Yvan Saint-Aubin
Luc Vinet
Editors

Algebraic Methods in Physics

A Symposium for the 60th Birthdays of Jiří Patera and Pavel Winternitz

With 17 Illustrations

Springer

Yvan Saint-Aubin
Département de mathématiques
et de statistique
CRM, Université de Montréal
C.P. 6128, succ. Centre-ville
Montréal, Québec H3C 3J7
Canada
saint@math.ias.edu

Luc Vinet
Département de mathématiques
et de statistique
CRM, Université de Montréal
C.P. 6128, succ. Centre-ville
Montréal, Québec H3C 3J7
Canada
vinet@crm.umontreal.ca

Library of Congress Cataloging-in-Publication Data
Algebraic methods in physics : A symposium for the 60th
birthdays of Jiří Patera and Pavel Winternitz
Yvan Saint-Aubin, Luc Vinet [editors].
p. cm.—(The CRM series in mathematical physics)
Includes bibliographical references and index.
ISBN 978-1-4612-6528-3 ISBN 978-1-4613-0119-6 (eBook)
DOI 10.1007/978-1-4613-0119-6
1. Mathematical physics—Congresses. 2. Algebra—
Congresses. I. Patera, Jiří. II. Winternitz, Pavel
III. Saint-Aubin, Yvan. IV. Vinet, Luc. V. Series: CRM series
in mathematical physics.
QC19 .2 .A44 2000
530 .15′2 21 00-61863

Printed on acid-free paper.

Production managed by A. Orrantia; manufacturing supervised by Jeffrey Taub.
Photocomposed copy prepared by the editors.

9 8 7 6 5 4 3 2 1

ISBN 978-1-4612-6528-3 SPIN 10778273

10

Conditions for the Existence of Higher Symmetries and Nonlinear Evolutionary Equations on the Lattice

D. Levi and R. Yamilov

ABSTRACT In this paper we derive a set of five conditions necessary for the existence of generalized symmetries. We apply them to a class of dynamical equations on the lattice, depending on nearest neighboring interaction, which has a four-dimensional Lie group of continuous point symmetries.

1 Introduction

When considering differential equations we have many effective tools to integrate them, for example, Lie Group techniques [2, 12, 16]. This is not the case for difference equations, where few results are known [5–8, 13]. Thus it is very important to introduce new tools to be able to treat difference equations, as often discrete equations present "discrete" features that are lost in the continuum limit approximation. This is especially the case for nonlinear differential-difference equations, which are important in applications.

As in the case of partial differential equations, there are a certain number of difference equations that are integrable [1, 4], as, for example, the Toda lattice, which has a Lax pair, Bäcklund transformations, an infinity of conserved quantities and symmetries, and an infinity of explicit exact solutions. However, in order to improve the techniques for studying difference equations we first need us to find new integrable difference equations. There are various possible approaches to this problem. One possibility is to construct integrable difference equations of a given form by starting from the integrability conditions (i.e., Lax pair). A second possibility is to start from a class of equations of the desired form and deduce the integrable cases by applying some integrability conditions.

This second approach will be considered in the following. This relies on the so-called formal symmetry approach introduced by A.B. Shabat and collaborators in Ufa (see, e.g., review articles [10, 11, 15]) by which the authors classified all equations of a certain class that possess a few generalized symmetries of a certain kind. Such an approach has been introduced at first to study partial differential equations but then the procedure has been extended to the case of differential-difference equations [14, 17, 18]. In such an approach, one introduces conditions under which one can prove the existence of at least one (or more) generalized symmetries.

The class of nonlinear differential-difference equations we will consider in the following is given by

$$u_{n,t}(t) = f_n\big(u_{n-1}(t), u_n(t), u_{n+1}(t)\big), \tag{1.1}$$

where $u_n(t)$ is a dependent field expressed in terms of its independent variables, t varying over the reals and n varying over the integers. Equation (1.1) is a differential functional relation that correlates the time evolution of a function calculated at the point n to its values in its nearest neighboring points $(n+1, n-1)$. A peculiarity of the choice of Eq. (1.1) is the fact that its right-hand side is not a function—i.e., it is not the same for all points in the lattice—but for each point of the lattice one has an a priori different right-hand side. In fact, we can think of Eq. (1.1) as an infinite system of different differential equations for the infinite number of functions u_n. By proper choices of the functions f_n, Eq. (1.1) can be reduced to a system of k coupled differential-difference equations for the k unknown u_m^k or to a system of dynamical equations on the lattice. Let us assume that f_n and u_n are periodic functions of n of period k, i.e.,

$$f_n\big(u_{n-1}(t), u_n(t), u_{n+1}(t)\big) = \sum_{j=0}^{k-1} P_{n-j}^k f^j\big(u_{m-1}(t), u_m(t), u_{m+1}(t)\big),$$
$$u_n = \sum_{j=0}^{k-1} P_{n-j}^k u_m^j,$$

where P_n^k is a projection operator such that for any integer m, such that $n = km + j$ with $0 \le j \le k-1$. The following relations are true:

$$P_{km}^k = 1, \quad P_{km+j}^k = 0, \quad (j = 1, 2, \ldots, k-1);$$

then Eq. (1.1) becomes, for example, in the case $k = 2$, the system

$$\begin{aligned} u_{m,t}^0 &= f^0(u_{m-1}^1, u_m^0, u_m^1), \\ u_{m,t}^1 &= f^1(u_m^0, u_m^1, u_{m+1}^0). \end{aligned} \tag{1.2}$$

A subclass of Eq. (1.2) of particular relevance for its physical applications is given by dynamical systems on the lattice, i.e., equations of the type

$$\chi_{n,tt} = g(\chi_{n+1} - \chi_n, \chi_n - \chi_{n-1}). \tag{1.3}$$

So part of the results presented in Ref. [8] can be further analyzedd for the existence of generalized symmetries and of their integrability using the conditions presented here as these equations can be written in the form (1.3).

Section 2 is devoted to the construction of a certain number of conditions (the simpler ones) necessary to prove that an equation of the class (1.1) has generalized symmetries and higher-order conservation laws. The obtained conditions thus obtained are applied in Section 3 to the case

$$u_{n,t} = P_{n+1}^2 e^{u_{n+1}} g_n(u_{n+1} - u_{n-1}) + P_{n}^2, \lambda_n(u_{n+1} - u_{n-1}), \tag{1.4}$$

where $g_n(\zeta_n)$ is a set of functions and λ_n is a set of constants. This equation describes a class of dynamical equations

$$\begin{aligned} v_{ktt} &= \exp\left(\frac{v_{k+1} - v_k}{\varepsilon_{k+1}}\right) G_k\left(\frac{v_{k+1} - v_k}{\varepsilon_{k+1}} - v_k - v_{k-1}\varepsilon_k\right), \\ u_{2k} &= \frac{v_{k+1} - v_k}{\varepsilon_{k+1}}, \quad u_{2k-1} = v_{kt}, \quad g_{2k-1} = G_k, \quad \lambda_{2k} = \varepsilon_{k+1}^{-1}, \end{aligned} \tag{1.5}$$

having a four-dimensional group of point symmetries and including the Toda lattice as one of its members [8]. A more complete set of results on the conditions associated with the class of Eqs. (1.1), together with other examples of equations that pass the test, can be found in Ref. [9].

2 Construction of the Classifying Conditions

Before considering in detail the problem of constructing generalized symmetries to Eq. (1.1) we will introduce a few definitions necessary for future calculations.

A function g_n depending on the set of fields u_n, for n varying on the lattice, will be called a *restricted function* and will be denoted by the symbol RF if it is defined on a compact support; i.e., if

$$g_n = g_n(u_{n+i}, u_{n+i-1}, \ldots, u_{n+j+1}, u_{n+j}), \quad i \geq j, \tag{2.1}$$

and i and j are finite integer numbers. If there exist, in the range of possible values of n, values k and m such that

$$\frac{\partial g_k}{\partial u_{k+i}} \neq 0, \quad \frac{\partial g_m}{\partial u_{m+j}} \neq 0, \tag{2.2}$$

then we say that the function g_n has a *length* $i - j + 1$.

Let us define the shift operator D by

$$D g_n(u_{n+i}, \ldots, u_{n+j}) = g_{n+1}(u_{n+i+1}, \ldots, u_{n+j+1}).$$

Then we can split the RF into equivalence classes.

Definition. Two RF $a_n(u_{n+i_a}, \ldots, u_{n+j_a})$ and $b_n(u_{n+i_b}, \ldots, u_{n+j_b})$ are said to be *equivalent*, denoted by $a_n \sim b_n$, if

$$a_n - b_n = (D-1)c_n, \tag{2.3}$$

where c_n is an RF.

Let us notice that any function that is equal to a total difference is equivalent to zero; i.e., $a_n = (D-1)c_n \sim 0$. If an RF a_n of length $i-j+1$ $(i > j)$ is equivalent to zero, then there will exist, by necessity, a RF c_n of length $i-j$ such that $a_n = (D-1)c_n$ and consequently

$$\frac{\partial^2 a_n}{\partial u_{n+i} \partial u_{n+j}} = 0. \tag{2.4}$$

In the case $i = j$, $(da_n)/(du_{n+i}) = 0$, i.e., a_n is an *invariant* function, that is a function which depends only on n.

We can moreover define the *"formal" variational derivative* of an RF a_n of length $i-j+1$ as

$$\frac{\delta a_n}{\delta u_n} = \sum_{k=n-i}^{n-j} \frac{\partial a_k}{\partial u_n}. \tag{2.5}$$

If a_n is linear in u_n, then $\delta a_n / \delta u_n$ is an invariant function. However, if it is nonlinear, then $\delta a_n/\delta u_n = \widetilde{g}_n(u_{n+N}, \ldots, u_{n-N})$, where $\partial \widetilde{g}_k / \partial u_{k+N} \neq 0$ for some k and $\partial \widetilde{g}_m / \partial u_{m-N} \neq 0$ for some m. It is easy to prove that if a_n is an RF equivalent to zero, then the formal variational derivative of a_n is zero. The inverse is also true; i.e., if $\delta a_n / \delta u_n = 0$, then a_n is equivalent to zero.

Given a nonlinear chain (1.1), we will say that the RF $g_n(u_{n+i}, \ldots, u_{n+j})$ is a generalized (or higher) local symmetry of *order* i (more precisely, of left order i) of our equation if

$$u_{n,\tau} = g_n(u_{n+i}, \ldots, u_{n+j}), \tag{2.6}$$

is compatible with (1.1), i.e., if

$$\partial_t \partial_\tau (u_n) = \partial_\tau \partial_t (u_n). \tag{2.7}$$

Making explicit condition (2.7), we get

$$(\partial_t - f_n^*) g_n = 0, \tag{2.8}$$

where by f_n^* we mean the Frechet derivative of the function f_n, given by

$$f_n^* = \frac{\partial f_n}{\partial u_{n+1}} D + \frac{\partial f_n}{\partial u_n} + \frac{\partial f_n}{\partial u_{n-1}} D^{-1} = f_n^{(1)} D + f_n^{(0)} + f_n^{(-1)} D^{-1}. \tag{2.9}$$

Equation (2.8) is an equation for the symmetries g_n once the function f_n is given.

Given a symmetry we can construct a new symmetry by applying a recursive operator, i.e., an operator that transforms symmetries into symmetries. Given a symmetry g_n of Eq. (1.1), an operator

$$L_n = \sum_{j=-\infty}^{m} l_n^{(j)}(t)D^j \tag{2.10}$$

is a recursive operator for Eq. (1.1) if $\widetilde{g}_n$ given by

$$\widetilde{g}_n = L_n g_n \tag{2.11}$$

is a new generalized symmetry associated with (1.1). Equations (2.8) and (2.11) imply that

$$A(L_n) = L_{n,t} - [f_n^*, L_n] = 0. \tag{2.12}$$

Moreover from (2.8) it follows that

$$A(g_n^*) = \partial_\tau(f_n^*). \tag{2.13}$$

Equation (2.13) implies that, as its right-hand side is an operator of order 1 (see (2.9)), the highest terms in the left-hand side must be zero.

Thus we can define as *approximate* symmetry of *order* i and *length* m, the operator $G_n = \sum_{k=i-m+1}^{i} g_n^{(k)}D^k$, such that the m terms of highest order of the operator $A(G_n) = \sum_{k=i-m}^{i+1} a_n^{(k)}D^k$ are zeros. Taking into account Eq. (2.13), we find that we must have $i - m + 2 > 1$ if the equation

$$A(G_n) = 0, \tag{2.14}$$

is to be satisfied. From this result we can derive the first integrability condition, which can be stated in the following theorem:

Theorem 2.1. *If Eq.* (1.1) *has a local generalized symmetry of order* $i \geq 2$, *then it must have a conservation law given by*

$$\partial_t \log f_n^{(1)} = (D-1)q_n^{(1)}, \tag{2.15}$$

where $q_n^{(1)}$ *is an RF.*

The next canonical conservation laws could be obtained in the same way, namely, by assuming the existence of a higher symmetry so that we may consider an approximate symmetry of higher length. This procedure can be carried out, and leads to the following theorem:

Theorem 2.2. *If Eq.* (1.1) *has two generalized local symmetries of order* i *and* $i+1$, *with* $i \geq 4$, *then the following conservation laws must be true:*

$$\begin{gathered} \partial_t p_n^{(k)} = (D-1)q_n^{(k)} \quad (k = 1, 2, 3), \\ p_n^{(1)} = \log \frac{\partial f_n}{\partial u_{n+1}}, \quad p_n^{(2)} = q_n^{(1)} + \frac{\partial f_n}{\partial u_n}, \\ p_n^{(3)} = q_n^{(2)} + \frac{1}{2}(p_n^{(2)})^2 + \frac{\partial f_n}{\partial u_{n+1}}\frac{\partial f_{n+1}}{\partial u_n}, \end{gathered} \tag{2.16}$$

where $q_n^{(k)}$ $(k = 1, 2, 3)$ *are some RFs.*

Thus, if Eq. (1.1) has local generalized symmetries of high enough order, we can construct a few conservation laws depending on the function on the right-hand side of Eq. (1.1).

One can divide the conservation laws into conjugacy classes under an equivalence condition. Two conservation laws $p_{n,t} = (D-1)q_n$ and $r_{n,t} = (D-1)s_n$ are *equivalent* if

$$p_n \sim r_n. \tag{2.17}$$

A local conservation law is *trivial* if $p_n \sim 0$. If $p_n \sim r_n(u_n)$, with $r_n' \neq 0$ for at least some n, then we have a conservation law of *zeroth order*, while if $p_n \sim r_n(u_{n+N}, \dots, u_n)$, $N > 0$, and $(\partial^2 r_n)/(\partial u_n \partial u_{n+N}) \neq 0$ for at least some n, the conservation law is of *order* N.

An alternative way to define equivalence classes of local conservation laws is via the formal variational derivative. Let us denote by $\widetilde{p}_n$ the formal variational derivative of the density p_n of a local conservation law, i.e.,

$$\widetilde{p}_n = \frac{\delta p_n}{\delta u_n}. \tag{2.18}$$

If the local conservation law is trivial, then $\widetilde{p}_n = 0$. If it is of order zero, then $\widetilde{p}_n = \widetilde{p}_n(u_n) \neq 0$ for at least some n. And if it is of order N, then $\widetilde{p}_n = \widetilde{p}_n(u_{n+N}, \dots, u_n, \dots, u_{n-N})$, where $\partial\widetilde{p}_n/\partial u_{n+N} \neq 0$, $(\partial\widetilde{p}_n)/(\partial u_{n-N}) \neq 0$ for at least some n. Then, for any conserved density p_n, by direct calculation, we derive the following relation:

$$p_{n,t} \sim \widetilde{p}_n f_n \sim 0. \tag{2.19}$$

By carrying out the formal variational derivative of Eq. 2.19 we get that the formal variational derivative $\widetilde{p}_n$ of a conserved density p_n satisfies the following equation:

$$(\partial_t + f_n^{*T})\widetilde{p}_n = 0, \tag{2.20}$$

where the transposed Frechet derivative of f_n is given by

$$f_n^{*T} = \frac{\partial f_{n+1}}{\partial u_n} D + \frac{\partial f_n}{\partial u_n} + \frac{\partial f_{n-1}}{\partial u_n} D^{-1} = f_{n+1}^{(-1)} D + f_n^{(0)} + f_{n-1}^{(1)} D^{-1}. \tag{2.21}$$

Let us consider the Frechet derivative of $\widetilde{p}_n$ for a local conservation law of order N. In this case, we have

$$\widetilde{p}_n^* = \sum_{k=-N}^{N} \widetilde{p}_n^{(k)} D^k, \quad \widetilde{p}_n^{(k)} = \frac{\partial \widetilde{p}_n}{\partial u_{n+k}}. \tag{2.22}$$

We can construct the following operator:

$$B(S_n) = S_{n,t} + S_n f_n^* + f_n^{*T} S_n, \tag{2.23}$$

where

$$S_n = \sum_{j=-\infty}^{l} s_n^{(j)}(t) D^j. \tag{2.24}$$

Taking into account (2.8) and (2.20), one can easily prove that

$$\widetilde{p}_n = S_n g_n. \tag{2.25}$$

Let us construct

$$B(\widetilde{p}_n^*) = \sum_k b_n^{(k)}(t) D^k, \tag{2.26}$$

where $b_n^{(k)}(t)$ are some RFs. It then follows from Eq. (2.23) that $b_n^{(k)}$ are different from zero only for $-2 \leq k \leq 2$.

In this way, for a sufficiently high order conserved density p_n, we can require that

$$B(\widetilde{p}_n^*) = 0 \tag{2.27}$$

is approximately solved. If the first $m < N-1$ terms of the Frechet derivative of $\widetilde{p}_n$ satisfy Eq. (2.27), then we say that we have an *approximate* conserved density of *order* N and *length* m. Taking into account all the results up to now, we can state the following theorem:

Theorem 2.3. *If the chain* (1.1) *has a conservation law of order* $N \geq 3$, *and condition* (2.16) *is satisfied, then the following conditions must take place:*

$$r_n^{(k)} = (D-1)s_n^{(k)} \quad (k = 1, 2), \tag{2.28a}$$

with

$$r_n^{(1)} = \log[-f_n^{(1)}/f_n^{(-1)}], \quad r_n^{(2)} = s_{n,t}^{(1)} + 2f_n^{(0)}, \tag{2.28b}$$

where $s_n^{(k)}$ *are RFs.*

Conditions (2.16) require only that $f_n^{(1)} \neq 0$. An analogous set of conditions could be derived if we required that just $f_n^{(-1)} \neq 0$ for all n. They can be derived in a straightforward way by considering expansions in negative powers of D instead of positive, as we have done up to now. This set of conditions also will have the form of canonical conservation laws,

$$\partial_t \widehat{p}_n^{(k)} = (D-1)\widehat{q}_n^{(k)}, \tag{2.29}$$

and conserved densities will be symmetric to those of (2.16). For example,

$$\widehat{p}_n^{(1)} = \log\left(-\frac{\partial f_n}{\partial u_{n-1}}\right). \tag{2.30}$$

Let us notice moreover that if $H_n^{(1)}$ and $H_n^{(2)}$ are two solutions of (2.27) of different order, the operator

$$K_n = (H_n^{(1)})^{-1} H_n^{(2)}, \tag{2.31}$$

satisfies (2.12) and thus it is a recursive operator. Consequently, if we start from two approximate solutions of (2.27), i.e., two Frechet derivatives of formal variational derivatives of conserved densities, we can, using (2.27), get an approximate symmetry.

If we compare conditions (2.16), (2.28), and (2.29), we can see, for example, that

$$r_n^{(1)} = p_n^{(1)} - \widehat{p}_n^{(1)}; \tag{2.32}$$

i.e., the first of conditions (2.28) implies that $p_n^{(1)} \sim \widehat{p}_n^{(1)}$. Thus the first canonical conservation laws of (2.16) and (2.29) are equivalent.

The solutions of the conditions, be they those obtained by requiring existence of the generalized symmetries or those of local conservation laws, provide the highest-order coefficients of the Frechet derivative of a symmetry or of the formal variational derivative of a conserved density. Those coefficients are the building blocks for the reconstruction of the symmetries or of the formal variational derivatives of the conserved densities. In fact, the knowledge of $g_n^{(k)} = (\partial g_n)/(\partial u_{n+k})$ with $k = i, i-1, \ldots$ for a few values of k, gives a set of partial right-hand equations for g_n with respect to its variables, whose solution provides the needed symmetry. In the same way we can reconstruct variational derivatives of conserved densities. There is, however, a more direct way to obtain conserved densities. In fact, if we know the highest coefficients of L_n, the solution of Eq. (2.12), we can obtain several conserved densities by computing $p_n^{(j)} = \mathrm{res}(L_n^j)$ for $j = 1, 2, \ldots$, where $\mathrm{res}(L_n^j)$ denotes the coefficient of D^0 in L_n^j.

It is worthwhile to show here how all five conditions (2.16), (2.28) can be rewritten in explicit form. Such explicit conditions can be easily verified using a computer and thus they can be the starting point for the construction of a program like DELIA [3] to check the integrability of differential-difference equations of the form (1.1).

A condition is *explicit* if it has the form $A_n = 0 \ \forall n$, where A_n is a function depending only on f_n and its partial derivatives with respect to all u_{n+i}. Let us define the functions $P_n^{(k)} = \delta/\delta u_n \partial_t p_n^{(k)}$, $R_n^{(k)} = \delta/\delta u_n r_n^{(k)}$ for $k = 1, 2$ and $P_n^{(3)} = \delta/\delta u_n \partial_t p_n^{(3)} - q_n^{(1)} P_n^{(2)}$. The five explicit conditions are given by

$$P_n^{(k)} = 0, \quad R_n^{(l)} = 0 \quad \forall n, \quad k = 1, 2, 3; \quad l = 1, 2. \tag{2.33}$$

The functions $p_n^{(1)}$, $r_n^{(1)}$ are already explicit, and from them one can derive all partial derivatives $\partial q_n^{(1)}/\partial u_{n+i}$, $\partial s_n^{(1)}/\partial u_{n+i}$ and then express $\partial_t p_n^{(2)}$, $r_n^{(2)}$ in an explicit form. For example, from (2.28) we have $r_n^{(2)} = \partial r_{n-1}^{(1)}/\partial u_n f_n - \partial r_n^{(1)}/\partial u_{n-1} f_{n-1} + 2\partial f_n/\partial u_n$. Let us now consider $P_n^{(3)}$. We have

$$\partial_t p_n^{(3)} = \partial_t \left(q_n^{(2)} + \frac{\partial f_n}{\partial u_{n+1}} \frac{\partial f_{n+1}}{\partial u_n} \right) + \left(q_n^{(1)} + \frac{\partial f_n}{\partial u_n} \right) \partial_t p_n^{(2)}.$$

Using (2.16) with $k = 2$, one can find all the partial derivatives of $q_n^{(2)}$ and consequently get an explicit expression for $\partial_t q_n^{(2)}$. Using (2.16) with $k = 1$, we can obtain not only the functions $\partial q_{n+i}^{(1)}/\partial u_n$ but also all differences $q_{n+i}^{(1)} - q_n^{(1)}$. Consequently, as

$$\begin{aligned}\frac{\delta}{\delta u_n}&(q_n^{(1)}\partial_t p_n^{(2)}) - q_n^{(1)} P_n^{(2)} \\ &= \sum_i \frac{\partial q_{n+i}^{(1)}}{\partial u_n}\partial_t p_{n+i}^{(2)} + \sum_i (q_{n+i}^{(1)} - q_n^{(1)})\frac{\partial}{\partial u_n}\partial_t p_{n+i}^{(2)}, \qquad (2.34)\end{aligned}$$

we can easily write down the explicit form of $P_n^{(3)}$.

3 The Toda Lattice Class

In the following we learn about integrability of differential-difference equations of the form (1.4). The following conditions must be imposed to ensure that (1.1) represents an evolutionary difference equation:

$$\frac{\partial f_n}{\partial u_{n+1}} = P_{n+1}^2 e^{u_{n+1}}(g_n + g_n') + P_n^2 \lambda_n \neq 0, \qquad (3.1)$$

$$-\frac{\partial f_n}{\partial u_{n-1}} = P_{n+1}^2 e^{u_{n+1}} g_n' + P_n^2 \lambda_n \neq 0. \qquad (3.2)$$

This means, in particular, that $\lambda_n \neq 0$ for n even. As the λ_n do not exist in our equation for n odd, we can make them arbitrary for n odd and then assume that $\lambda_n \neq 0$ for all n. Analogously, we have to require that $g_n' \neq 0$, $g_n + g_n' \neq 0$ for all n. We can than formulate the following theorem:

Theorem. *A chain of the form* (1.5) *satisfies the classifying conditions* (2.16), (2.28) *iff it is related by a point transformation of the form* $\widetilde{u}_n = \alpha_n u_n + \beta_n$ *to one of two following chains:*

$$u_{n,t} = P_{n+1}^2(\exp u_{n+1} - \exp u_{n-1}) + P_n^2(u_{n+1} - u_{n-1}), \qquad (3.3a)$$

$$u_{n,t} = P_{n+1}^2 \exp\left(\frac{u_{n+1} - u_{n-1}}{a_n}\right) + P_n^2 a_{n+1} a_{n-1}(u_{n+1} - u_{n-1}), \qquad (3.3b)$$

with $a_n = \alpha n + \beta$ *where* α *and* β *are arbitrary constants.*

To prove this theorem we first consider the conditions (2.28). As we have (2.32),

$$p_n^{(1)} = \log\frac{\partial f_n}{\partial u_{n+1}} = P_{n+1}^2\bigl(u_{n+1} + \log(g_n + g_n')\bigr) + P_n^2 \log\lambda_n, \qquad (3.4)$$

$$\widetilde{p}_n^{(1)} = \log\left(-\frac{\partial f_n}{\partial u_{n-1}}\right) = P_{n+1}^2(u_{n+1} + \log g_n') + P_n^2 \log\lambda_n, \qquad (3.5)$$

and consequently $r_n^{(1)} = P_{n+1}^2 H_n$, where $H_n = \log(g_n + g_n') - \log g_n'$. Hence

$$P_{n+1}^2 H_n'' = 0, \quad \text{i.e., } P_{n+1}^2 H_n = a_n v_n + b_n, \tag{3.6}$$

where $v_n = u_{n+1} - u_{n-1}$, and a_n and b_n are constants that depend on n. Consequently we must have $a_{n-1} = a_{n+1}$ for all n. Thus from (3.6) we get that

$$a_n = P_{n+1}^2 a, \tag{3.7}$$

where a is a pure constant. So, the condition $r_n^{(1)} \sim 0$ implies (3.6) and (3.7). We find moreover that $r_n^{(2)} = \partial_t s_n^{(1)}$. Consequently

$$a P_{n+1}^2 (\lambda_{n-1} - \lambda_{n+1}) = 0. \tag{3.8}$$

Let us consider the first canonical conservation law. It follows from (2.32) that this condition is equivalent to $\partial_t \widetilde{p}_n^{(1)} \sim 0$. Since $\widetilde{p}_n^{(1)} \sim P_n^2 u_n + P_{n+1}^2 \log g_n'$ (see (3.5)). Then we must have

$$P_{n+1}^2 (\log g_n')'' = 0, \quad \text{i.e., } P_{n+1}^2 \log g_n' = c_n v_n + d_n, \tag{3.9}$$

where c_n and d_n are constants depending on n.

By comparing (3.6) and (3.9) we obtain an explicit formula for g_n. Since for any function θ_n the formula $P_{n+1}^2 \exp(P_{n+1}^2 \theta_n) = P_{n+1}^2 \exp \theta_n$ is valid, we get from Eq. (3.9)

$$P_{n+1}^2 g_n' = P_{n+1}^2 \exp(c_n v_n + d_n). \tag{3.10}$$

Using (3.6) we are led to the following formula for g_n:

$$P_{n+1}^2 g_n = P_{n+1}^2 \exp\big((a + c_n) v_n + b_n + d_n\big) - P_{n+1}^2 \exp(c_n v_n + d_n). \tag{3.11}$$

Moreover the consistency between (3.10) and (3.11) implies that we must have

$$a(a-1) = 0, \tag{3.12}$$

$$P_{n+1}^2 (1 + c_n) a = 0. \tag{3.13}$$

Let us first consider the case $a = 1$. Condition (3.13) gives $P_{n+1}^2 c_n = -P_{n+1}^2$ and it then follows from (3.8) that $\lambda_{2k} = \lambda_{2k-2} = \lambda$, where λ is a constant different from zero. Taking into account formula (3.11), we get that the chain thus obtained is of the form

$$u_{n,t} = P_{n+1}^2 \big(\exp(u_{n+1} + \alpha_n) - \exp(u_{n-1} + \beta_n)\big) + (u_{n+1} - u_{n-1}) P_n^2 \lambda,$$

where α_n, β_n are some constants depending on n. This chain can be further simplified, using simple point transformations. If we apply first the transformation $\widetilde{u}_n = (P^2_{n+1}\lambda + P^2_n)u_n$ and then $\widetilde{u}_n = u_n + P^2_n\alpha_{n-1}$, we can reduce it to the form

$$u_{n,t} = P^2_{n+1}\big(\exp u_{n+1} - \exp(u_{n-1} + \gamma_n)\big) + (u_{n+1} - u_{n-1})P^2_n,$$

where γ_n are some n-dependent constants. Moreover, we have $\partial_t p^{(2)}_n \sim 2P^2_n \exp(u_n)(1 - \exp\gamma_{n+1})$. Since $P^2_{n+1}\exp\gamma_n = P^2_{n+1}$ for all n, the chain takes the form (3.3a).

Let us consider now the case $a = 0$. It follows from (3.11) that $P^2_{n+1}g_n = P^2_{n+1}\exp(c_n v_n + d_n)(\exp b_n - 1)$ so that the function $\exp b_n - 1$ cannot be zero for odd n, as $g_n \neq 0$. This means we can redefine d_n so that (1.4) takes the form

$$u_{n,t} = P^2_{n+1}\exp\big[(1 + c_n)u_{n+1} - c_n u_{n-1} + d_n\big] + (u_{n+1} - u_{n-1})P^2_n\lambda_n. \quad (3.14)$$

As in the previous case, the chain (3.14) can be simplified, using point transformations of the form $\widetilde{u}_n = \alpha_n u_n + \beta_n$, and we get the following chain:

$$u_{n,t} = f_n = P^2_{n+1}\exp\frac{v_n}{a_n} + P^2_n b_n v_n, \quad (3.15)$$

where $v_n = u_{n+1} - u_{n-1}$ and a_n and b_n are some nonzero constants depending on n. To fix a_n and b_n we use two of the conditions (2.16) that will give us two other constraints. It is easy to see that $p^{(1)}_n = P^2_{n+1}(v_n)/(a_n) + \delta_n$, and that

$$\partial_t p^{(2)}_n \sim 2P^2_{n+1}\frac{b_{n-1}}{a_{n-2}} - \frac{b_{n+1}}{a_{n+2}}\exp\frac{v_n}{a_n}.$$

Thus we get the constraint

$$P^2_{n+1}A_n = 0. \quad (3.16)$$

Introducing $\tilde{b}_n$ such that $b_n = \tilde{b}_n a_{n+1}a_{n-1}$, we obtain from (3.16) that $P^2_n\tilde{b}_n = P^2_n b$, where b is a constant different from zero. Therefore

$$P^2_n b_n = bP^2_n a_{n+1}a_{n-1}, \quad (3.17)$$

and we obtain

$$P^2_{n+1}(a_{n+2} - 2a_n + a_{n-2}) = 0.$$

Taking into account the form of (3.15), it follows that we can set $a_n = cn+d$ for all n. The chain (3.15) with b_n satisfying (3.17) has the form

$$u_{n,t} = P^2_{n+1}\exp\frac{v_n}{a_n} + bP^2_n a_{n+1}a_{n-1}v_n,$$

i.e., coincides with (3.3b) up to the constant b. This constant, however, can be easily removed, using an obvious point transformation.

If we go over to the class (1.5), we see that, in case (3.3a),

$$\varepsilon_k = 1, \quad G_k = 1 - \exp\big((v_k - v_{k-1}) - (v_{k+1} - v_k)\big),$$

and this is the Toda model for the function v_k. The chain (3.3b) is a new example of an integrable (and n-dependent) equation. In this case, the chain equation can be rewritten, setting, for simplicity, $c_k = a_{2k-1}$ as

$$v_{k,tt} = \exp\big[c_{k+1}(v_{k+1} - v_k) - c_{k-1}(v_k - v_{k-1})\big]. \tag{3.18}$$

It belongs to the class (1.5), as

$$G_k(\zeta_k) = \exp(\delta_k \zeta_k), \quad \delta_k = c_{k-1}\varepsilon_k, \quad c_{k+1}\varepsilon_{k+1} - c_{k-1}\varepsilon_k = 1.$$

As c_k is linear in k, Eq. (3.18) can be written as

$$v_{k,tt} = \exp(c_{k+1}v_{k+1} - 2c_k v_k + c_{k-1}v_{k-1})$$

and by an obvious point transformation, we can remove the c_k and obtain the potential Toda equation:

$$v_{k,tt} = \exp(v_{k+1} - 2v_k + v_{k-1}). \tag{3.19}$$

This reduces to the Toda equation by the transformation

$$\widetilde{v}_k = v_{k+1} - v_k.$$

This implies that Eq. (3.18) is completely integrable.

In conclusion we state that the only equation in the class (1.4)–(1.5) that possesses generalized symmetries characterized by RF is the Toda lattice. Point symmetries by themselves are not sufficient to discriminate between integrable and nonintegrable equations. At least in the case of dynamical systems on the lattice depending on nearest neighbor interaction, the existence of a large group of continuous Lie point symmetries is not an indication that an equation is more integrable.

References

1. M.J. Ablowitz and J. Ladik, *Nonlinear differential-difference equations*, J. Math. Phys. **16** (1975), 598–603; *Nonlinear differential-difference equations and Fourier transform*, J. Math. Phys. **17** (1976), 1011–1018; *A nonlinear difference scheme and inverse scattering*, Stud. Appl. Math. **55** (1976) 213–229; *On the solution of a class of nonlinear partial difference equations*, Stud. Appl. Math. **57** (1976/77), No. 1, 1–12.

2. G.W. Bluman and S. Kumei, *Symmetries and differential equations*, Springer, New York, 1989.

3. A.V. Bocharov, *DEliA: A system of exact analysis of differential equations using S. Lie approach*, Report by Joint Venture OWIMEX Program Systems Institute of the USSR, Academy of Sciences, Pereslavl-Zalessky, USSR, 1989; *DEliA: project presentation*, SIGSAM Bulletin **24** (1990), 37–38; *Will DEliA grow into an expert system?*, Design and Implementation of Symbolic Computation Systems (A. Miola, ed.) (Capri, 1990), Lecture Notes in Comput. Sci., Vol. 429, Springer Verlag, Berlin, 1990, pp. 266–267.

4. D. Levi and O. Ragnisco, *Nonlinear differential-difference equations with n-dependent coefficients.* I, J. Phys. A **12** (1979), L157–L162; *Nonlinear differential-difference equations with n-dependent coefficients.* II, J. Phys. A **12** (1979), L163–L167; *The inhomogeneous Toda Lattice: Its hierarchy and Darboux–Bäcklund transformations*, J. Phys. A **24** (1991), 1729–1739; R. Benguria and D. Levi, *Bäcklund transformations and nonlinear differential-difference equations*, Proc. Nat. Acad. Sci. USA **77** (1980), No. 9, part 1, 5025–5027; D. Levi, L. Pilloni, and P.M. Santini, *Integrable three-dimensional lattices*, J. Phys. A **14** (1981), 1567–1575; M. Bruschi, D. Levi, and O.Ragnisco, *Evolution equations associated with the discrete analogue of the matrix Schrödinger spectral problem solvable by the inverse spectral transform*, J. Math. Phys. **22** (1981), 2463–2471; *Toda lattice and generalized Wronskian technique*, J. Phys. A **13** (1980), 2531–2533; *The discrete chiral field hierarchy*, Lett. Nuovo Cimento **33** (1982), 284–288; *Continuous and discrete matrix Burgers hierarchies*, Nuovo Cimento B (11) **74** (1983), 33–51.

5. D. Levi, L. Vinet, and P. Winternitz, *Lie group formalism for difference equations*, J. Phys. A **30** (1997), 633–649.

6. D. Levi and P. Winternitz, *Continuous symmetries of discrete equations*, Phys. Lett. A **152** (1991), 335–338.

7. D. Levi and P. Winternitz, *Symmetries and conditional symmetries of differential-difference equations*, J. Math. Phys. **34** (1993), 3713–3730.

8. D. Levi and P. Winternitz, *Symmetries of discrete dynamical systems*, J. Math. Phys. **37** (1996), 5551–5576.

9. D. Levi and R. Yamilov, *Conditions for the existence of higher symmetries of evolutionary equations on the lattice*, J. Math. Phys. **38** (1997), 6648–6674.

10. A.V. Mikhailov, A.B. Shabat, and V.V. Sokolov, *The symmetry approach to classification of integrable equations*, What is Integrability?, Springer, Berlin, 1991, pp. 115–184.

11. A.V. Mikhailov, A.B. Shabat, and R.I. Yamilov, *The symmetry approach to the classification of nonlinear equations. Complete lists of integrable systems*, Uspekhi Mat. Nauk **42** (1987), No. 4, 3–53 (Russian).

12. P.J. Olver, *Applications of Lie group to differential equations*, Springer, New York, 1986.

13. G.R. Quispel, H.W. Capel, and R. Sahavedan, *Continuous symmetries of difference equations; the Kac-van Moerbeke equation and Painlevé reduction*, Phys. Lett. A **170** (1992), 379–383.

14. A.B. Shabat and R.I. Yamilov, *Symmetries of nonlinear chains*, Algebra i Analiz **2** (1990), No. 2, 183–208 (Russian); English transl. Leningrad Math. J. **2** (1991), No. 2, 377–400.

15. V.V. Sokolov and A.B. Shabat, *Classification of integrable evolution equations*, Mathematical Physics Reviews, Vol. 4, Soviet Sci. Rev., Sect. C, Vol. 4, 1984, pp. 221–280.

16. P. Winternitz, *Lie groups and solutions of nonlinear partial differential equations*, Integrable Systems, Quantum groups and Quantum Field theories (L.A. Ibort and M.A. Rodriguez, eds.) Kluwer, Dordrecht, 1993, pp. 429–495.

17. R.I. Yamilov, *Classification of discrete evolution equations*, Uspekhi Mat. Nauk **38** (1983), No. 6, 155–156 (Russian).

18. R.I. Yamilov, *Generalizations of the Toda model and conservation laws*, Preprint, Inst. of Mathematics, Ufa, 1989 (Russian); *Classification of Toda type scalar lattices*, Proc. Internat. Conf. NEEDS'92, World Scientific, 1993, pp. 423-431.