}{\partial u_{n+1}}f_{n+1}+\frac{\partial p_n^{(1)}}{\partial u_n}f_n+\frac{\partial p_{n+1}^{(1)}}{\partial u_n}f_n=\Phi_n\sim 0,$$

$$\Phi_n=\Phi_n(u_{n+2},u_{n+1},u_n,u_{n-1}).$$

As Φ_n is a RF equivalent to zero, we have

$$\frac{\partial^2\Phi_n}{\partial u_{n+2}\,\partial u_{n-1}}=\rho_n\frac{\partial f_{n+1}}{\partial u_{n+2}}+\rho_{n+1}\frac{\partial f_n}{\partial u_{n-1}}=0, \quad \forall n, \tag{3.5}$$

where

$$\rho_n=\frac{\partial^2 p_n^{(1)}}{\partial u_{n+1}\,\partial u_{n-1}}.$$

Conditions (2.1) and (3.5) imply that there are only two possibilities:

$$\rho_n = 0, \quad \forall n, \tag{A}$$

$$\rho_n \neq 0, \quad \forall n. \tag{B}$$

Both classes are not empty. The Toda and Volterra equation belong to class (A) while the discrete analogue of Krichever Novikov equation[14] belongs to class (B). One can prove the following statement for chains of the class (B): if a chain (1.1) satisfies (2.1) and (2.21), then this chain has conservation laws of the orders 2, 3, 4, which will be given just by the canonical conservation laws (2.21). In fact, it is obvious that the first of conservation laws (2.21) has order 2. So, let us consider (2.21) with $k=1, 2$ and use them to obtain informations about $q_n^{(1)}$, $q_n^{(2)}$. The function $q_n^{(1)}$ depends on $u_{n+1},\ldots,u_{n-2}$ and

$$\frac{\partial q_n^{(1)}}{\partial u_{n-2}} = -\frac{\partial p_n^{(1)}}{\partial u_{n-1}} f_{n-1}^{(-1)}. \tag{3.6}$$

The function $q_n^{(2)}$ depends on $u_{n+1},\ldots,u_{n-3}$ and

$$\frac{\partial q_n^{(2)}}{\partial u_{n-3}} = -\frac{\partial p_n^{(2)}}{\partial u_{n-2}} f_{n-2}^{(-1)} = \frac{\partial p_n^{(1)}}{\partial u_{n-1}} f_{n-1}^{(-1)} f_{n-2}^{(-1)}.$$

Now one easily can show that

$$\frac{\partial^2 p_n^{(2)}}{\partial u_{n+1}\,\partial u_{n-2}} = -\rho_n f_{n-1}^{(-1)}, \quad \frac{\partial^2 r_n}{\partial u_{n+1}\,\partial u_{n-3}} = \rho_n f_{n-1}^{(-1)} f_{n-2}^{(-1)}, \tag{3.7}$$

where $r_n = q_n^{(2)} + \frac{1}{2}(p_n^{(2)})^2 + (\partial f_{n-1}/\partial u_n)\ (\partial f_n/\partial u_{n-1}) \sim p_n^{(3)}$. Taking into account Eq. (2.1) the functions (3.7) are different from zero for any n, thus showing that the conservation laws (2.21) for $k=2, 3$ are of the orders 3 and 4, respectively.

The same formulas (3.7) show that in the case of chains of the class (A) canonical conservation laws (2.21) have orders less than 2, 3, 4, respectively. For example, for the Toda chain (1.6), (1.7), formulas (2.21) give three inequivalent nontrivial conservation laws of order 0. In the case of the chain

$$u_{n,t} = (u_{n+1} - u_n)^{1/2}(u_n - u_{n-1})^{1/2}, \tag{3.8}$$

all three canonical conservation laws are trivial.

If the chain satisfies all five conditions and the conservation laws are all of low order, than in case (A) the chain might be linearizable. In the case of Eq. (3.8), such a transformation is $v_n = (u_{n+1} - u_n)^{1/2}$ and leads to the linear equation

$$2v_{n,t} = v_{n+1} - v_{n-1}. \tag{3.9}$$

It is worthwhile to show here how all five conditions (2.21), (2.36) can be rewritten in explicit form. Such explicit conditions can be easily verified using the computer and thus they can be the starting point for the construction of a program of the kind of DELIA[10] to check the integrability of differential-difference equations of the form (1.1).

A condition is *explicit* if it has the form $A_n = 0$, $\forall n$, where A_n is a function depending only on f_n and its partial derivatives with respect to all u_{n+i}. Let us define the functions

$$P_n^{(k)} = \frac{\delta}{\delta u_n} \partial_t p_n^{(k)}, \quad R_n^{(k)} = \frac{\delta}{\delta u_n} r_n^{(k)}, \quad k=1, 2,$$

and

$$P_n^{(3)}=\frac{\delta}{\delta u_n}\,\partial_t p_n^{(3)}-q_n^{(1)}P_n^{(2)}.$$

The five explicit conditions are given by

$$P_n^{(k)}=0,\quad R_n^{(l)}=0,\quad \forall_n,\quad k=1,\,2,\,3;\quad l=1,\,2. \tag{3.10}$$

The functions $p_n^{(1)}$, $r_n^{(1)}$ are already explicit and from them one can derive all partial derivatives $\partial q_n^{(1)}/\partial u_{n+i}$, $\partial s_n^{(1)}/\partial u_{n+i}$ [see, e.g., (3.6)] and then express $\partial_t p_n^{(2)}$, $r_n^{(2)}$ in an explicit form. For example, from (2.36) we have

$$r_n^{(2)}=\frac{\partial r_{n-1}^{(1)}}{\partial u_n}\,f_n-\frac{\partial r_n^{(1)}}{\partial u_{n-1}}\,f_{n-1}+2\,\frac{\partial f_n}{\partial u_n}.$$

Let us now consider $P_n^{(3)}$. On one hand we have

$$\partial_t p_n^{(3)}=\partial_t\left(q_n^{(2)}+\frac{\partial f_n}{\partial u_{n+1}}\,\frac{\partial f_{n+1}}{\partial u_n}\right)+\left(q_n^{(1)}+\frac{\partial f_n}{\partial u_n}\right)\partial_t p_n^{(2)}.$$

Using (2.21) with $k=2$, one can find all the partial derivatives of $q_n^{(2)}$ and consequently get an explicit expression for $\partial_t q_n^{(2)}$. Using (2.21) with $k=1$, we can obtain not only the functions $\partial q_{n+i}^{(1)}/\partial u_n$ but also all differences $q_{n+i}^{(1)}-q_n^{(1)}$. Consequently, as

$$\frac{\delta}{\delta u_n}\,(q_n^{(1)}\partial_t p_n^{(2)})-q_n^{(1)}P_n^{(2)}=\sum_i\,\frac{\partial q_{n+i}^{(1)}}{\partial u_n}\,\partial_t p_{n+i}^{(2)}+\sum_i\,(q_{n+i}^{(1)}-q_n^{(1)})\,\frac{\partial}{\partial u_n}\,\partial_t p_{n+i}^{(2)}, \tag{3.11}$$

we can easily write down the explicit form of $P_n^{(3)}$.

Let us end this discussion by looking into the connection between the symmetries and conservation laws for Eq. (1.1) and those belonging to the reduced case (1.3). For simplicity of exposition we just present the results in the case of periodicity 2, where Eq. (1.1) reduces to Eq. (1.4).

Generalized symmetries and conservation laws for Eq. (1.4) can be defined in the same way as those for Eq. (1.1). For example, a local conservation law of Eq. (1.4) is of the form

$$C_{n,t}=\Delta_{n+1}-\Delta_n, \tag{3.12}$$

where C_n, Δ_n are RFs of variables u_n^0, u_n^1, $u_{n\pm1}^0$, $u_{n\pm1}^1,\ldots$. Let us consider conservation laws of order N for $N\geqslant 1$. For such a conservation law to exist, we need it to be represented by $C_n\sim h_n(u_n^0,\,u_n^1,\ldots,\,u_{n+N}^0,\,u_{n+N}^1)$, with

$$\begin{pmatrix} \dfrac{\partial^2 h_n}{\partial u_n^0\,\partial u_{n+N}^0} & \dfrac{\partial^2 h_n}{\partial u_n^0\,\partial u_{n+N}^1} \\ \dfrac{\partial^2 h_n}{\partial u_n^1\,\partial u_{n+N}^0} & \dfrac{\partial^2 h_n}{\partial u_n^1\,\partial u_{n+N}^1} \end{pmatrix}\neq 0.$$

There is the following one-to-one correspondence between Eqs. (1.1) and the system (1.4): $u_n^0=u_{2n+1}$, $u_n^1=u_{2n}$, $f_n^0=f_{2n+1}$, $f_n^1=f_{2n}$.

The same transformation allows one to rewrite the generalized symmetries. Let us see what happens to the conservation laws. If $p_{n,t}=q_{n+1}-q_n$ is a conservation law of (1.1), then

$$C_n = p_{2n+1} + p_{2n}, \quad \Delta_n = q_{2n}, \tag{3.13}$$

gives a conservation law for (1.4). Given the conservation law (3.12), we have

$$\overline{p_{2n+1}} = C_n, \quad \overline{p_{2n}} = 0, \quad \overline{q_{2n+1}} - \overline{q_{2n}} = \Delta_n, \tag{3.14}$$

and thus equivalent conservation laws are turned into equivalent ones. In fact, if we pass from p_n to C_n and then back to $\overline{p_n}$, we have

$$\overline{p_n} = P_{n+1}^2 (p_n + p_{n-1}) \sim (P_{n+1}^2 + P_n^2) p_n = p_n .$$

It can be checked that if we use formulas (3.13), then ord $C_n \approx 1/2$ ord p_n; in the case of transition (3.14), ord $\overline{p_n} \approx 2$ ord C_n. For example, the Toda chain (1.6), written in the form (1.4), has conserved densities,

$$\log u_n^0, \quad u_n^1, \quad 2u_n^0 + (u_n^1)^2, \quad u_n^0 (u_{n+1}^1 + u_n^1) + \tfrac{1}{3} (u_n^1)^3,$$

while in the form (1.17) the conserved densities are

$$P_{n+1} \log u_n, \quad P_n u_n, \quad 2P_{n+1} u_n + P_n u_n^2, \quad u_{n+1} u_n + \tfrac{1}{3} P_n u_n^3.$$

IV. APPLICATIONS

In the following we will find out about the integrability of differential difference equations of the form (1.1) by going through all examples considered in the Introduction, following the enumeration given there.

(1) In the case (1.10) the first canonical conservation law (2.18) implies that

$$\partial_t \log \beta_n \sim 0, \tag{4.1}$$

i.e.,

$$\beta_n' (u_{n+1} - u_{n-1}) \sim \beta_n' u_{n+1} - \beta_{n+1}' u_n \sim 0. \tag{4.2}$$

So β_n'' must be an n-independent constant:

$$\beta_n = A u_n^2 + B_n u_n + C_n . \tag{4.3}$$

Inserting this result into (4.2), we get that

$$(B_{n-1} - B_{n+1}) u_n \sim 0, \quad \text{i.e.} \quad B_n = B + (-1)^n \widetilde{B},$$

and thus the first canonical conservation law gives

$$q_n^{(1)} = 2A u_n u_{n-1} + B(u_n + u_{n-1}) - \widetilde{B}[(-1)^n u_n + (-1)^{n-1} u_{n-1}]. \tag{4.4}$$

Introducing (1.10) and (4.4) into the second canonical conservation law, we get, after a straightforward but lengthy calculation, that

$$p_{n,t}^{(2)} \sim (C_{n-1} - C_{n+1})(A u_n^2 + B_n u_n).$$

It follows, in particular, that if $A \neq 0$, then

$$C_n = C + (-1)^n \widetilde{C}. \tag{4.5}$$

Using the last canonical conservation law, one can prove that C_n must have always the form (4.5), and thus the most general chain of the form (1.10) that satisfies all five conditions is

$$u_{n,t}=[Au_n^2+(B+(-1)^n\tilde{B})u_n+C+(-1)^n\tilde{C}](u_{n+1}-u_{n-1}), \tag{4.6}$$

which depends on five arbitrary complex constants.

By obvious point transformations, we can reduce any nonlinear chain of the form (4.6) to one of the following chains: the Toda chain (1.7), or

$$u_{n,t}=u_n(u_{n+1}-u_{n-1}), \tag{4.7}$$

the Volterra equation, or

$$u_{n,t}=(C_n-u_n^2)(u_{n+1}-u_{n-1}), \tag{4.8}$$

where

$$C_n=1 \quad \text{or} \quad C_n=0 \quad \text{or} \quad C_n=P_{n+1}^2, \tag{4.9}$$

corresponding to three modifications of the Volterra equation. Unlike the discrete version of the Krichever–Novikov equation (1.21), (1.22), all the chains (4.7)–(4.9) can be reduced to the Toda chain by Miura transformations. For example, in the case of the Volterra equation, we have the transformation

$$\tilde{u}_n=P_{n+1}^2u_{n+1}u_n+P_n^2(u_{n+1}+u_n), \tag{4.10}$$

which brings any solution u_n of the Volterra equation into a solution $\tilde{u}_n$ of the Toda chain. Transformations of the modified Volterra equations (4.8), (4.9) into the Volterra equation are given by the formula

$$\tilde{u}_n=(C_n+u_n)(C_{n+1}-u_{n+1}). \tag{4.11}$$

Consequently, due to transformations (4.10), (4.11), together with point transformations, any nonlinear chain of the form (4.7)–(4.9) possesses local conservation laws of an arbitrary high order. This means, in particular, that the chains (4.7)–(4.9) satisfy not only classifying conditions (2.21), (2.36) but also all other conditions of higher order we could derive using approximate symmetries and conserved densities.

(2) We now classify chains of the form (1.11). Equation (2.1) reduces to the following conditions:

$$\frac{\partial f_n}{\partial u_{n+1}}=P_{n+1}^2e^{u_{n+1}}(g_n+g_n')+P_n^2\lambda_n\neq 0, \tag{4.12}$$

$$-\frac{\partial f_n}{\partial u_{n-1}}=P_{n+1}^2e^{u_{n+1}}g_n'+P_n^2\lambda_n\neq 0. \tag{4.13}$$

This means, in particular, that $\lambda_n\neq 0$ for even n. As λ_n do not exist in our equation for n odd, we can take them arbitrary for n odd and then assume that $\lambda_n\neq 0$ for all n. Analogously, we have to require that $g_n'\neq 0$, $g_n+g_n'\neq 0$ for all n. We can then formulate the following theorem.

Theorem: A chain of the form (1.11) satisfies the classifying conditions (2.21), (2.36) iff it is related by a point transformation of the form $\tilde{u}_n=\alpha_n u_n+\beta_n$ to one of two following chains:

$$u_{n,t}=P_{n+1}^2(\exp u_{n+1}-\exp u_{n-1})+P_n^2(u_{n+1}-u_{n-1}), \tag{4.14a}$$

$$u_{n,t}=P^2_{n+1}\ \exp\left(\frac{u_{n+1}-u_{n-1}}{a_n}\right)+P^2_n a_{n+1}a_{n-1}(u_{n+1}-u_{n-1}), \tag{4.14b}$$

with $a_n=\alpha n+\beta$, where α and β are arbitrary constants.

To prove this theorem we consider at first the conditions (2.36). As we have (3.4),

$$p^{(1)}_n=\log\frac{\partial f_n}{\partial u_{n+1}}=P^2_{n+1}(u_{n+1}+\log(g_n+g'_n))+P^2_n\ \log\lambda_n, \tag{4.15}$$

$$\widetilde{p}^{(1)}_n=\log\left(-\frac{\partial f_n}{\partial u_{n-1}}\right)=P^2_{n+1}(u_{n+1}+\log g'_n)+P^2_n\ \log\lambda_n, \tag{4.16}$$

then

$$r^{(1)}_n=P^2_{n+1}H_n,\quad H_n=\log(g_n+g'_n)-\log g'_n.$$

Hence

$$P^2_{n+1}H''_n=0,\quad \text{i.e.,}\ \ P^2_{n+1}H_n=a_n v_n+b_n, \tag{4.17}$$

where $v_n=u_{n+1}-u_{n-1}$, and a_n, b_n are some n-dependent constants. Consequently, we must have $a_{n-1}=a_{n+1}$ for all n. As $a_{2k}=0$ [see (4.17)], and $a_{2k-1}=a_{2k+1}$, we have that

$$a_n=P^2_{n+1}a, \tag{4.18}$$

where a is a pure constant. So, the condition $r^{(1)}_n\sim 0$ implies (4.17) and (4.18). We find, moreover, that

$$s^{(1)}_n=c_n+(D-1)(aP^2_{n+1}u_{n-1})\sim 2aP^2_n u_n,$$

and as in this case $\partial f_n/\partial u_n=0$, then $r^{(2)}_n=\partial_t s^{(1)}_n$. Consequently,

$$r^{(2)}_n\sim 2aP^2_n\ f_n=2aP^2_n\lambda_n u_n\sim 2aP^2_{n+1}(\lambda_{n-1}-\lambda_{n+1})u_n,$$

and the last condition gives, for all n,

$$aP^2_{n+1}(\lambda_{n-1}-\lambda_{n+1})=0. \tag{4.19}$$

Let us pass over to the first canonical conservation law. It follows from (3.4) that this condition is equivalent to the condition $\partial_t\widetilde{p}^{(1)}_n\sim 0$. Since $\widetilde{p}^{(1)}_n\sim P^2_n u_n+P^2_{n+1}\ \log g'_n$ [see (4.16)], then

$$\begin{aligned}\partial_t\widetilde{p}^{(1)}_n&\sim P^2_n f_n+P^2_{n+1}(\log g'_n)'(f_{n+1}-f_{n-1})\\&\sim P^2_n\lambda_n v_n+P^2_{n+1}\lambda_{n+1}v_{n+1}(\log g'_n)'+P^2_n\lambda_n v_n(\log g'_{n+1})'.\end{aligned}$$

Applying the operator $\partial^2/\partial u_{n+2}\partial u_{n-1}$ to it, we obtain

$$P^2_{n+1}\lambda_{n+1}(\log g'_n)''-P^2_n\lambda_n(\log g'_{n+1})''=0,$$

which means that for all n we must have

$$P^2_{n+1}(\log g'_n)''=0,\quad \text{i.e.}\quad P^2_{n+1}\ \log g'_n=c_n v_n+d_n, \tag{4.20}$$

where c_n, d_n are some n-dependent constants.

By comparing (4.17) and (4.20) we obtain an explicit formula for g_n. As for any function θ_n the formula $P_{n+1}^2 \exp(P_{n+1}^2\theta_n)=P_{n+1}^2 \exp \theta_n$ is valid:

$$P_{n+1}^2 g_n' = P_{n+1}^2 \exp(\log g_n') = P_{n+1}^2 \exp(P_{n+1}^2 \log g_n'),$$

and consequently from Eq. (4.20) we get

$$P_{n+1}^2 g_n' = P_{n+1}^2 \exp(c_n v_n + d_n). \tag{4.21}$$

Using (4.17) we are led to the following formula for g_n:

$$P_{n+1}^2 g_n = P_{n+1}^2 \exp((a+c_n)v_n + b_n + d_n) - P_{n+1}^2 \exp(c_n v_n + d_n). \tag{4.22}$$

Moreover, the consistency between (4.21) and (4.22) implies that we must have

$$P_{n+1}^2(1+c_n) = P_{n+1}^2(a+c_n)\exp(a v_n + b_n),$$

from which it follows that

$$a(a-1)=0, \tag{4.23}$$

$$P_{n+1}^2(1+c_n)a=0. \tag{4.24}$$

Condition (4.23) implies that either $a=1$ or $a=0$. Let us, at first, consider the case $a=1$. Condition (4.24) gives $P_{n+1}^2 c_n = -P_{n+1}^2$, and then it follows from (4.19) that $\lambda_{2k}=\lambda_{2k-2}=\lambda$, where λ is a constant different from zero. Taking into account formula (4.22), we get that the obtained chain is of the form

$$u_{n,t} = P_{n+1}^2(\exp(u_{n+1}+\alpha_n) - \exp(u_{n-1}+\beta_n)) + (u_{n+1}-u_{n-1})P_n^2\lambda,$$

where α_n, β_n are some n-dependent constants.

This chain can be further simplified, using simple point transformations. If we apply first the transformation $\tilde{u}_n = (P_{n+1}^2\lambda + P_n^2)u_n$ and then $\tilde{u}_n = u_n + P_n^2\alpha_{n-1}$, we can reduce it to the form

$$u_{n,t} = P_{n+1}^2(\exp u_{n+1} - \exp(u_{n-1}+\gamma_n)) + (u_{n+1}-u_{n-1})P_n^2,$$

where γ_n are some n-dependent constants. Moreover, we have

$$\partial_t p_n^{(2)} \sim 2P_n^2 \exp(u_n)(1-\exp \gamma_{n+1}).$$

As $P_{n+1}^2 \exp \gamma_n = P_{n+1}^2$ for all n, the chain takes the form (4.14a). There are no problems to check that all the five classifying conditions are satisfied for (4.14a). Let us consider now the case $a=0$. It follows from (4.22) that

$$P_{n+1}^2 g_n = P_{n+1}^2 \exp(c_n v_n + d_n)(\exp b_n - 1),$$

so that the function $\exp b_n - 1$ cannot be zero for odd n, as $g_n \neq 0$. This means we can redefine d_n so that (1.11) takes the form

$$u_{n,t} = P_{n+1}^2 \exp[(1+c_n)u_{n+1} - c_n u_{n-1} + d_n] + (u_{n+1}-u_{n-1})P_n^2\lambda_n. \tag{4.25}$$

Let us notice that $1+c_n$ and c_n are different from zero for all n, as (2.1) must be valid.

As in the previous case, the chain (4.25) can be simplified, using point transformations of the form $\tilde{u}_n = \alpha_n u_n + \beta_n$, and we get the following chain:

$$u_{n,t}=f_n=P^2_{n+1}\ \exp\frac{v_n}{a_n}+P^2_n b_n v_n\,, \tag{4.26}$$

where $v_n=u_{n+1}-u_{n-1}$ and a_n, b_n are some n-dependent constants different from zero for any n.

To fix a_n and b_n we will use two of the conditions (2.21) that will give us two other constraints. It is easy to see that $p_n^{(1)}=P^2_{n+1}(v_n/a_n)+\delta_n$, and then

$$\partial_t p_n^{(1)}\sim u_n P^2_{n+1}(B_{n-1}-B_{n+1}),\quad B_n=b_n\left(\frac{1}{a_{n-1}}-\frac{1}{a_{n+1}}\right),$$

from which it follows that

$$P^2_{n+1}(B_{n-1}-B_{n+1})=0. \tag{4.27}$$

Now

$$q_n^{(1)}=\text{const}+P^2_n\,\frac{b_n}{a_{n-1}}\,v_n+P^2_{n+1}\,\frac{b_{n-1}}{a_n}\,v_{n-1}+P^2_{n+1}B_{n-1}u_n+P^2_n B_n u_{n-1}\sim 2u_n P^2_{n+1}A_n\,,$$

$$A_n=\frac{b_{n-1}}{a_{n-2}}-\frac{b_{n+1}}{a_{n+2}},$$

and thus

$$\partial_t p_n^{(2)}\sim 2P^2_{n+1}A_n\ \exp\frac{v_n}{a_n}.$$

So we get the second constraint:

$$P^2_{n+1}A_n=0. \tag{4.28}$$

Introducing $\widetilde{b}_n$ such that $b_n=\widetilde{b}_n a_{n+1}a_{n-1}$, we obtain from (4.28) that $P^2_n\widetilde{b}_n=P^2_n b$, where b is a constant different from zero. Therefore

$$P^2_n b_n=bP^2_n a_{n+1}a_{n-1}. \tag{4.29}$$

Taking into account (4.27) and using (4.29), we obtain

$$P^2_{n+1}(a_{n+2}-2a_n+a_{n-2})=0.$$

From this it follows that, for n odd, a_n will have the form

$$a_{2k-1}=c(2k-1)+d,$$

where c, d are constants. As our chain (4.26) does not contain any a_n with even n, we can set $a_n=cn+d$ for all n.

The chain (4.26) with b_n satisfying (4.29) has the form

$$u_{n,t}=P^2_{n+1}\ \exp\frac{v_n}{a_n}+bP^2_n a_{n+1}a_{n-1}v_n\,,$$

i.e. coincides with (4.14b) up to the constant b. This constant, however, can be easily removed, using an obvious point transformation.

If we go over to the class (1.12), we see that in case (4.14a)

$$\epsilon_k=1, \quad G_k=1-\exp((v_k-v_{k-1})-(v_{k+1}-v_k)),$$

and this is nothing but the Toda model (1.6) for the function v_k. The chain (4.14b) is a new example of integrable (and n-dependent) equation. In this case, the chain equation can be rewritten as, setting for semplicity, $c_k=a_{2k-1}$)

$$v_{k,tt}=\exp[c_{k+1}(v_{k+1}-v_k)-c_{k-1}(v_k-v_{k-1})]. \tag{4.30}$$

It belongs to the class (1.12), as

$$G_k(\zeta_k)=\exp(\delta_k\zeta_k), \quad \delta_k=c_{k-1}\epsilon_k, \quad c_{k+1}\epsilon_{k+1}-c_{k-1}\epsilon_k=1.$$

As c_k is linear in k, Eq. (4.30) can be written as

$$v_{k,tt}=\exp(c_{k+1}v_{k+1}-2c_kv_k+c_{k-1}v_{k-1}),$$

and by an obvious point transformation, we can remove the c_k and will have the potential Toda equation:

$$v_{k,tt}=\exp(v_{k+1}-2v_k+v_{k-1}), \tag{4.31}$$

which reduces to the Toda by the following transformation:

$$\tilde{v}_k=v_{k+1}-v_k.$$

This implies that Eq. (4.30) is completely integrable.

(3) In the case of the classification problem (1.14) we present here just the final results. If integrability conditions (2.21), (2.36) are satisfied for a chain of the form (1.14), then such a chain, up to a point transformation of the form $\tilde{u}_n=au_n+b_n$, $\tilde{t}=ct$, must have the form

$$u_{n,t}=P_{n+1}^2\left(\exp\left(\frac{v_n}{a_n}\right)+\frac{\lambda_n}{a_n}\right)+P_n^2a_{n+1}a_{n-1}v_n, \tag{4.32a}$$

where

$$v_n=u_{n+1}-u_{n-1}, \tag{4.32b}$$

$$a_n=\alpha n+\beta\neq 0, \quad \forall n, \quad \lambda_n=\gamma n+\delta. \tag{4.32c}$$

It turns out that there exists a complicated and not obvious transformation:

$$\tilde{u}_n=P_{n+1}^2\epsilon_n \exp\left(\frac{v_n}{\epsilon_n}\right)+P_n^2(\epsilon_{n+1}u_{n+1}-\epsilon_{n-1}u_{n-1}-2\gamma t), \tag{4.33}$$

which turns (4.32) into the polynomial Toda chain (1.7). This shows that (4.32) is integrable. Two of its three canonical conservation laws are nontrivial. More precisely,

$$p_n^{(1)}\sim 2\alpha\rho_n^{(1)}, \quad \rho_n^{(1)}=\frac{P_n^2}{a_{n+1}a_{n-1}}u_n, \quad p_n^{(2)}\sim 0,$$

$$p_n^{(3)}\sim\rho_n^{(2)}=2P_{n+1}^2a_n \exp\left(\frac{v_n}{\epsilon_n}\right)+P_n^2(a_{n+1}u_{n+1}-a_{n-1}u_{n-1})^2,$$

where $\rho_n^{(1)}$ and $\rho_n^{(2)}$ are densities of conservation laws of the orders 0 and 2.

The chain (4.32) depends essentially on the spatial variable n, as point transformations do not allow us to remove this dependence. This dependence is nonlinear (unlike local master symmetries[15]).

Local conservation laws of (4.32) are constructed using the transformation (4.33), which is of the form $\widetilde{u}_n = \psi_n(u_{n+1}, u_{n-1})$. If $p_{n,t} = (D-1)q_n$ is of (1.7), via (4.33) we obtain a local conservation law of (4.32). As a result, we have local conservation laws of (4.32) of orders $m \geqslant 3$ (we already have written down two local conservation laws of the orders 0 and 2). Indeed, let us consider a conserved density of (1.7) of order M,

$$p_n = p_n(u_{n+i}, \ldots, u_{n+j}), \quad i - j = M \geqslant 1,$$

where $\partial^2 p_n / \partial u_{n+i} \, \partial u_{n+j} \neq 0$ for at least some n. Using (4.33), we are led to the conserved density,

$$\hat{p}_n(u_{n+i+1}, \ldots, u_{n+j-1}) = p_n(\psi_{n+i}, \ldots, \psi_{n+j}),$$

of (4.32). It is easy to see that, if $M \geqslant 1$, then

$$\frac{\partial^2 \hat{p}_n}{\partial u_{n+i+1} \, \partial u_{n+j-1}} = \frac{\partial^2 p_n}{\partial \psi_{n+i} \, \partial \psi_{n+j}} \frac{\partial \psi_{n+i}}{\partial u_{n+i+1}} \frac{\partial \psi_{n+j}}{\partial u_{n+j-1}} \neq 0, \quad \forall n.$$

Then the local conservation law of (4.32) is of the order $m = M+2$. So, the new chain (4.32) has local conservation laws of an arbitrary high order. In general, these local conservation laws depend on the time t. If $\gamma = 0$, the transformation (4.33) does not depend on t but, however, still depends on n.

Let us rewrite Eq. (4.32) as a dynamical system. If we introduce

$$\widetilde{u}_k = u_{2k} + (\alpha\delta - \beta\gamma)t^2,$$

and denote $c_k = a_{2k-1}$, we are led to an integrable (in the sense that we can construct solutions) lattice equation of the form (1.17)

$$\frac{u_{k,tt}}{c_{k+1}c_k} = \exp \frac{u_{k+1} - u_k}{c_{k+1}} - \exp \frac{u_k - u_{k-1}}{c_k}, \quad c_k = ak + b \neq 0, \quad \forall k. \tag{4.34}$$

Equation (4.34) can be reduced directly to the potential Toda equation (4.31) by the following transformation:

$$\frac{u_k}{c_k c_{k+1}} = (D-1)\left(\frac{v_k}{c_k} + \lambda_k\right).$$

Such a transformation is not invertible and transform point symmetries in potential symmetries (i.e., it does not provide local conservation law). One can see that the chain (4.34) is a direct and very close generalization of the exponential Toda model. Surely it has physical applications and, in any case, this chain seems interesting in itself.

Let us now consider the following generalization of the discrete analog of the Krichever–Novikov equation (1.21), (1.22), obtained by introducing into Eq. (1.22) arbitrary n-dependent coefficients, i.e.

$$p_n = \alpha_n u_n^2 + 2\beta_n u_n + \gamma_n, \tag{4.35a}$$

$$q_n = \widetilde{\beta}_n u_n^2 + \lambda_n u_n + \widetilde{\delta}_n, \tag{4.35b}$$

$$r_n=\tilde{\gamma}_n u_n^2+2\,\delta_n u_n+\omega_n\,. \tag{4.35c}$$

Let us define f_n, the rhs of (1.21), as

$$f_n=\frac{Q_n}{v_n}-\frac{1}{2}\,\frac{\partial Q_n}{\partial u_{n+1}}=\frac{\tilde{Q}_{n-1}}{v_n}+\frac{1}{2}\,\frac{\partial\tilde{Q}_{n-1}}{\partial u_{n-1}},$$

where $v_n=u_{n+1}-u_{n-1}$ is the denominator of (1.21), and

$$Q_n=u_{n+1}^2p_n+2u_{n+1}q_n+r_n\,,\quad \tilde{Q}_n=p_{n+1}u_n^2+2q_{n+1}u_n+r_{n+1}\,.$$

One can easily prove that

$$\partial_t p_n^{(1)}\sim 2\,\frac{Q_n-\tilde{Q}_n}{v_{n+1}v_n}+h_n(u_{n+1},u_n,u_{n-1})\sim 0,$$

and consequently

$$Q_n=\tilde{Q}_n\,.$$

This condition can be rewritten as a condition for the coefficients appearing in the equation; its solution gives

$$\alpha_{n+1}=\alpha_n=\alpha,\quad \lambda_{n+1}=\lambda_n=\lambda,\quad \omega_{n+1}=\omega_n=\omega, \tag{4.36a}$$

$$\beta_{n+2}=\beta_n\,,\quad \gamma_{n+2}=\gamma_n\,,\quad \delta_{n+2}=\delta_n\,, \tag{4.36b}$$

$$\tilde{\beta}_n=\beta_{n+1}\,,\quad \tilde{\gamma}_n=\gamma_{n+1}\,,\quad \tilde{\delta}_n=\delta_{n+1}\,. \tag{4.36c}$$

As for this chain, the conditions

$$\partial_t p_n^{(1)}\sim\partial_t p_n^{(2)}\sim r_n^{(1)}\sim r_n^{(2)}\sim 0,$$

are identically satisfied, we can say that it is integrable. We have conservation laws of the orders 2 and 3 with the following densities:

$$p_n^{(1)}\sim\log\,Q_n-2\,\log\,v_n\,,\quad p_n^{(2)}\sim -2\,\frac{Q_n}{v_{n+1}v_n}-\frac{1}{2}\,\frac{\partial^2 Q_n}{\partial u_{n+1}\,\partial u_n}.$$

Transformations of the type

$$\tilde{u}_n=a_nU_n\,,\quad \tilde{u}_n=u_n+a_n\,,\quad a_{n+2}=a_n\,,$$

and

$$\tilde{u}_n=1/u_n$$

(and, therefore, any linear–fractional transformation with two-periodic coefficients) do not change the form (1.21), (4.35), and the conditions (4.36), but, in general, would allow us to remove only one of three two-periodic constants β_n, γ_n, δ_n. This implies that one has written down an integrable two-field extension of the Krichever–Novikov equation.

V. CONCLUSIONS

In this paper we have constructed a set of five conditions necessary for the existence of higher symmetries and conservation laws for differential difference equations of the class (1.1). By applying these conditions to a few subcases of particular interest, we have been able to prove that this class of equations contains new integrable nonlinear equations related to the Toda (1.7) or to the discrete Krichever–Novikov equation (1.21). In this way we have proved the validity of these conditions for stating the integrability of equations of the form (1.1). We have, moreover, shown that these conditions are, in a certain sense, not only necessary but also sufficient as, whenever they are satisfied the equation is integrable. So they can be used as a very convenient test for the integrability of equations of the form (1.1). The explicit form of these conditions, presented in Sec. III, allows us to check them easily, even using a computer.

The complete classification of the equations of the form (1.1) is left to a future work together with the extension of the method for the case of difference–difference equations, the extension of the class of symmetries from that of the restricted function to unrestricted ones and to the case of potential symmetries.

ACKNOWLEDGMENTS

The research of R.Y. is partially supported by grants from INTAS, Russian Foundation for Fundamental Research and NATO—Royal Society Fellowship Program (NATO/JS/96A). The visits to Rome of R.Y. had been possible thanks to a fellowship from GNFM of CNR.

APPENDIX A: PROOF OF THEOREM 1

For a sufficiently high-order symmetry, i.e. $i \gg 1$, the highest terms of

$$g_n^* = \sum_{k=j}^{i} g_n^{(k)} D^k,$$

will satisfy the following equation:

$$\sum_{l=2}^{i} g_{n,t}^{(l)} D^l + \sum_{k}^{i} \sum_{m=-1}^{1} g_n^{(k)} f_{n+k}^{(m)} D^{k+m} - \sum_{k}^{i} \sum_{m=-1}^{1} f_n^{(m)} g_{n+m}^{(k)} D^{k+m} = 0, \tag{A1}$$

where the sum is over those k such that $k+m>1$, as otherwise the lhs of (A1) is different from zero. In (A1) the coefficients of any power of D must vanish; so the highest coefficient, that of D^{i+1}, reads as

$$g_n^{(i)} f_{n+i}^{(1)} - f_n^{(1)} g_{n+1}^{(i)} = 0. \tag{A2}$$

As, due to (2.1), $f_n^{(1)} \neq 0,\ \forall n,$ we have

$$g_n^{(i)} = \prod_{k=n}^{n+i-1} f_k^{(1)}, \tag{A3}$$

where we have, with no restriction, set to unity the arbitrary integration constant.

Let us consider now the coefficient of D^i; this comes from more than one term ($k=i$, $m=0$ or $k=i-1$, $m=1$) and involves the time evolution of $g_n^{(i)}$. It can be cast in the following form:

$$\frac{g_{n,t}^{(i)}}{g_n^{(i)}}=\frac{g_{n+1}^{(i-1)}}{\Pi_{k=n+1}^{n+i-1}f_k^{(1)}}-\frac{g_n^{(i-1)}}{\Pi_{k=n}^{n+i-2}f_k^{(1)}}-(D-1)\sum_{k=n}^{n+i-1} f_k^{(0)}, \tag{A4}$$

from which we derive

$$\partial_t \log g_n^{(i)}=(D-1)\left[\frac{g_n^{(i-1)}}{\Pi_{k=n}^{n+i-2}f_k^{(1)}}-\sum_{k=n}^{n+i-1} f_k^{(0)}\right]. \tag{A5}$$

Introducing (A3) onto the lhs of (A5), we get

$$\partial_t \log \prod_{k=n}^{n+i-1} f_k^{(1)}=\sum_{k=n}^{n+i-1} \partial \log f_k^{(1)}\sim i\partial_t \log f_n^{(1)}\sim 0 \quad \text{c.v.d.}$$

APPENDIX B: PROOF OF THEOREM 2

From Theorem I, we deduce that we have an approximate symmetry of order $i=1$,

$$G_n=g_n^{(1)}D+g_n^{(0)}+g_n^{(-1)}D^{-1}+g_n^{(-2)}D^{-2}+\cdots , \tag{B1}$$

where, from (A3),

$$g_n^{(1)}=f_n^{(1)}. \tag{B2}$$

Instead of (A5) we have

$$\partial_t \log f_n^{(1)}=(D-1)(g_n^{(0)}-f_n^{(0)}) \tag{B3}$$

[see the coefficient of D in the equation $A(G_n)=0$ with A defined by (2.14)]. Consequently, the function on the rhs of the first canonical conservation law is

$$q_n^{(1)}=g_n^{(0)}-f_n^{(0)},$$

from which we get

$$g_n^{(0)}=f_n^{(0)}+q_n^{(1)}. \tag{B4}$$

Let us now consider the coefficient of D^0 in the equation $A(G_n)=0$:

$$\partial_t g_n^{(0)}=(D-1)[f_{n-1}^{(1)}(g_n^{(-1)}-f_n^{(-1)})]. \tag{B5}$$

Equation (B5) is the second canonical conservation law with the lhs given by (B4):

$$p_n^{(2)}=g_n^{(0)}=f_n^{(0)}+q_n^{(1)}, \quad q_n^{(2)}=f_{n-1}^{(1)}(g_n^{(-1)}-f_n^{(-1)}). \tag{B6}$$

From (B6) we get

$$g_n^{(-1)}=f_n^{(-1)}+q_n^{(2)}/f_{n-1}^{(1)}. \tag{B7}$$

This last relation is obtained in a simpler way using the following lemma.

If $H_n=h_n^{(i)}D^i+h_n^{(i-1)}D^{i-1}+\cdots$ is an approximate symmetry, which satisfies the first $m\geqslant i+2$ terms of the equation $H_{n,t}=[f_n^*,H_n]$, then

$$\text{res}(H_n)\equiv h_n^{(0)}, \tag{B8}$$

will be a conserved density.

In fact,

$$\partial_t \operatorname{res}(H_n) = \operatorname{res}(H_{n,t}) = \operatorname{res}[f_n^*, H_n]. \tag{B9}$$

The coefficient of D^0 of $[f_n^*, H_n]$ can be obtained only from terms of the type

$$[r_n D^m, s_n D^{-m}],$$

which are equivalent to zero.

As any power of an approximate symmetry is also an approximate symmetry of the same length, we can construct a new conserved density, calculating the residue of G_n^2. In such a case, after a long but straightforward calculation, we get

$$\begin{aligned}\operatorname{res}(G_n^2) &= \operatorname{res}[(g_n^{(1)}D + g_n^{(0)} + g_n^{(-1)}D^{-1} + \cdots)^2]\\ &= g_n^{(1)}g_{n+1}^{(-1)} + (g_n^{(0)})^2 + g_n^{(-1)}g_{n-1}^{(1)} \sim 2g_n^{(1)}g_{n+1}^{(-1)}\\ &\quad + (g_n^{(0)})^2 \sim 2q_n^{(2)} + (p_n^{(2)})^2 + 2f_n^{(1)}f_{n+1}^{(-1)} = 2p_n^{(3)}.\end{aligned}$$

As, from the previous lemma, ∂_t res $G_n^2 \sim 0$,

$$\partial_t p_n^{(3)} \sim 0 \text{ c.v.d.}$$

APPENDIX C: PROOF OF THEOREM 3

Let us assume that we have a solution of

$$\partial_t S_n + S_n\, f_n^* + f_n^{*T} S_n = 0, \tag{C1}$$

where S_n is an approximate conserved density,

$$S_n = \sum_{k=N-m+1}^{N} s_n^{(k)} D^k, \tag{C2}$$

of order $N \geqslant 3$ and length $m \geqslant 2$. In such a case, introducing (C2) into (C1), the coefficient of D^{N+1} in (C1) reads as

$$s_n^{(N)} f_{n+N}^{(1)} + f_{n+1}^{(-1)} s_{n+1}^{(N)} = 0. \tag{C3}$$

As $f_n^{(\pm 1)} \neq 0$ for any n, it follows that $s_n^{(N)} \neq 0$ for any n. Then we get

$$-f_{n+N}^{(1)}/f_{n+1}^{(-1)} = s_{n+1}^{(N)}/s_n^{(N)}, \tag{C4}$$

and thus, by taking the logarithm of both sides, we are led to

$$r_n^{(1)} = \log[-f_n^{(1)}/f_n^{(-1)}]. \tag{C5}$$

From (C4) and (C5) we get

$$s_n^{(N)} = \tilde{s}_{n+1} f_{n+1}^{(1)} f_{n+2}^{(1)} \cdots f_{n+N-1}^{(1)}, \tag{C6}$$

where $\tilde{s}_n$ is such that

$$\tilde{s}_n\, f_n^{(1)} + f_n^{(-1)} \tilde{s}_{n+1} = 0. \tag{C7}$$

The coefficient at order D^N of (C1) gives

$$\partial_t \log (\tilde{s}_{n+1} f_{n+1}^{(1)} \cdots f_{n+N-1}^{(1)}) + f_n^{(0)} + f_{n+N}^{(0)} \sim 0,$$

from which

$$(N-1)\partial_t \log f_n^{(1)} + \partial_t \log \tilde{s}_n + 2 f_n^{(0)} \sim 0.$$

As condition (2.18) is satisfied, and $\log \tilde{s}_n = \tilde{s}_n^{(1)}$ [compare (C7) and the first of the conditions (2.32)], we are led to

$$r_n^{(2)} = \partial_t \tilde{s}_n^{(1)} + 2 f_n^{(0)} \quad \text{c.v.d.}$$

[1] M. J. Ablowitz and J. Ladik, ''Nonlinear differential-difference equations,'' J. Math. Phys. **16**, 598–603 (1975); ''Nonlinear differential-difference equations and Fourier transform,'' *ibid.* **17**, 1011–1018 (1976); ''On the solution of a class of nonlinear partial difference equations,'' Stud. Appl. Math. **55**, 213–229 (1976); ''On the solution of a class of nonlinear partial difference equations,'' *ibid.* **57**, 1 (1976).
[2] R. I. Yamilov, ''Classification of discrete evolution equations,'' Usp. Mat. Nauk **38**, 155–156 (1983) (in Russian).
[3] R. I. Yamilov, ''Generalizations of the Toda model, and conservation laws,'' preprint, Institute of Mathematics, Ufa, 1989 (in Russian); English version: R. I. Yamilov, *Classification of Toda Type Scalar Lattices*, *Proceedings of the International Conference NEEDS '92* (World Scientific, Singapore, 1993).
[4] A. B. Shabat and R. I. Yamilov, ''Symmetries of nonlinear chains,'' Alg. Anal. **2**, 183–208 (1990) (in Russian); English translation in Leningrad Math. J. **2**, 377–400 (1991).
[5] D. Levi and O. Ragnisco, ''Nonlinear differential difference equations with n-dependent coefficients I,'' J. Phys. A **12**, L157–L162 (1979); ''Nonlinear differential difference equations with n-dependent coefficients II,'' *ibid.* **12**, L163–L167 (1979); ''The inhomogeneous Toda Lattice: Its hierarchy and Darboux–Backlund transformations,'' *ibid.* **24**, 1729–39 (1991); R. Benguria and D. Levi, ''Backlund transformations and nonlinear differential difference equations,'' Proc. Natl. Acad. Sci. USA **77**, 5025–5027 (1980); D. Levi, L. Pilloni, and P. M. Santini, ''Integrable three dimensional lattices,'' J. Phys. A **14**, 1567–1575 (1981); M. Bruschi, D. Levi, and O. Ragnisco, ''Evolution equations associated with the discrete analogue of the matrix Schroedinger spectral problem solvable by the inverse spectral transform,'' J. Math. Phys. **22**, 2463–2471 (1981); ''Toda lattice and generalized Wronskian technique,'' J. Phys. A **13**, 2531–2533 (1980); ''The discrete chiral field hierarchy,'' Lett. Nuovo Cimento **33**, 284–288 (1982); ''Continuous and discrete matrix Burgers hierarchies,'' Nuovo Cimento B **74**, 33–51 (1983).
[6] C. Claude, Yu. S. Kivshar, O. Kluth, and K.H. Spatschek, ''Moving localized modes in nonlinear lattices,'' Phys. Rev. B **47**, 14 228 (1993).
[7] V. V. Sokolov and A. B. Shabat, ''Classification of integrable evolution equations,'' Space Sci. Rev. **4**, 221–280 (1984).
[8] A. V. Mikhailov, A. B. Shabat, and R. I. Yamilov, ''The symmetry approach to the classification of nonlinear equations. Complete lists of integrable systems,'' Usp. Mat. Nauk **42**, 3–53 (1987) (in Russian); English transl, in Russ. Math Surv. **42**, 1–63 (1987).
[9] A. V. Mikhailov, A. B. Shabat, and V. V. Sokolov, ''The symmetry approach to classification of integrable guations,'' in *What is Integrability*?, Springer Series in Nonlinear Dynamics (Springer-Verlag, Berlin, 1991), pp. 115–184.
[10] A. V. Bocharov, ''DEliA: A system of exact analysis of differential equations using S. Lie approach,'' report by Joint Venture OWIMEX Program Systems Institute of the USSR (Academy of Sciences, PereslavlZalessky, USSR, 1989); ''DEliA: project presentation,'' SIGSAM Bull. **24**, 37–38 (1990); ''Will DEliA grow into an expert system?,'' in *Design and Implementation of Symbolic Computation Systems*, *Proceedings of the International Symposium*, Capri, Italy, 1990, edited by A. Miola, Lecture Notes in Computer Science 429 (Springer-Verlag, Berlin, 1990), pp. 266–267.
[11] R. I. Yamilov, ''Classification of discrete integrable equations,'' Ph.D. thesis, Ufa, (1984) (in Russian).
[12] D. Levi and P. Winternitz, ''Symmetries of discrete dynamical systems,'' J. Math. Phys. **37**, 5551–5576 (1996).
[13] A. Campa, A. Giansanti, A. Tenenbaum, D. Levi, and O. Ragnisco, ''Quasi solitons on a diatomic chain at room temperature,'' Phys. Rev. B **48**, 10 168–10 182 (1993).
[14] I. M. Krichever and S. P. Novikov, ''Holomorphic bundles over algebraic curves and non-linear equations,'' Usp. Mat. Nauk. 35, 47–68 (1980) (in Russian); English translation in Russ. Math Surv. **35**, 53–80 (1980).
[15] B. Fuchssteiner and A. S. Fokas, ''Symplectic structures, their Bäcklund transformations and hereditary symmetries,'' Physica D **4**, 47–66 (1981).