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Darboux integrability of trapezoidal H^4 and H^4 families of lattice equations I: first integrals

G Gubbiotti[1] and R I Yamilov[2]

[1] Dipartimento di Matematica e Fisica, Università degli Studi Roma Tre and Sezione INFN di Roma Tre, Via della Vasca Navale 84, 00146 Roma, Italy

[2] Institute of Mathematics, Ufa Scientific Center, Russian Academy of Sciences, 112 Chernyshevsky Street, Ufa 450008, Russia

E-mail: gubbiotti@mat.uniroma3.it and RvlYamilov@matem.anrb.ru

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Abstract

In this paper we prove that the trapezoidal H^4 and the H^6 families of quad-equations are Darboux integrable by constructing their first integrals. This result explains why the rate of growth of the degrees of the iterates of these equations is linear (Gubbiotti *et al* 2016 *J. Nonlinear Math. Phys.* **23** 507–43), which according to the algebraic entropy conjecture implies linearizability. We conclude by showing how first integrals can be used to obtain general solutions.

Keywords: Darboux integrability, CAC, linearizable discrete equations, integrable discrete equations, general solutions

(Some figures may appear in colour only in the online journal)

1. Introduction

Since its introduction the integrability criterion denoted Consistency Around the Cube (CAC) has been a source of many results in the classification of nonlinear partial difference equations on a quad graph. The importance of this criterion relies on the fact that it ensures the existence of Bäcklund transformations [1–5] and, as a consequence, of Lax pairs. As it is well known [6], Lax pairs and Bäcklund transforms are associated with both linearizable and integrable equations. We point out that to be a *bona fide* Lax pair it has to give rise to a genuine spectral problem [7], otherwise the Lax pair is called *fake Lax pair* [8–12]. A fake Lax pair is useless in proving (or disproving) the integrability, since it can be equally found for integrable and non-integrable equations. In the linearizable case Lax pairs must be then fake ones, even if proving it is usually a highly nontrivial task [13].

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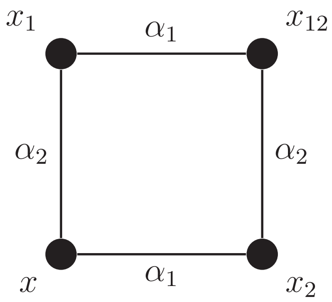

Figure 1. The purely geometric quad graph not embedded in any lattice.

A partial difference equations on a quad graph for an unknown function $u_{n,m}$, with $(n,m) \in \mathbb{Z}^2$, is a relation of the form:

$$f\left(u_{n,m}, u_{n+1,m}, u_{n,m+1}, u_{n+1,m+1}\right) = 0, \quad (n,m) \in \mathbb{Z}^2, \tag{1}$$

where $f = f(x, y, z, w)$ is a well defined function, i.e. *analytic* and *single-valued*, of its arguments. Partial difference equations on a quad graph are the discrete analogue of *hyperbolic equations*. Indeed to describe an arbitrary second order PDE a lattice depending on at least six points is needed. A possible choice is to take these points to be $u_{n,m}$, $u_{n+1,m}$, $u_{n,m+1}$, $u_{n+1,m+1}$, $u_{n+2,m}$, $u_{n,m+2}$. For PDEs which can be reduced to the form $u_{xy} = F(x, y, u, u_x, u_y)$, i.e. not involving u_{xx} and u_{yy}, the minimum number of points involved in a discretization is four, and it is sufficient to take these four points to be $u_{n,m}$, $u_{n+1,m}$, $u_{n,m+1}$, $u_{n+1,m+1}$ [14]. This means that the 'minimal' discretization of a hyperbolic equation is an equation of the form (1). The simplest possible equation of the form (1) is the *discrete wave* equation:

$$u_{n,m} - u_{n+1,m} - u_{n,m+1} + u_{n+1,m+1} = 0, \tag{2}$$

which arises from the discretization of the *wave equation* $u_{xy} = 0$ on an uniform grid.

The first attempt to classify all the multi-affine partial difference equations defined on the quad graph and possessing CAC was carried out in [15]. In [15] the quad graph was treated as a geometric object not embedded in any $\mathbb{Z}^2$-lattice, as displayed in figure 1. Then the quad-equation is an expression of the form:

$$Q\left(x, x_1, x_2, x_{12}; \alpha_1, \alpha_2\right) = 0, \tag{3}$$

connecting some *a priori* independent fields x, x_1, x_2, x_{12} assigned to the vertices of the quad graph, see figure 1. Q is assumed to be a *multi-affine* polynomial in x, x_1, x_2, x_{12} and, as shown in figure 1, α_1 and α_2 are parameters assigned to the edges of the quad graph.

In this setting, we define the consistency around the cube as follows: assume we are given six quad-equations:

$$A\left(x, x_1, x_2, x_{12}; \alpha_1, \alpha_2\right) = 0, \tag{4a}$$

$$\bar{A}\left(x_3, x_{13}, x_{23}, x_{123}; \alpha_1, \alpha_2\right) = 0, \tag{4b}$$

$$B\left(x, x_2, x_3, x_{23}; \alpha_3, \alpha_2\right) = 0, \tag{4c}$$

$$\bar{B}\left(x_1, x_{12}, x_{13}, x_{123}; \alpha_3, \alpha_2\right) = 0, \tag{4d}$$

$$C\left(x, x_1, x_3, x_{13}; \alpha_1, \alpha_3\right) = 0, \tag{4e}$$

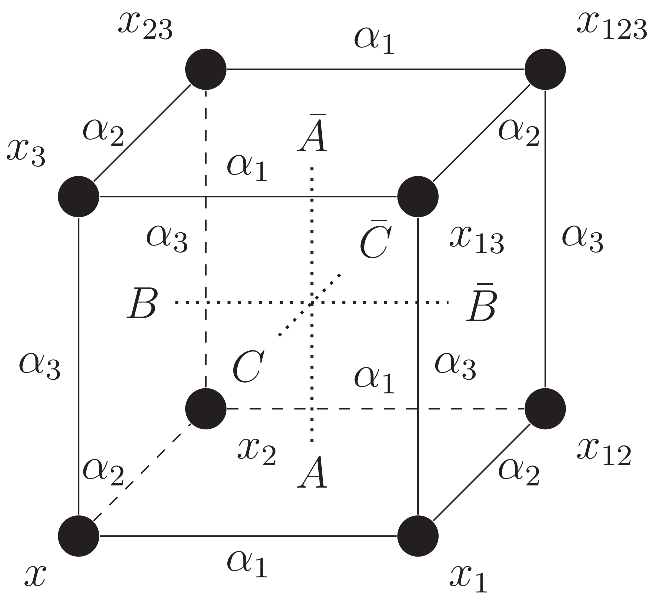

Figure 2. Equations on a cube.

$$\bar{C}\left(x_2,x_{12},x_{23},x_{123};\alpha_1,\alpha_3\right)=0, \tag{4f}$$

arranged on the faces of a cube as in figure 2. Then if x_{123} computed from (4*b*), (4*d*) and (4*f*) coincide we say that the system (4) possesses the consistency around the cube property.

In [15] the classification was carried out up to the action of a general Möbius transformation and up to point transformations of the edge parameters, with the additional assumptions:

(i) All the faces of the cube in figure 2 carry the same equation up to the edge parameters.
(ii) The quad-equation (3) possesses the D_4 discrete symmetries:

$$\begin{aligned}Q\left(x,x_1,x_2,x_{12};\alpha_1,\alpha_2\right)&=\mu Q\left(x,x_2,x_1,x_{12};\alpha_2,\alpha_1\right)\\&=\mu' Q\left(x_1,x,x_{12},x_2;\alpha_1,\alpha_2\right),\end{aligned} \tag{5}$$

where $\mu,\mu'\in\pm 1$.
(iii) The system (4) possesses the *tetrahedron property*, i.e. x_{123} is independent of x:

$$x_{123}=x_{123}\left(x,x_1,x_2,x_3;\alpha_1,\alpha_2,\alpha_3\right)\Longrightarrow\frac{\partial x_{123}}{\partial x}=0. \tag{6}$$

The result was the existence of two classes of discrete autonomous equations: the H and Q equations.

Releasing the hypothesis that every face of the cube carried the same equation, the same authors in [16] presented some new equations without classification purposes.

A complete classification in this extended setting was then accomplished by Boll in a series of papers culminating in his PhD thesis [17–19]. In these papers the classification of all the consistent sextuples of partial difference equations on the quad graph, i.e. systems of the form (4), has been carried out. The only technical assumption used in [17–19] is the tetrahedron property. The obtained equations may fall into three disjoint families depending on their bi-quadratics:

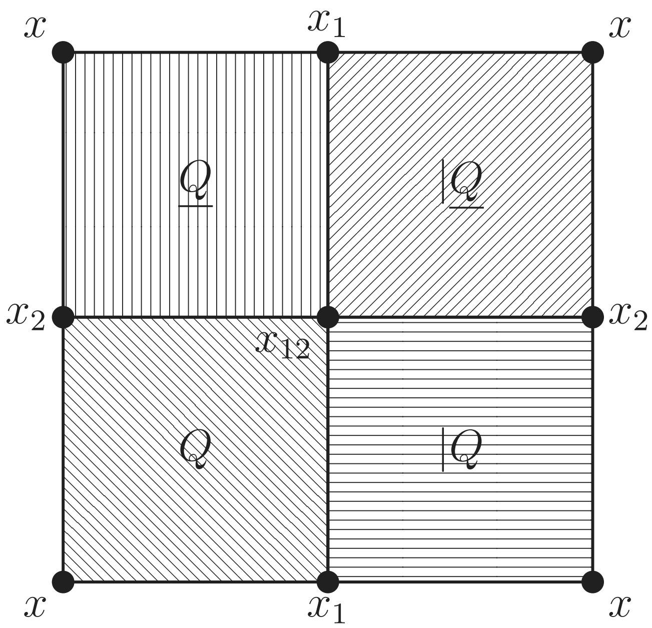

Figure 3. The 'four stripe' lattice.

$$
h_{ij} = \frac{\partial Q}{\partial y_k}\frac{\partial Q}{\partial y_l} - Q\frac{\partial^2 Q}{\partial y_k\,\partial y_l}, \qquad Q = Q\left(y_1, y_2, y_3, y_4; \alpha_1, \alpha_2\right), \tag{7}
$$

where we use a special notation for variables of Q, and the pair $\{k, l\}$ is the complement of the pair $\{i, j\}$ in $\{1, 2, 3, 4\}$. A bi-quadratic is called *degenerate* if it contains linear factors of the form $y_i - c$, where c is a constant, otherwise a bi-quadratic is called *non-degenerate*. The three families are classified depending on how many bi-quadratics are degenerate:

- Q-type equations: all the bi-quadratics are nondegenerate,
- H^4-type equations: four bi-quadratics are degenerate,
- H^6-type equations: all of the six bi-quadratics are degenerate.

Let us notice that the Q family is the same as that introduced in [15]. The H^4 equations are divided into two subclasses: *rhombic* and *trapezoidal*, depending on their discrete symmetries.

We remark that all classification results hold locally in the sense that they relate to a single quadrilateral cell or a single cube displayed in figures 1 and 2. The important problem of embedding these results into a two- or three-dimensional lattice, with preservation of the three-dimensional consistency condition, was already discussed in [16, 20] by using the concept of a Black and White lattice. One way to solve this problem is to embed (3) into a $\mathbb{Z}^2$-lattice with an elementary cell of size greater than one. In this case, the quad-equation (3) can be extended to a lattice, and the lattice equation becomes integrable or linearizable. To this end, following [17–19], we reflect the square with respect to the normal to its right and top sides and then complete a 2×2 lattice by again reflecting one of the obtained squares in the other direction. Such procedure is graphically described in figure 3.

This corresponds to constructing three equations obtained from (3) by flipping its arguments:

$$
Q = Q(x, x_1, x_2, x_{12}; \alpha_1, \alpha_2) = 0, \tag{8a}
$$

$$|Q = Q(x_1, x, x_{12}, x_2; \alpha_1, \alpha_2) = 0, \tag{8b}$$

$$\underline{Q} = Q(x_2, x_{12}, x, x_1; \alpha_1, \alpha_2) = 0, \tag{8c}$$

$$|\underline{Q} = Q(x_{12}, x_2, x_1, x; \alpha_1, \alpha_2) = 0. \tag{8d}$$

By paving the whole $\mathbb{Z}^2$ with such equations, we get a partial difference equation which can be in principle studied using known methods. Since *a priori* $Q \neq |Q \neq \underline{Q} \neq |\underline{Q}$, the obtained lattice will be a four stripe lattice, i.e. an extension of the Black and White lattice considered, for example, in [16, 20, 21]. This gives rise to lattice equations with two-periodic coefficients for an unknown function $u_{n,m}$, with $(n,m) \in \mathbb{Z}^2$:

$$\begin{aligned} & F_n^{(+)} F_m^{(+)} Q(u_{n,m}, u_{n+1,m}, u_{n,m+1}, u_{n+1,m+1}; \alpha_1, \alpha_2) \\ & + F_n^{(-)} F_m^{(+)} |Q(u_{n,m}, u_{n+1,m}, u_{n,m+1}, u_{n+1,m+1}; \alpha_1, \alpha_2) \\ & + F_n^{(+)} F_m^{(-)} \underline{Q}(u_{n,m}, u_{n+1,m}, u_{n,m+1}, u_{n+1,m+1}; \alpha_1, \alpha_2) \\ & + F_n^{(-)} F_m^{(-)} |\underline{Q}(u_{n,m}, u_{n+1,m}, u_{n,m+1}, u_{n+1,m+1}; \alpha_1, \alpha_2) = 0, \end{aligned} \tag{9}$$

where

$$F_k^{(\pm)} = \frac{1 \pm (-1)^k}{2}. \tag{10}$$

This explicit formula was first presented in [13]. For more details on the construction of equations on the lattice from the single cell equations, we refer to [17–20] and to the Appendix in [22].

A detailed study of all the lattice equations derived from the *rhombic* H^4 family, including the construction of their three-leg forms, Lax pairs, Bäcklund transformations and infinite hierarchies of generalized symmetries, has been presented in [20]. However, besides the CAC property, little was known about the integrability features of the *trapezoidal* H^4 equations and of the H^6 equations. These equations where thoroughly studied in a series of papers [13, 22–25] with some unexpected results. First in [22] their explicit non-autonomous form was presented, which was constructed using the rules above given. Indeed it was shown that on the $\mathbb{Z}^2$ lattice with coordinates (n,m) the trapezoidal H^4 equations had the following expression:

$$\begin{aligned} {}_tH_1: \ & (u_{n,m} - u_{n+1,m})(u_{n,m+1} - u_{n+1,m+1}) \\ & - \alpha_2 \varepsilon^2 \left(F_m^{(+)} u_{n,m+1} u_{n+1,m+1} + F_m^{(-)} u_{n,m} u_{n+1,m} \right) - \alpha_2 = 0, \end{aligned} \tag{11a}$$

$$\begin{aligned} {}_tH_2: \ & (u_{n,m} - u_{n+1,m})(u_{n,m+1} - u_{n+1,m+1}) \\ & + \alpha_2 (u_{n,m} + u_{n+1,m} + u_{n,m+1} + u_{n+1,m+1}) \\ & + \frac{\varepsilon \alpha_2}{2} \left(2F_m^{(+)} u_{n,m+1} + 2\alpha_3 + \alpha_2 \right) \left(2F_m^{(+)} u_{n+1,m+1} + 2\alpha_3 + \alpha_2 \right) \\ & + \frac{\varepsilon \alpha_2}{2} \left(2F_m^{(-)} u_{n,m} + 2\alpha_3 + \alpha_2 \right) \left(2F_m^{(-)} u_{n+1,m} + 2\alpha_3 + \alpha_2 \right) \\ & + (\alpha_3 + \alpha_2)^2 - \alpha_3^2 - 2\varepsilon \alpha_2 \alpha_3 (\alpha_3 + \alpha_2) = 0, \end{aligned} \tag{11b}$$

$$\begin{aligned} {}_tH_3: \ & \alpha_2 (u_{n,m} u_{n+1,m+1} + u_{n+1,m} u_{n,m+1}) \\ & - (u_{n,m} u_{n,m+1} + u_{n+1,m} u_{n+1,m+1}) - \alpha_3 (\alpha_2^2 - 1) \delta^2 \\ & - \frac{\varepsilon^2 (\alpha_2^2 - 1)}{\alpha_3 \alpha_2} \left(F_m^{(+)} u_{n,m+1} u_{n+1,m+1} + F_m^{(-)} u_{n,m} u_{n+1,m} \right) = 0, \end{aligned} \tag{11c}$$

and the H^6 equations had the following expression:

$$
\begin{aligned}
{}_1D_2\colon\ & \left(F_{n+m}^{(-)}-\delta_1 F_n^{(+)}F_m^{(-)}+\delta_2 F_n^{(+)}F_m^{(+)}\right)u_{n,m}\\
&+\left(F_{n+m}^{(+)}-\delta_1 F_n^{(-)}F_m^{(-)}+\delta_2 F_n^{(-)}F_m^{(+)}\right)u_{n+1,m}\\
&+\left(F_{n+m}^{(+)}-\delta_1 F_n^{(+)}F_m^{(+)}+\delta_2 F_n^{(+)}F_m^{(-)}\right)u_{n,m+1}\\
&+\left(F_{n+m}^{(-)}-\delta_1 F_n^{(-)}F_m^{(+)}+\delta_2 F_n^{(-)}F_m^{(-)}\right)u_{n+1,m+1}\\
&+\delta_1\left(F_m^{(-)}u_{n,m}u_{n+1,m}+F_m^{(+)}u_{n,m+1}u_{n+1,m+1}\right)\\
&+F_{n+m}^{(+)}u_{n,m}u_{n+1,m+1}+F_{n+m}^{(-)}u_{n+1,m}u_{n,m+1}=0,
\end{aligned}
\tag{12a}
$$

$$
\begin{aligned}
{}_2D_2\colon\ & \left(F_m^{(-)}-\delta_1 F_n^{(+)}F_m^{(-)}+\delta_2 F_n^{(+)}F_m^{(+)}-\delta_1\lambda F_n^{(-)}F_m^{(+)}\right)u_{n,m}\\
&+\left(F_m^{(-)}-\delta_1 F_n^{(-)}F_m^{(-)}+\delta_2 F_n^{(-)}F_m^{(+)}-\delta_1\lambda F_n^{(+)}F_m^{(+)}\right)u_{n+1,m}\\
&+\left(F_m^{(+)}-\delta_1 F_n^{(+)}F_m^{(+)}+\delta_2 F_n^{(+)}F_m^{(-)}-\delta_1\lambda F_n^{(-)}F_m^{(-)}\right)u_{n,m+1}\\
&+\left(F_m^{(+)}-\delta_1 F_n^{(-)}F_m^{(+)}+\delta_2 F_n^{(-)}F_m^{(-)}-\delta_1\lambda F_n^{(+)}F_m^{(-)}\right)u_{n+1,m+1}\\
&+\delta_1\left(F_{n+m}^{(-)}u_{n,m}u_{n+1,m+1}+F_{n+m}^{(+)}u_{n+1,m}u_{n,m+1}\right)\\
&+F_m^{(+)}u_{n,m}u_{n+1,m}+F_m^{(-)}u_{n,m+1}u_{n+1,m+1}-\delta_1\delta_2\lambda=0,
\end{aligned}
\tag{12b}
$$

$$
\begin{aligned}
{}_3D_2\colon\ & \left(F_m^{(-)}-\delta_1 F_n^{(-)}F_m^{(-)}+\delta_2 F_n^{(+)}F_m^{(+)}-\delta_1\lambda F_n^{(-)}F_m^{(+)}\right)u_{n,m}\\
&+\left(F_m^{(-)}-\delta_1 F_n^{(+)}F_m^{(-)}+\delta_2 F_n^{(-)}F_m^{(+)}-\delta_1\lambda F_n^{(+)}F_m^{(+)}\right)u_{n+1,m}\\
&+\left(F_m^{(+)}-\delta_1 F_n^{(-)}F_m^{(+)}+\delta_2 F_n^{(+)}F_m^{(-)}-\delta_1\lambda F_n^{(-)}F_m^{(-)}\right)u_{n,m+1}\\
&+\left(F_m^{(+)}-\delta_1 F_n^{(+)}F_m^{(+)}+\delta_2 F_n^{(-)}F_m^{(-)}-\delta_1\lambda F_n^{(+)}F_m^{(-)}\right)u_{n+1,m+1}\\
&+\delta_1\left(F_n^{(-)}u_{n,m}u_{n,m+1}+F_n^{(+)}u_{n+1,m}u_{n+1,m+1}\right)\\
&+F_m^{(-)}u_{n,m+1}u_{n+1,m+1}+F_m^{(+)}u_{n,m}u_{n+1,m}-\delta_1\delta_2\lambda=0,
\end{aligned}
\tag{12c}
$$

$$
\begin{aligned}
D_3\colon\ & F_n^{(+)}F_m^{(+)}u_{n,m}+F_n^{(-)}F_m^{(+)}u_{n+1,m}+F_n^{(+)}F_m^{(-)}u_{n,m+1}\\
&+F_n^{(-)}F_m^{(-)}u_{n+1,m+1}+F_m^{(-)}u_{n,m}u_{n+1,m}\\
&+F_n^{(-)}u_{n,m}u_{n,m+1}+F_{n+m}^{(-)}u_{n,m}u_{n+1,m+1}\\
&+F_{n+m}^{(+)}u_{n+1,m}u_{n,m+1}+F_n^{(+)}u_{n+1,m}u_{n+1,m+1}\\
&+F_m^{(+)}u_{n,m+1}u_{n+1,m+1}=0,
\end{aligned}
\tag{12d}
$$

$$
\begin{aligned}
{}_1D_4\colon\ & \delta_1\left(F_n^{(-)}u_{n,m}u_{n,m+1}+F_n^{(+)}u_{n+1,m}u_{n+1,m+1}\right)\\
&+\delta_2\left(F_m^{(-)}u_{n,m}u_{n+1,m}+F_m^{(+)}u_{n,m+1}u_{n+1,m+1}\right)\\
&+u_{n,m}u_{n+1,m+1}+u_{n+1,m}u_{n,m+1}+\delta_3=0,
\end{aligned}
\tag{12e}
$$

$$
\begin{aligned}
{}_2D_4\colon\ & \delta_1 \left(F_n^{(-)} u_{n,m} u_{n,m+1} + F_n^{(+)} u_{n+1,m} u_{n+1,m+1} \right) \\
& + \delta_2 \left(F_{n+m}^{(-)} u_{n,m} u_{n+1,m+1} + F_{n+m}^{(+)} u_{n+1,m} u_{n,m+1} \right) \\
& + u_{n,m} u_{n+1,m} + u_{n,m+1} u_{n+1,m+1} + \delta_3 = 0,
\end{aligned} \tag{12f}
$$

where the coefficients $F_k^{(\pm)}$ are given by (10). Then algebraic entropy [26–29] was computed. The result of this computation showed that the *rate of growth of the degrees of the iterates* of all the trapezoidal H^4 (11) and of all H^6 equation (12) is *linear*. This fact according to the *algebraic entropy conjecture* [28, 30] implies the linearizability. To support this result two explicit examples of linearization were given.

In [13] a particular example, the ${}_tH_1^\varepsilon$ equation, was studied and it was found that it possessed three-point generalized symmetries depending on arbitrary functions. This property was later linked in [25] to the fact that the ${}_tH_1^\varepsilon$ was *Darboux integrable*. In addition in [25] it was proved that some other consistend around the Cube linearizable quad-equations [31, 32], were in fact Darboux integrable. These facts provide some evidence of an intimate connection between linearizable equations with the consistency around the cube property and Darboux integrability.

The scope of this paper is to generalize the result obtained for the ${}_tH_1^\varepsilon$ equation in [13, 25]. Our main statement is enclosed in the following theorem:

Theorem 1.1. *Every trapezoidal H^4 equation (11) and every H^6 equation (12) is Darboux integrable.*

The fact that an equation is Darboux integrable is a *formal* proof that it is linearizable, as it will be discussed in more detail in section 2. In this paper we relate different forms of integrability, namely the CAC the algebraic entropy test and the Darboux integrability. Comparing various definitions of integrability and relating the outcome of different integrability tests is very important in the theory of integrable systems, and in mathematical physics in general. Indeed such kind of study provides an understanding of the limitation and of the benefits of the various integrability tests which can be applied to relevant physical models.

The plan of the paper is the following: in section 2 we recall the basic facts about Darboux integrability and discuss the methodologies employed in the case of non-autonomous, two-periodic quad-equations. In section 3 we give the proof of theorem 1.1 presenting the first integrals of the trapezoidal H^4 equation (11) and of the H^6 equation (12). In the section 4 we present some conclusions and provide an outlook on how first integrals can be used to obtain general solutions.

2. Darboux integrability

We start by giving the following definition:

Definition 2.1. A hyperbolic partial differential equation (PDE) in two variables

$$u_{xt} = f\left(x, t, u, u_t, u_x\right) \tag{13}$$

is said to be *Darboux integrable* if it possesses two independent first integrals T, X depending only on derivatives with respect to one variable:

$$T = T\left(x, t, u, u_t, \ldots, u_{nt}\right), \quad \left.\frac{\mathrm{d}T}{\mathrm{d}x}\right|_{u_{xt}=f} \equiv 0, \tag{14a}$$

$$X = X\left(x, t, u, u_x, \ldots, u_{mx}\right), \quad \left.\frac{\mathrm{d}X}{\mathrm{d}t}\right|_{u_{xt}=f} \equiv 0, \tag{14b}$$

where $u_{kt} = \partial^k u/\partial t^k$ and $u_{kx} = \partial^k u/\partial x^k$ for every $k \in \mathbb{N}$.

Definition 2.1 was given by Euler and Laplace [33, 34] in the linear case. Definition 2.1 was extended to nonlinear hyperbolic equation (13) in the 19th and early 20th centuries [35–39]. The method was then used at the end of the 20th century mainly by Russian mathematicians as a source of new exactly solvable PDEs in two variables [40–46]. We note that in many papers Darboux integrability is defined as the stabilization to zero of the so-called Laplace chain of the linearized equation. However it can be proved that the two definitions are equivalent [43, 47–49].

The most famous equation belonging to the class of Darboux integrable equations is the Liouville equation [50]:

$$u_{xt} = e^u \tag{15}$$

which possesses the two following first integrals:

$$X = u_{xx} - \frac{1}{2}u_x^2, \quad T = u_{tt} - \frac{1}{2}u_t^2. \tag{16}$$

In the case of quad-equations we can state the following definition:

Definition 2.2. A quad-equation, possibly non-autonomous:

$$Q_{n,m}\left(u_{n,m}, u_{n+1,m}, u_{n,m+1}, u_{n+1,m+1}\right) = 0, \tag{17}$$

is Darboux integrable if there exist two independent *first integrals*, one containing only shifts in the first direction and the other containing only shifts in the second direction, i.e. that there exist two functions:

$$W_1 = W_{1,n,m}(u_{n+l_1,m}, u_{n+l_1+1,m}, \ldots, u_{n+k_1,m}), \tag{18a}$$

$$W_2 = W_{2,n,m}(u_{n,m+l_2}, u_{n,m+l_2+1}, \ldots, u_{n,m+k_2}), \tag{18b}$$

where $l_1 < k_1$ and $l_2 < k_2$ are integers, and $T_n h_{n,m} = h_{n+1,m}$, $T_m h_{n,m} = h_{n,m+1}$, $\mathrm{Id} h_{n,m} = h_{n,m}$ such that:

$$(T_n - \mathrm{Id})W_{2,n,m} = 0, \tag{19a}$$

$$(T_m - \mathrm{Id})W_{1,n,m} = 0 \tag{19b}$$

hold true identically on the solutions of (17).

Remark 2.1. The numbers $k_i - l_i$ in (18), where $i = 1, 2$, are called the *orders* of the first integrals W_i.

Definition 2.2 was introduced in [51], where it was used to obtain a discrete analogue of the Liouville equation (15).

In [51] was proved the following proposition:

Proposition 2.1. *Assume we are given a Darboux integrable quad-equation* (17). *Then this equation is linearizable in two different ways.*

Proof. Since the quad-equation (17) is Darboux integrable according to definition 2.2 it possesses two first integrals of the form (18). Define the following transformations:

$$u_{n,m} \to \tilde{u}_{n,m} = W_{1,n,m}, \tag{20a}$$

$$u_{n,m} \to \hat{u}_{n,m} = W_{2,n,m}. \tag{20b}$$

Then due to equation (19) the two functions $\tilde{u}_{n,m}$ and $\hat{u}_{n,m}$ satisfy the following *trivial linear equations*:

$$\tilde{u}_{n,m+1} - \tilde{u}_{n,m} = 0, \tag{21a}$$

$$\hat{u}_{n+1,m} - \hat{u}_{n,m} = 0. \tag{21b}$$

Equation (21) gives the desired linearization. □

Proposition 2.1 gives the relationship between the Darboux integrability and linearization.

Corollary 2.2. *Assume we are given a Darboux integrable quad-equation* (17) *whose first integrals are given by* (18)*. Then the following equations hold true:*

$$W_{1,n,m} = \lambda_n, \tag{22a}$$

$$W_{2,n,m} = \rho_m, \tag{22b}$$

where λ_n and ρ_m are arbitrary functions of the lattice variables n and m, respectively.

Proof. It follows trivially by applying the transformations (20) along with the relations (21). □

Remark 2.2. The relations (22) obtained in corollary 2.2 can be seen as *ordinary difference equations* which must be satisfied by any solution $u_{n,m}$ of a Darboux integrable quad-equation (17). Therefore we can say that a Darboux integrable quad-equation is in fact an *over-determined system* consisting of a quad-equation and of two ordinary difference equation. This observation was first made by S. Lie in the case of PDEs [52]. The transformations (20) and the ordinary difference equation (22) may be quite complicated. However in case of the trapezoidal H^4 (11) and the H^6 equation (12), one can prove [53] that also (22), defined by the first integrals, are *linearizable*. Therefore we can use Darboux integrability in order to obtain the *general solutions* of these equations. In section 4 we will present an example of this procedure for the ${}_tH_1^\varepsilon$ equation.

After the introduction of definition 2.2 in [51], various papers were devoted to the study of Darboux integrability for quad-equations [54–58]. Computational methods to establish the existence of first integrals were developed in [54, 56, 58]. A method to find first integrals with fixed l_i, k_i of a given autonomous equation was presented in [54]. A slight modification of this method was applied in [56] to autonomous equations with non-autonomous first integrals. Finally in [58] it was applied to equations with two-periodic coefficients. In the present paper, we present a further modification of this method aimed to deal with non-autonomous equations with two-periodic coefficients straightforwardly.

Let us consider the operator

$$Y_{-1} = T_m \frac{\partial}{\partial u_{n,m-1}} T_m^{-1} \tag{23}$$

and apply it to the definition of first integral in the n-direction (19*b*):

$$Y_{-1} W_1 \equiv 0. \tag{24}$$

The application of the operator Y_{-1} is to be understood in the following sense: first we must apply T_m^{-1} and then we should express, using the equation (17), $u_{n+i,m-1}$ in terms of the functions $u_{n+j,m}$ and $u_{n,m-1}$ which will be considered in this problem as independent variables. Then we can differentiate in (24) with respect to $u_{n,m-1}$ and apply T_m [56].

Taking in (24) the coefficients at powers of the independent variable $u_{n,m+1}$, we obtain a system of PDEs for W_1. If this is sufficient to determine W_1 up to arbitrary functions of a single variable, then we are done, otherwise we can add other equations by considering the 'higher-order' operators

$$Y_{-k} = T_m^k \frac{\partial}{\partial u_{n,m-1}} T_m^{-k}, \quad k \in \mathbb{N}. \tag{25}$$

Equation (19*b*) implies that *the infinite family of equations*:

$$T_m^k W_1 = W_1, \quad \forall k \in \mathbb{Z} \tag{26}$$

holds true. The members of this infinite family of difference equation are called the *difference consequences of (*19*b*). Applying the operator (25) to equation (26) we have:

$$Y_{-k} W_1 \equiv 0, \quad k \in \mathbb{N}, \tag{27}$$

with the same computational prescriptions as above. So we can add equations until we find a non-constant function[3] W_1 which depends on a *single* combination of the variables $u_{n,m+l_1}, \ldots, u_{n,m+k_1}$. If we find a non-constant solution W_1 of the equations generated by (23) and possibly (25), then we must insert it into (19*b*) to specify it.

In the same way first integrals in the m-direction W_2 can be found by considering the operators

$$Z_{-k} = T_n^k \frac{\partial}{\partial u_{n-1,m}} T_n^{-k}, \quad k \in \mathbb{N}, \tag{28}$$

which provide the equations

$$Z_{-k} W_2 \equiv 0, \quad k \in \mathbb{N}, \tag{29}$$

through the difference consequences of (19*a*):

$$T_n^k W_2 = W_2, \quad \forall k \in \mathbb{Z}. \tag{30}$$

In the case of non-autonomous equations with two-periodic coefficients, we can assume that a decomposition analogue of the quad-equation (9) holds for the first integrals:

$$\begin{aligned} W_i = {} & F_n^{(+)} F_m^{(+)} W_i^{(+,+)} + F_n^{(-)} F_m^{(+)} W_i^{(-,+)} \\ & + F_n^{(+)} F_m^{(-)} W_i^{(+,-)} + F_n^{(-)} F_m^{(-)} W_i^{(-,-)}, \end{aligned} \tag{31}$$

with $F_k^{(\pm)}$ given by (10). We can then derive from (27, 29) a set of equations for the functions $W_i^{(\pm,\pm)}$ by considering the even/odd points on the lattice. The final form of the functions W_i will be then fixed by substituting in (19).

When successful, the above procedure gives first integrals depending on arbitrary functions. This fact has to be understood as a restatement of the trivial property that any autonomous function of a first integral is again a first integral. So, in general, one does not need first integrals depending on arbitrary functions. Therefore we can take these arbitrary functions in the first integrals to be *linear* function in their arguments.

[3] Obviously constant functions are trivial first integrals.

As an example let us consider the problem of finding the first integrals of the ${}_tH_1^\varepsilon$ equation (11*a*). We have the following proposition:

Proposition 2.3. *The ${}_tH_1^\varepsilon$ equation (11a) is Darboux integrable since it possesses the following first integrals:*

$$W_1 = F_m^{(+)} \frac{\alpha_2}{u_{n+1,m} - u_{n,m}} + F_m^{(-)} \frac{u_{n+1,m} - u_{n,m}}{1 + \varepsilon^2 u_{n,m} u_{n+1,m}}, \tag{32a}$$

$$W_2 = F_m^{(+)} \alpha \frac{1 + \varepsilon^2 u_{n,m+1} u_{n,m-1}}{u_{n,m+1} - u_{n,m-1}} + F_m^{(-)} \beta \left(u_{n,m+1} - u_{n,m-1}\right), \tag{32b}$$

where α and β are two arbitrary constants.

Remark 2.3. The first integrals (32) of the ${}_tH_1^\varepsilon$ equation (11*a*) were first presented in [25]. Therein they were found by direct inspection.

Proof. Since all the H^4 equations and the ${}_tH_1^\varepsilon$, in particular, are non-autonomous only in the direction m, we can consider a simplified version of (31):

$$W_i = F_m^{(+)} W_i^{(+)} + F_m^{(-)} W_i^{(-)}. \tag{33}$$

If we assume that $W_1 = W_{1,n,m}(u_{n,m}, u_{n+1,m})$, then, separating the even and odd terms with respect to m in (24), we find the following equations:

$$\frac{\partial W_1^{(+)}}{\partial u_{n+1,m}} + \frac{\partial W_1^{(+)}}{\partial u_{n,m}} = 0, \tag{34a}$$

$$\left(1 + \varepsilon^2 u_{n+1,m}^2\right) \frac{\partial W_1^{(-)}}{\partial u_{n+1,m}} + \left(1 + \varepsilon^2 u_{n,m}^2\right) \frac{\partial W_1^{(-)}}{\partial u_{n,m}} = 0. \tag{34b}$$

Their solution is:

$$W_1 = F_m^{(+)} F\left(u_{n+1,m} - u_{n,m}\right) + F_m^{(-)} G\left(\frac{u_{n+1,m} - u_{n,m}}{1 + \varepsilon^2 u_{n,m} u_{n+1,m}}\right), \tag{35}$$

where F and G are arbitrary functions. Inserting (35) into the difference equation (19*b*), we obtain that F and G must satisfy the following identity:

$$G(\xi) = F\left(\frac{\alpha_2}{\xi}\right). \tag{36}$$

This yields the first integral

$$W_1 = F_m^{(+)} F\left(\frac{\alpha_2}{u_{n+1,m} - u_{n,m}}\right) + F_m^{(-)} F\left(\frac{u_{n+1,m} - u_{n,m}}{1 + \varepsilon^2 u_{n,m} u_{n+1,m}}\right). \tag{37}$$

For the m-direction we may also suppose that our first integral $W_2 = W_{2,m}(u_{n,m}, u_{n,m+1})$ is of the first order or a *two-point* first integral. It easy to see from (29) with $k = 1$ that this yields the trivial solution $W_2 = \text{constant}$. Therefore we consider the case of a second order, *three-point* first integral: $W_2 = W_{2,m}(u_{n,m-1}, u_{n,m}, u_{n,m+1})$. From (29) with $k = 1$, separating the even and odd terms with respect to m, we obtain:

$$\alpha_2\left(1+\varepsilon^2 u_{n,m+1}^2\right)\frac{\partial W_2^{(+)}}{\partial u_{n,m+1}}-\left[\left(u_{n,m}-u_{n+1,m}\right)^2+\varepsilon^2\alpha_2^2\right]\frac{\partial W_2^{(+)}}{\partial u_{n,m}} + \alpha_2\left(1+\varepsilon^2 u_{n,m-1}^2\right)\frac{\partial W_2^{(+)}}{\partial u_{n,m-1}}=0, \tag{38a}$$

$$\alpha_2\left(1+\varepsilon^2 u_{n+1,m}^2\right)\frac{\partial W_2^{(-)}}{\partial u_{n,m+1}}-\left(u_{n,m}-u_{n+1,m}\right)^2\frac{\partial W_2^{(-)}}{\partial u_{n,m}} + \alpha_2\left(1+\varepsilon^2 u_{n+1,m}^2\right)\frac{\partial W_2^{(-)}}{\partial u_{n,m-1}}=0. \tag{38b}$$

Taking the coefficients with respect to $u_{n+1,m}$ and then solving, we have:

$$W_2 = F_m^{(+)}F\left(\frac{1+\varepsilon^2 u_{n,m+1}u_{n,m-1}}{u_{n,m+1}-u_{n,m-1}}\right)+F_m^{(-)}G\left(u_{n,m+1}-u_{n,m-1}\right). \tag{39}$$

Inserting (39) into (19*a*) we do not have any further restriction on the form of the first integral. So we conclude that we have two independent first integrals in the m-direction, as it was observed in [25].

Now we can take the first integrals to be linear in the arguments of the arbitrary functions. Therefore from equations (37) and (39) we have that the first integrals of the ${}_tH_1^\varepsilon$ equation are given by (32). □

In next section we present the explicit form of first integrals for the remaining H^4 and H^6 equations computed by the method presented in this section. We will not present the details of the calculations as we did in the proof of proposition 2.3. Indeed such computations are algorithmic and can be implemented in any computer algebra system available. The result of this computations will be the proof of theorem 1.1.

3. First integrals for the H^4 and H^6 equations

In this section we present the first integrals of the ${}_tH_2^\varepsilon$ equation (11*b*), the ${}_tH_3^\varepsilon$ equation (11*c*) and the whole family of the H^6 equation (12).

3.1. Trapezoidal H^4 equations

Let us consider the ${}_tH_2^\varepsilon$ equation (11*b*). We have the following proposition:

Proposition 3.1. *The ${}_tH_2^\varepsilon$ equation (11b) is Darboux integrable. If $\varepsilon \neq 0$ it possesses a four-point, third order first integral in the n-direction:*

$$W_1 = F_m^{(+)}\frac{\left(-u_{n+1,m}+u_{n-1,m}\right)\left(u_{n,m}-u_{n+2,m}\right)}{\varepsilon^2\alpha_2^4+4\varepsilon\alpha_2^3+\left[\left(8\alpha_3-2u_{n,m}-2u_{n+1,m}\right)\varepsilon-1\right]\alpha_2^2+\left(u_{n,m}-u_{n+1,m}\right)^2} - F_m^{(-)}\frac{\left(-u_{n+1,m}+u_{n-1,m}\right)\left(u_{n,m}-u_{n+2,m}\right)}{\left(-u_{n-1,m}+u_{n,m}+\alpha_2\right)\left(u_{n+1,m}+\alpha_2-u_{n+2,m}\right)} \tag{40a}$$

and a five-point, fourth order first integral in the m-direction:

$$W_2 = F_m^{(+)} \alpha \frac{\left(u_{n,m-1} - u_{n,m+1}\right)^2 \left(u_{n,m+2} - u_{n,m}\right) \left(u_{n,m} - u_{n,m-2}\right)}{\begin{array}{l}\left(\left(\alpha_2 + \alpha_3 + u_{n,m-1}\right)^2 \varepsilon - u_{n,m-1} + \alpha_3 - u_{n,m}\right) \cdot \\ \left(\left(\alpha_3 + \alpha_2 + u_{n,m+1}\right)^2 \varepsilon - u_{n,m+1} + \alpha_3 - u_{n,m}\right)\end{array}} + F_m^{(-)} \beta \frac{\left[\begin{array}{c} -\varepsilon \left(u_{n,m-2} - u_{n,m+2}\right) u_{n,m}^2 - \left(\alpha_3 + \alpha_2\right)^2 \left(u_{n,m-2} - u_{n,m+2}\right) \varepsilon \\ + \left(-2 \left(u_{n,m-2} - u_{n,m+2}\right) \left(\alpha_3 + \alpha_2\right) \varepsilon + u_{n,m-1} - u_{n,m+2} - u_{n,m+1} + u_{n,m-2}\right) u_{n,m} \\ + \left(-\alpha_3 + u_{n,m+1}\right) u_{n,m-2} + u_{n,m+2} \left(\alpha_3 - u_{n,m-1}\right) \end{array}\right]}{\left(-u_{n,m+2} + u_{n,m}\right) \left(-u_{n,m-2} + u_{n,m}\right) \left(u_{n,m-1} - u_{n,m+1}\right)}. \tag{40b}$$

If $\varepsilon = 0$ its first integrals are given by:

$$W_1^{\varepsilon=0} = (-1)^m \frac{2\alpha_2 - u_{n-1,m} + 2u_{n,m} - u_{n+1,m}}{u_{n-1,m} - u_{n+1,m}}, \tag{41a}$$

$$W_2^{\varepsilon=0} = \frac{\left(u_{n,m+1} - u_{n,m-1}\right) \left(u_{n,m+2} - u_{n,m}\right)}{u_{n,m} + u_{n,m+1} - \alpha_3}. \tag{41b}$$

The first integral in the n-direction (41a) it is a three-point, second order first integral. On the contrary, the first integral in the m-direction (41b) is a four-point, third order first integral.

Proof. By direct computation using the method explained in section 2. □

Remark 3.1. We note that if $\varepsilon = 0$ the ${}_tH_2^\varepsilon$ equation (11b) is autonomous. We denote this sub-case by ${}_tH_2^{\varepsilon=0}$. Despite the ${}_tH_2^{\varepsilon=0}$ equation is autonomous its first integral in the n direction (41a) is still non-autonomous.

Moreover the ${}_tH_2^{\varepsilon=0}$ equation is related to the equation (1) from *List 3* in [56]:

$$\left(\hat{u}_{n+1,m+1} - \hat{u}_{n+1,m}\right) \left(\hat{u}_{n,m} - \hat{u}_{n,m+1}\right) + \hat{u}_{n,m} + \hat{u}_{n+1,m} + \hat{u}_{n,m+1} + \hat{u}_{n+1,m+1} = 0 \tag{42}$$

through the transformation:

$$u_{n,m} = -\alpha_2 \hat{u}_{m,n} + \frac{1}{4}\alpha_2 + \frac{1}{2}\alpha_3. \tag{43}$$

Note that in this formula (43) the two lattice variables are *exchanged*. So it was already known in the literature that the ${}_tH_2^{\varepsilon=0}$ equation was Darboux integrable.

Let us consider the ${}_tH_3^\varepsilon$ equation (11c). We have the following proposition:

Proposition 3.2. *The ${}_tH_3^\varepsilon$ equation (11c) is Darboux integrable. If $\varepsilon \neq 0$ it possesses the following four-point, third order first integral in the n-direction:*

$$W_1 = F_m^{(+)} \frac{\left(u_{n-1,m} - u_{n+1,m}\right) \left(-u_{n+2,m} + u_{n,m}\right)}{{\alpha_2}^4 \varepsilon^2 \delta^2 - {\alpha_2}^3 u_{n+1,m} u_{n,m} + \left({u_{n,m}}^2 + u_{n+1,m}^2 - 2\varepsilon^2\delta^2\right) {\alpha_2}^2 - \alpha_2 u_{n,m} u_{n+1,m} + \varepsilon^2 \delta^2} - F_m^{(-)} \frac{\left(u_{n+1,m} - u_{n-1,m}\right) \left(u_{n+2,m} - u_{n,m}\right)}{\alpha_2 \left(-u_{n-1,m} + \alpha_2 u_{n,m}\right) \left(-u_{n+2,m} + u_{n+1,m} \alpha_2\right)} \tag{44a}$$

and a five-point, fourth order first integral in the m-direction:

$$W_2 = F_m^{(+)} \alpha \frac{\left(u_{n,m-1} - u_{n,m+1}\right)^2 \left(u_{n,m+2} - u_{n,m}\right)\left(u_{n,m} - u_{n,m-2}\right)}{\left(\delta^2\alpha_3^2 + u_{n,m-1}^2\varepsilon^2 - \alpha_3 u_{n,m-1}u_{n,m}\right)\left(\delta^2\alpha_3^2 + u_{n,m+1}^2\varepsilon^2 - \alpha_3 u_{n,m}u_{n,m+1}\right)}$$
$$- F_m^{(-)}\beta \frac{\begin{bmatrix} -u_{n,m}^2\alpha_3 u_{n,m-1} + u_{n,m}^2\alpha_3 u_{n,m+1} - u_{n,m}^2\varepsilon^2 u_{n,m+2} \\ +u_{n,m}^2\varepsilon^2 u_{n,m-2} + \alpha_3 u_{n,m}u_{n,m-1}u_{n,m+2} - \alpha_3 u_{n,m}u_{n,m-2}u_{n,m+1} \\ -\delta^2{\alpha_3}^2 u_{n,m+2} + \delta^2{\alpha_3}^2 u_{n,m-2} \end{bmatrix}}{\left(u_{n,m} - u_{n,m+2}\right)\left(-u_{n,m-2} + u_{n,m}\right)\left(u_{n,m-1} - u_{n,m+1}\right)}. \qquad (44b)$$

If $\varepsilon = 0$ it possesses the following first integrals:

$$W_1^{\varepsilon=0} = (-1)^m \left[\frac{\alpha_2 u_{n,m} - u_{n-1,m}}{u_{n+1,m} - u_{n-1,m}} + \frac{1}{2}\right], \qquad (45a)$$

$$W_2^{\varepsilon=0} = \frac{\left(u_{n,m+2} - u_{n,m}\right)u_{n,m-1} - u_{n,m+1}u_{n,m+2} + \alpha_3\delta^2}{\alpha_3\delta^2 - u_{n,m}u_{n,m+1}}. \qquad (45b)$$

The first integral in the n-direction (45a) it is a three-point, second order first integral. The first integral in the m-direction (45b) is a four-point, third order first integral.

Proof. By direct computation using the method explained in section 2. □

Remark 3.2. With the same notation as in remark 3.1, we note that also the ${}_tH_3^\varepsilon$ equation (11c) if $\varepsilon = 0$ becomes autonomous. Despite the ${}_tH_3^{\varepsilon=0}$ equation is autonomous its first integral in the n direction (45a) is still non-autonomous.

Moreover the ${}_tH_3^{\varepsilon=0}$ equation is related through the inversion of two lattice parameters $u_{n,m} = \hat{u}_{m,n}$ and the choice of parameters:

$$b_2 = -\frac{1}{\alpha_2}, \quad c_4 = \frac{\delta^2\alpha_3\left(1 - \alpha_2^2\right)}{\alpha_2} \qquad (46)$$

to equation (2) from *List 3* in [56]:

$$\hat{u}_{n+1,m+1}\left(\hat{u}_{n,m} + b_2\hat{u}_{n,m+1}\right) + \hat{u}_{n+1,m}\left(b_2\hat{u}_{n,m} + \hat{u}_{n,m+1}\right) + c_4 = 0. \qquad (47)$$

So it was already known in the literature that the ${}_tH_3^{\varepsilon=0}$ equation was Darboux integrable.

We can state the following:

Theorem 3.3. *The trapezoidal H^4 equation (11) are Darboux integrable.*

Proof. Just use propositions 2.3, 3.1 and 3.2. □

Remark 3.3. As a final remark we can say that the first integrals of the ${}_tH_2^\varepsilon$ and ${}_tH_3^\varepsilon$ equations have the same order in each direction. Furthermore, they share the important property that in the direction m, which is the direction of the non-autonomous factors $F_m^{(\pm)}$, the W_2 integrals are built up from two different 'sub'-integrals as in the known case of the ${}_tH_1^\varepsilon$ equation.

3.2. H^6 equations

We now present the first integrals of the H^6 equation (12) in both directions.

We consider the ${}_1D_2$ equation (12a). We have the following proposition:

Proposition 3.4. *The ${}_1D_2$ equation (12a) is Darboux integrable. If $\delta_1 \neq 0$ and $\delta_1 \neq (1 + \delta_2)^{-1}$ it possesses the following three-point, second order first integrals:*

$$
\begin{aligned}
W_1 = {} & F_n^{(+)} F_m^{(+)} \alpha \frac{\left[\left(1+\delta_2\right) u_{n,m} + u_{n+1,m}\right] \delta_1 - u_{n,m}}{\left[\left(1+\delta_2\right) u_{n,m} + u_{n-1,m}\right] \delta_1 - u_{n,m}} \\
& + F_n^{(+)} F_m^{(-)} \alpha \frac{1 + \left(u_{n+1,m} - 1\right) \delta_1}{1 + \left(u_{n-1,m} - 1\right) \delta_1} \\
& + F_n^{(-)} F_m^{(+)} \beta \left(u_{n+1,m} - u_{n-1,m}\right) \\
& - F_n^{(-)} F_m^{(-)} \beta \frac{\left(u_{n+1,m} - u_{n-1,m}\right) \left[1 - \left(1 - u_{n,m}\right) \delta_1\right]}{\delta_2 + u_{n,m}},
\end{aligned} \tag{48a}
$$

$$
\begin{aligned}
W_2 = {} & F_n^{(+)} F_m^{(+)} \alpha \frac{u_{n,m+1} - u_{n,m-1}}{u_{n,m} + \delta_1 u_{n,m-1}} \\
& + F_n^{(+)} F_m^{(-)} \beta \left(u_{n,m+1} - u_{n,m-1}\right) \\
& - F_n^{(-)} F_m^{(+)} \alpha \frac{u_{n,m+1} - u_{n,m-1}}{1 + \delta_1 \left(u_{n,m+1} - 1\right)} \\
& - F_n^{(-)} F_m^{(-)} \beta \frac{u_{n,m+1} - u_{n,m-1}}{\delta_2 + u_{n,m}}.
\end{aligned} \tag{48b}
$$

If $\delta_1 = 0$ it possesses the following three-point, second order first integrals:

$$
\begin{aligned}
W_1^{(0,\delta_2)} = {} & F_n^{(+)} F_m^{(+)} \alpha \frac{u_{n+1,m} - u_{n-1,m}}{u_{n,m}} - F_n^{(+)} F_m^{(-)} \alpha \left(u_{n+1,m} - u_{n-1,m}\right) \\
& + F_n^{(-)} F_m^{(+)} \beta \left(u_{n+1,m} - u_{n-1,m}\right) + F_n^{(-)} F_m^{(-)} \beta \frac{u_{n-1,m} - u_{n+1,m}}{\delta_2 + u_{n,m}},
\end{aligned} \tag{49a}
$$

$$
\begin{aligned}
W_2^{(0,\delta_2)} = {} & F_n^{(+)} F_m^{(+)} \alpha \frac{u_{n,m+1} - u_{n,m-1}}{u_{n,m}} + F_n^{(+)} F_m^{(-)} \beta \left(u_{n,m+1} - u_{n,m-1}\right) \\
& - F_n^{(-)} F_m^{(+)} \alpha \left(u_{n,m+1} - u_{n,m-1}\right) - F_n^{(-)} F_m^{(-)} \beta \frac{u_{n,m+1} - u_{n,m-1}}{\delta_2 + u_{n,m}}.
\end{aligned} \tag{49b}
$$

If $\delta_1 = \left(1+\delta_2\right)^{-1}$ it possesses the following first integrals:

$$
\begin{aligned}
W_1^{\left(\left(1+\delta_2\right)^{-1},\delta_2\right)} = {} & F_n^{(+)} F_m^{(+)} \alpha \frac{u_{n+1,m}}{u_{n-1,m}} + F_n^{(+)} F_m^{(-)} \alpha \frac{\delta_2 + u_{n+1,m}}{\delta_2 + u_{n-1,m}} \\
& + F_n^{(-)} F_m^{(+)} \beta \left(u_{n+1,m} - u_{n-1,m}\right) - F_n^{(-)} F_m^{(-)} \beta \frac{\left(u_{n+1,m} - u_{n-1,m}\right)}{\delta_2 + 1},
\end{aligned} \tag{50a}
$$

$$
\begin{aligned}
W_2^{\left(\left(1+\delta_2\right)^{-1},\delta_2\right)} = {} & F_n^{(+)} F_m^{(+)} \alpha \left[\left(1+\delta_2\right) u_{n,m} + u_{n,m+1}\right] \\
& + F_n^{(+)} F_m^{(-)} \beta \frac{u_{n,m} + \left(1+\delta_2\right) u_{n,m+1}}{\delta_2 + 1} \\
& - F_n^{(-)} F_m^{(+)} \alpha \frac{\left(\delta_2 + 1\right) u_{n,m}}{\delta_2 + u_{n,m+1}} - F_n^{(-)} F_m^{(-)} \beta \frac{u_{n,m+1}}{\delta_2 + u_{n,m}}.
\end{aligned} \tag{50b}
$$

In this case the first integral in the m-direction (50b) is a two-point, first order first integral. On the contrary, the first integral in the n-direction (50a) is still a three-point, second order first integral.

Proof. By direct computation using the method explained in section 2. □

We consider the ${}_2D_2$ equation (12b). We have the following proposition:

Proposition 3.5. *The* ${}_2D_2$ *equation* (12*b*) *is Darboux integrable. If* $\delta_1 \neq 0$ *and* $\delta_1 \neq (1+\delta_2)^{-1}$ *it possesses the following three-point, second order first integrals:*

$$\begin{aligned} W_1 = {} & F_n^{(+)} F_m^{(+)} \alpha \frac{\delta_2 + u_{n+1,m}}{\delta_2 + u_{n-1,m}} \\ & + F_n^{(+)} F_m^{(-)} \alpha \frac{(1-(1+\delta_2)\,\delta_1)\,u_{n,m} + u_{n+1,m}}{(1-(1+\delta_2)\,\delta_1)\,u_{n,m} + u_{n-1,m}} \\ & + F_n^{(-)} F_m^{(+)} \beta \frac{(u_{n+1,m} - u_{n-1,m})\,(u_{n,m} + \delta_2)}{1 + (-1 + u_{n,m})\,\delta_1} \\ & - F_n^{(-)} F_m^{(-)} \beta\,(u_{n+1,m} - u_{n-1,m}), \end{aligned} \tag{51a}$$

$$\begin{aligned} W_2 = {} & F_n^{(+)} F_m^{(+)} \alpha\,(u_{n,m+1} - u_{n,m-1}) \\ & - F_n^{(+)} F_m^{(-)} \beta \frac{u_{n,m+1} - u_{n,m-1}}{(\lambda - u_{n,m})\,\delta_1 - u_{n,m-1}} \\ & - F_n^{(-)} F_m^{(+)} \alpha \frac{u_{n,m+1} - u_{n,m-1}}{1 + (-1 + u_{n,m})\,\delta_1} \\ & - F_n^{(-)} F_m^{(-)} \beta \frac{u_{n,m+1} - u_{n,m-1}}{u_{n,m+1} + \delta_2}. \end{aligned} \tag{51b}$$

If $\delta_1 = 0$ *it possesses the following first integrals:*

$$\begin{aligned} W_1^{(0,\delta_2)} = {} & F_n^{(+)} F_m^{(+)} \alpha\,(\delta_2 + u_{n+1,m})\,u_{n,m} - F_n^{(+)} F_m^{(-)} \alpha\,(u_{n+1,m} + u_{n,m}) \\ & + F_n^{(-)} F_m^{(+)} \beta\,(\delta_2 + u_{n,m})\,u_{n+1,m} - F_n^{(-)} F_m^{(-)} \beta\,(u_{n+1,m} + u_{n,m}), \end{aligned} \tag{52a}$$

$$\begin{aligned} W_2^{(0,\delta_2)} = {} & F_n^{(+)} F_m^{(+)} \alpha\,(u_{n,m+1} - u_{n,m-1}) + F_n^{(+)} F_m^{(-)} \beta \frac{u_{n,m+1}}{u_{n,m-1}} \\ & - F_n^{(-)} F_m^{(+)} \alpha\,(u_{n,m+1} - u_{n,m-1}) - F_n^{(-)} F_m^{(-)} \beta \frac{u_{n,m+1} - u_{n,m-1}}{u_{n,m+1} + \delta_2}. \end{aligned} \tag{52b}$$

In this case the first integral in the n-direction (52*a*) *is a two-point, first order first integral. The first integral in the m-direction* (52*b*) *is a three-point, second order first integral. If* $\delta_1 = (1+\delta_2)^{-1}$ *it possesses the following first integrals:*

$$\begin{aligned} W_1^{\left((1+\delta_2)^{-1},\delta_2\right)} = {} & F_n^{(+)} F_m^{(+)} \alpha \frac{\delta_2 + u_{n+1,m}}{\delta_2 + u_{n-1,m}} + F_n^{(+)} F_m^{(-)} \alpha \frac{u_{n+1,m}}{u_{n-1,m}} \\ & - F_n^{(-)} F_m^{(+)} \beta\,(u_{n-1,m} - u_{n+1,m}) - F_n^{(-)} F_m^{(-)} \beta \frac{u_{n+1,m} - u_{n-1,m}}{1+\delta_2}, \end{aligned} \tag{53a}$$

$$\begin{aligned} W_2^{\left((1+\delta_2)^{-1},\delta_2\right)} = {} & F_n^{(+)} F_m^{(+)} \alpha\,[(1+\delta_2)\,u_{n,m} + u_{n,m+1}] \\ & + F_n^{(+)} F_m^{(-)} \beta\,[u_{n,m} + (1+\delta_2)\,u_{n,m+1}] \\ & + F_n^{(-)} F_m^{(+)} \alpha \frac{\delta_2\lambda - (1+\delta_2)\,u_{n,m+1} + \lambda u_{n,m}}{u_{n,m} + \delta_2} \\ & + F_n^{(-)} F_m^{(-)} \beta \frac{\delta_2\lambda - (1+\delta_2)\,u_{n,m} + \lambda u_{n,m+1}}{u_{n,m+1} + \delta_2}. \end{aligned} \tag{53b}$$

In this case the first integral in the m-direction (53*b*) *is a two-point, first order first integral. The first integral in the n-direction* (53*a*) *is still a three-point, second order first integral.*

Proof. By direct computation using the method explained in section 2. □

We consider the ${}_3D_2$ equation (12c). We have the following proposition:

Proposition 3.6. *The ${}_3D_2$ equation (12c) is Darboux integrable. If $\delta_1 \neq 0$ and $\delta_1 \neq (1+\delta_2)^{-1}$ it possesses the following three-point, second order first integrals:*

$$\begin{aligned} W_1 = {} & F_n^{(+)} F_m^{(+)} \alpha \frac{(u_{n-1,m}+\delta_2)\left[1+(u_{n+1,m}-1)\,\delta_1\right]}{(u_{n+1,m}+\delta_2)\left[1+(u_{n-1,m}-1)\,\delta_1\right]} \\ & + F_n^{(+)} F_m^{(-)} \alpha \frac{u_{n,m}+(1-\delta_1-\delta_1\delta_2)\,u_{n-1,m}}{u_{n,m}+(1-\delta_1-\delta_1\delta_2)\,u_{n+1,m}} \\ & + F_n^{(-)} F_m^{(+)} \beta \left(u_{n+1,m}-u_{n-1,m}\right)(\delta_2+u_{n,m}) \\ & - F_n^{(-)} F_m^{(-)} \beta \left(u_{n+1,m}-u_{n-1,m}\right), \end{aligned} \tag{54a}$$

$$\begin{aligned} W_2 = {} & F_n^{(+)} F_m^{(+)} \alpha \left(u_{n,m+1}-u_{n,m-1}\right) \\ & - F_n^{(+)} F_m^{(-)} \beta \frac{u_{n,m+1}-u_{n,m-1}}{\lambda\,(1+\delta_2)\,{\delta_1}^2-\left[(1+\delta_2)\,u_{n,m-1}+u_{n,m}+\lambda\right]\delta_1+u_{n,m-1}} \\ & + F_n^{(-)} F_m^{(+)} \alpha \left(u_{n,m-1}-u_{n,m+1}\right)\left[1+(u_{n,m}-1)\,\delta_1\right] \\ & + F_n^{(-)} F_m^{(-)} \beta \frac{u_{n,m+1}-u_{n,m-1}}{(\delta_2+u_{n,m+1})\left[1+(1-\delta_1)\,u_{n,m-1}\right]}. \end{aligned} \tag{54b}$$

If $\delta_1 = 0$ it possesses the following first integrals:

$$\begin{aligned} W_1^{(0,\delta_2)} = {} & F_n^{(+)} F_m^{(+)} \alpha u_{n,m}\left(\delta_2+u_{n+1,m}\right) - F_n^{(+)} F_m^{(-)} \alpha \left(u_{n+1,m}+u_{n,m}\right) \\ & + F_n^{(-)} F_m^{(+)} \beta u_{n+1,m}\left(\delta_2+u_{n,m}\right) - F_n^{(-)} F_m^{(-)} \beta \left(u_{n+1,m}+u_{n,m}\right), \end{aligned} \tag{55a}$$

$$\begin{aligned} W_2^{(0,\delta_2)} = {} & F_n^{(+)} F_m^{(+)} \alpha \left(u_{n,m+1}-u_{n,m-1}\right) + F_n^{(+)} F_m^{(-)} \beta \frac{u_{n,m+1}}{u_{n,m-1}} \\ & - F_n^{(-)} F_m^{(+)} \alpha \left(u_{n,m+1}-u_{n,m-1}\right) + F_n^{(-)} F_m^{(-)} \beta \frac{\delta_2+u_{n,m-1}}{\delta_2+u_{n,m+1}}. \end{aligned} \tag{55b}$$

In this case the first integral in the n-direction (55a) is a two-point, first order first integral. The first integral in the m-direction (55b) is a three-point, second order first integral. If $\delta_1 = (1+\delta_2)^{-1}$, it possesses the following three-point, second order first integrals:

$$\begin{aligned} W_1^{\left((1+\delta_2)^{-1},\delta_2\right)} = {} & F_n^{(+)} F_m^{(+)} \alpha \frac{u_{n+1,m}-u_{n-1,m}}{(\delta_2+u_{n+1,m})(\delta_2+u_{n-1,m})} \\ & + F_n^{(+)} F_m^{(-)} \alpha \frac{u_{n+1,m}-u_{n-1,m}}{(\delta_2+1)\,u_{n,m}} \\ & - F_n^{(-)} F_m^{(+)} \beta \left(u_{n-1,m}-u_{n+1,m}\right)(\delta_2+u_{n,m}) \\ & - F_n^{(-)} F_m^{(-)} \beta \left(u_{n+1,m}-u_{n-1,m}\right), \end{aligned} \tag{56a}$$

$$\begin{aligned} W_2^{\left((1+\delta_2)^{-1},\delta_2\right)} = {} & F_n^{(+)} F_m^{(+)} \alpha \left(u_{n,m+1}-u_{n,m-1}\right) + F_n^{(+)} F_m^{(-)} \beta \frac{u_{n,m+1}-u_{n,m-1}}{u_{n,m}} \\ & - F_n^{(-)} F_m^{(+)} \alpha \frac{(u_{n,m+1}-u_{n,m-1})(\delta_2+u_{n,m})}{\delta_2+1} \\ & + F_n^{(-)} F_m^{(-)} \beta \frac{u_{n,m+1}-u_{n,m-1}}{(\delta_2+u_{n,m+1})(u_{n,m-1}+\delta_2)}. \end{aligned} \tag{56b}$$

Proof. By direct computation using the method explained in section 2. □

We consider the D_3 equation (12*d*). We have the following proposition:

Proposition 3.7. *The D_3 equation (12d) is Darboux integrable. It possesses the following four-point, third order first integrals:*

$$\begin{aligned} W_1 = {} & F_n^{(+)} F_m^{(+)} \alpha \frac{(u_{n+1,m} - u_{n-1,m})(u_{n+2,m} - u_{n,m})}{u_{n+1,m}^2 - u_{n,m}} \\ & + F_n^{(+)} F_m^{(-)} \alpha \frac{(u_{n+1,m} - u_{n-1,m})(u_{n+2,m - u_{n,m}})}{u_{n,m} + u_{n-1,m}} \\ & - F_n^{(-)} F_m^{(+)} \beta \frac{(u_{n+1,m} - u_{n-1,m})(u_{n+2,m} - u_{n,m})}{u_{n+1,m} - u_{n,m}^2} \\ & + F_n^{(-)} F_m^{(-)} \beta \frac{(u_{n+1,m} - u_{n-1,m})(u_{n+2,m} - u_{n,m})}{u_{n+1,m} + u_{n+2,m}}, \end{aligned} \tag{57a}$$

$$\begin{aligned} W_2 = {} & F_n^{(+)} F_m^{(+)} \alpha \frac{(u_{n,m+1} - u_{n,m-1})(u_{n,m+2} - u_{n,m})}{u_{n,m+1}^2 - u_{n,m}} \\ & - F_n^{(+)} F_m^{(-)} \beta \frac{(u_{n,m+1} - u_{n,m-1})(u_{n,m+2} - u_{n,m})}{u_{n,m+1} - u_{n,m}^2} \\ & + F_n^{(-)} F_m^{(+)} \alpha \frac{(u_{n,m+1} - u_{n,m-1})(u_{n,m+2} - u_{n,m})}{u_{n,m} + u_{n,m-1}} \\ & + F_n^{(-)} F_m^{(-)} \beta \frac{(u_{n,m+1} - u_{n,m-1})(u_{n,m+2} - u_{n,m})}{u_{n,m+1} + u_{n,m+2}}. \end{aligned} \tag{57b}$$

Proof. By direct computation using the method explained in section 2. Since the D_3 equation (12*d*) is invariant under the exchange of lattice variables $n \leftrightarrow m$ its W_2 first integral (57*b*) can be obtained from the W_1 one (57*a*) simply by exchanging the indices n and m. □

We consider the ${}_1D_4$ equation (12*e*). We have the following proposition:

Proposition 3.8. *The ${}_1D_4$ equation (12e) is Darboux integrable. If $\delta_1, \delta_2, \delta_3 \neq 0$ we have the following four-point, third order first integrals:*

$$\begin{aligned} W_1 = {} & F_n^{(+)} F_m^{(+)} \alpha \frac{u_{n+1,m}^2 \delta_1 + u_{n+1,m} u_{n+2,m} + u_{n-1,m}(u_{n,m} - u_{n+2,m}) - \delta_2 \delta_3}{u_{n+1,m}(\delta_1 + u_{n,m}) - \delta_2 \delta_3} \\ & + F_n^{(+)} F_m^{(-)} \alpha \frac{(u_{n,m} - u_{n+2,m} + \delta_1 u_{n+1,m}) u_{n-1,m} + u_{n+1,m} u_{n+2,m}}{(u_{n,m} + \delta_1 u_{n-1,m}) u_{n+1,m}} \\ & + F_n^{(-)} F_m^{(+)} \beta \frac{(u_{n+1,m} - u_{n-1,m})(u_{n+2,m} - u_{n,m})}{u_{n,m}^2 \delta_1 + u_{n+1,m} u_{n,m} - \delta_2 \delta_3} \\ & + F_n^{(-)} F_m^{(-)} \beta \frac{(u_{n+1,m} - u_{n-1,m})(u_{n+2,m} - u_{n,m})}{u_{n,m}(u_{n+2,m} \delta_1 + u_{n+1,m})}, \end{aligned} \tag{58a}$$

$$
\begin{aligned}
W_2 = {} & F_n^{(+)}F_m^{(+)}\alpha\frac{(u_{n,m+2}-u_{n,m})\,u_{n,m-1}+\delta_1\delta_3-\delta_2u_{n,m+1}^2-u_{n,m+1}u_{n,m+2}}{\delta_1\delta_3-u_{n,m}u_{n,m+1}-\delta_2u_{n,m+1}^2} \\
& -F_n^{(+)}F_m^{(-)}\beta\frac{(u_{n,m+1}-u_{n,m-1})\,(u_{n,m+2}-u_{n,m})}{\delta_1\delta_3-\delta_2{u_{n,m}}^2-u_{n,m}u_{n,m+1}} \\
& +F_n^{(-)}F_m^{(+)}\alpha\frac{(u_{n,m}-u_{n,m+2}+\delta_2u_{n,m+1})\,u_{n,m-1}+u_{n,m+1}u_{n,m+2}}{(u_{n,m}+\delta_2u_{n,m-1})\,u_{n,m+1}} \\
& +F_n^{(-)}F_m^{(-)}\beta\frac{(u_{n,m+1}-u_{n,m-1})\,(u_{n,m+2}-u_{n,m})}{u_{n,m}\,(u_{n,m+2}\delta_2+u_{n,m+1})}.
\end{aligned}
\tag{58b}
$$

If $\delta_1=\delta_2=0$, but $\delta_3\neq 0$ it possesses the following three-point, second order first integrals:

$$
W_1^{(0,0,\delta_3)}=(-1)^m\,\frac{u_{n+1,m}-u_{n-1,m}}{u_{n,m}}, \tag{59a}
$$

$$
W_2^{(0,0,\delta_3)}=(-1)^n\,\frac{u_{n,m+1}-u_{n,m-1}}{u_{n,m}}. \tag{59b}
$$

If $\delta_1=\delta_2=\delta_3=0$ it possesses the following two-point, first order, non-autonomous first integrals:

$$
W_1^{(0,0,0)}=(-1)^m\,\frac{u_{n+1,m}}{u_{n,m}},\quad W_2^{(0,0,0)}=(-1)^n\,\frac{u_{n,m+1}}{u_{n,m}}. \tag{60}
$$

Proof. By direct computation using the method explained in section 2. Notice that if $\delta_1=\delta_2=0$ and δ_3 is arbitrary the equation has the discrete symmetry $n\leftrightarrow m$, therefore the first integral in the m direction can be obtained from the first integral in the n direction using such transformation. □

Remark 3.4. If $\delta_1=\delta_2=0$ the ${}_1D_4$ equation (12*e*) becomes autonomous. Despite the equation being autonomous its first integrals (59) and (60) are non-autonomous.

In particular we notice that the sub-case $\delta_1=\delta_2=0$ $\delta_3\neq 0$ is linked to the equation (4) with $b_3=1$ of *List 3* in [56]:

$$
\hat{u}_{n+1,m+1}\hat{u}_{n,m}+\hat{u}_{n+1,m}\hat{u}_{n,m+1}+1=0 \tag{61}
$$

through the transformation $u_{n,m}=\sqrt{\delta_3}\hat{u}_{n,m}$. Moreover the sub-case with $\delta_1=\delta_2=\delta_3=0$ is linked to one of the linearizable and Darboux integrable cases presented in [25, 32, 59].

We consider the ${}_2D_4$ equation (12*f*). We have the following proposition:

Proposition 3.9. *The ${}_2D_4$ equation* (12f) *is Darboux integrable. If $\delta_1,\delta_2\neq 0$ it possesses the following four-point, third order first integrals:*

$$
\begin{aligned}
W_1 = {} & F_n^{(+)}F_m^{(+)}\alpha\frac{\begin{bmatrix}(u_{n,m}-u_{n+2,m}-\delta_1\delta_2u_{n-1,m})\,u_{n+1,m}^2\\ +u_{n+1,m}u_{n+2,m}u_{n-1,m}+\delta_3u_{n-1,m}\end{bmatrix}}{(\delta_2u_{n+1,m}^2\delta_1-\delta_3-u_{n,m}u_{n+1,m})\,u_{n-1,m}} \\
& -F_n^{(+)}F_m^{(-)}\alpha\frac{u_{n+2,m}u_{n-1,m}+(-u_{n+2,m}+u_{n,m})\,u_{n+1,m}+\delta_3}{u_{n-1,m}u_{n,m}+\delta_3} \\
& -F_n^{(-)}F_m^{(+)}\beta\frac{(u_{n+1,m}-u_{n-1,m})\,(u_{n+2,m}-u_{n,m})\,u_{n,m}}{u_{n+2,m}\,(\delta_2\delta_1{u_{n,m}}^2-u_{n,m}u_{n+1,m}-\delta_3)} \\
& +F_n^{(-)}F_m^{(-)}\beta\frac{(u_{n+1,m}-u_{n-1,m})\,(u_{n+2,m}-u_{n,m})}{u_{n+1,m}u_{n+2,m}+\delta_3},
\end{aligned}
\tag{62a}
$$

$$
\begin{aligned}
W_2 = {} & F_n^{(+)} F_m^{(+)} \alpha \frac{\left(u_{n,m+2} - u_{n,m}\right) u_{n,m-1} + \delta_1 \delta_3 - \delta_2 u_{n,m+1}^2 - u_{n,m+1} u_{n,m+2}}{\delta_1 \delta_3 - \delta_2 u_{n,m+1}^2 - u_{n,m} u_{n,m+1}} \\
& - F_n^{(+)} F_m^{(-)} \beta \frac{\left(u_{n,m+1} - u_{n,m-1}\right)\left(u_{n,m+2} - u_{n,m}\right)}{\delta_1 \delta_3 - \delta_2 u_{n,m}^2 - u_{n,m} u_{n,m+1}} \\
& + F_n^{(-)} F_m^{(+)} \alpha \frac{u_{n,m+2} \delta_2 u_{n,m} + u_{n,m-1} u_{n,m} + u_{n,m+1} u_{n,m+2} - u_{n,m} u_{n,m+1}}{u_{n,m+2}\left(\delta_2 u_{n,m} + u_{n,m-1}\right)} \\
& + F_n^{(-)} F_m^{(-)} \beta \frac{\left(u_{n,m+1} - u_{n,m-1}\right)\left(u_{n,m+2} - u_{n,m}\right)}{\left(\delta_2 u_{n,m+1} + u_{n,m+2}\right) u_{n,m-1}}.
\end{aligned}
\tag{62b}
$$

If $\delta_1 = \delta_2 = 0$ it possesses the following first integrals:

$$
W_1^{(0,0,\delta_3)} = (-1)^m \left(u_{n,m} u_{n+1,m} + \frac{\delta_3}{2} \right), \quad W_2^{(0,0,\delta_3)} = \left(\frac{u_{n,m+1}}{u_{n,m-1}} \right)^{(-1)^n}. \tag{63}
$$

In this case the W_1 first integral is two-point first order and the W_2 first integral is three-point, second order.

Proof. By direct computation using the method explained in section 2. Notice that if $\delta_1 = \delta_2 = 0$ and δ_3 is arbitrary we have the equation has the discrete symmetry $n \leftrightarrow m$, therefore the first integral in the m direction can from the first integral in the n direction using such transformation. If also $\delta_3 = 0$ the computations do not change. □

Remark 3.5. If $\delta_1 = \delta_2 = 0$ the ${}_2D_4$ equation (12*f*) becomes autonomous. Despite the equation being autonomous its first integrals (59) are non-autonomous. This case $\delta_1 = \delta_2 = 0$ corresponds to the equation (9) of *List 4* in [56]:

$$
\hat{u}_{n,m} \hat{u}_{n+1,m} + \hat{u}_{n,m+1} \hat{u}_{n+1,m+1} + c_4 = 0, \tag{64}
$$

with the identification $u_{n,m} = \hat{u}_{n,m}$ and $c_4 = \delta_3$.

We can state the following theorem:

Theorem 3.10. *The H^6 equation* (11) *are Darboux integrable.*

Proof. Just use propositions 3.4–3.9. □

Remark 3.6. The first integrals of the H^6 equations are rather peculiar. Excluding the autonomous particular cases given in remarks 3.4 and 3.5, we have that all the H^6 equations possess two different integrals in every direction. This is due to the presence of two arbitrary constants α and β in the expressions of the first integrals. We believe that this reflects the fact that the H^6 equations on the lattice have two-periodic coefficients in both directions.

Therefore we can now give:

Proof of theorem 1.1. Apply theorems 3.3 and 3.10. □

4. Conclusions and outlook

In this paper we have presented a non-autonomous modified version of the algorithm developed in [54, 56, 58] to compute the first integrals of two-dimensional partial difference equations. Applying this algorithm, we were able to prove theorem 1.1 which states that the trapezoidal H^4 equation (11) and the H^6 equation (12), are Darboux integrable. This proof is carried out by constructing explicitly the first integrals of those equations as required by definition 2.2. This result confirms the outcome of the Algebraic Entropy test presented in [22].

Furthermore the first integrals, even those of higher order, can be used to find the *general solutions* of these equations. Since this procedure is not trivial and not standard, we leave its application to a future work [53]. To be concrete, we will give an example on how this procedure can be carried out in the case of the ${}_tH_1^\varepsilon$ equation given by (11*a*), whose first integrals are given by (32) and have been first presented in [25].

We wish to solve the ${}_tH_1^\varepsilon$ equation using both first integrals. We are going to construct those general solutions, slightly modifying the construction scheme presented in [58]. In particular we will prove the following proposition:

Proposition 4.1. *The ${}_tH_1^\varepsilon$ equation (11a) is exactly solvable using both its first integrals (32). These solutions are defined up to a discrete integration, i.e. up to the solution of the difference equation:*

$$u_{k+1} - u_k = v_k, \tag{65}$$

where u_k is the unknown function and v_k is an assigned function.

Proof. Let us start from the integral W_1 (32*a*). This is a two-point, first order integral. From corollary 2.2 we have that the ${}_tH_1^\varepsilon$ equation (11*a*) can be rewritten as the relation (19*b*) for the first integral W_1:

$$(T_m - \mathrm{Id})\left(F_m^{(+)}\frac{\alpha_2}{u_{n+1,m}-u_{n,m}} + F_m^{(-)}\frac{u_{n+1,m}-u_{n,m}}{1+\varepsilon^2 u_{n,m}u_{n+1,m}}\right) = 0. \tag{66}$$

From (66) we can derive the general solution of (11*a*) itself. In fact (66) implies:

$$F_m^{(+)}\frac{\alpha_2}{u_{n+1,m}-u_{n,m}} + F_m^{(-)}\frac{u_{n+1,m}-u_{n,m}}{1+\varepsilon^2 u_{n,m}u_{n+1,m}} = \lambda_n, \tag{67}$$

where λ_n is an arbitrary function of n. This is a first order difference equation in the n-direction in which m plays the role of a parameter. For this reason we can safely separate the two cases: m even and m odd.

Case $m = 2k$ In this case (67) is reduced to the linear equation

$$u_{n+1,2k} - u_{n,2k} = \frac{\alpha_2}{\lambda_n} \tag{68}$$

which has the solution

$$u_{n,2k} = \theta_{2k} + \omega_n, \tag{69}$$

where θ_{2k} is an arbitrary function and ω_n is the solution of the simple ordinary difference equation

$$\omega_{n+1} - \omega_n = \frac{\alpha_2}{\lambda_n}, \quad \omega_0 = 0. \tag{70}$$

Case $m = 2k + 1$ In this case (67) is reduced to the discrete Riccati equation:

$$\lambda_n\varepsilon^2 u_{n,2k+1}u_{n+1,2k+1} - u_{n+1,2k+1} + u_{n,2k+1} + \lambda_n = 0. \tag{71}$$

By using the Möbius transformation

$$u_{n,2k+1} = \frac{\mathrm{i}}{\varepsilon}\frac{1 - v_{n,2k+1}}{1 + v_{n,2k+1}}, \tag{72}$$

this equation can be recast into the linear equation

$$(\mathrm{i} + \varepsilon\lambda_\mathrm{n})\, v_{n+1,2k+1} - (\mathrm{i} - \varepsilon\lambda_\mathrm{n})\, v_{n,2k+1} = 0. \tag{73}$$

If we introduce a new function κ_n, such that

$$\frac{\kappa_{n+1}}{\kappa_n} = \frac{\mathrm{i} - \varepsilon\lambda_\mathrm{n}}{\mathrm{i} + \varepsilon\lambda_\mathrm{n}}, \tag{74}$$

then we have that the general solution of (73) is:

$$v_{n,2k+1} = \kappa_n \theta_{2k+1}, \tag{75}$$

where θ_{2k+1} is an arbitrary function. Using (72) and (74) we then obtain:

$$u_{n,2k+1} = \frac{\mathrm{i}}{\varepsilon}\frac{1 - \kappa_n\theta_{2k+1}}{1 + \kappa_n\theta_{2k+1}}, \quad \lambda_n = \frac{\mathrm{i}}{\varepsilon}\frac{\kappa_n - \kappa_{n+1}}{\kappa_n + \kappa_{n+1}}. \tag{76}$$

So we have the general solution of (11*a*) in the form:

$$u_{n,m} = F_m^{(+)}(\theta_m + \omega_n) + F_m^{(-)}\frac{\mathrm{i}}{\varepsilon}\frac{1 - \kappa_n\theta_m}{1 + \kappa_n\theta_m}, \tag{77}$$

where θ_m,κ_n are arbitrary functions, ω_n is defined via λ_n by (70), and λ_n is defined via κ_n by (76).

Now we pass to the integral in the direction m, namely, W_2 given by (32*b*). This case is more interesting, as now we are dealing with a three-point, second order integral. We can choose without loss of generality $\alpha = \beta = 1$. From corollary 2.2 we have the relation (22*b*), i.e. $W_2 = \rho_m$, from which we can derive two different equations, one for the even and one for the odd m. This gives *a priori* a coupled system. However in this case, choosing $m = 2k$ and $m = 2k + 1$, we obtain the following two equations:

$$1 + \varepsilon^2 u_{n,2k+1}u_{n,2k-1} = \rho_{2k}\left(u_{n,2k+1} - u_{n,2k-1}\right), \tag{78a}$$

$$u_{n,2k+2} - u_{n,2k} = \rho_{2k+1}. \tag{78b}$$

So the system consists of two *uncoupled* equations.

The first one (78*a*) is a discrete Riccati equation which can be linearized through the non-autonomous Möbius transformation:

$$u_{n,2k-1} = \frac{1}{v_{n,k}} + \alpha_k, \quad \rho_{2k} = \frac{1 + \varepsilon^2\alpha_{k+1}\alpha_k}{\alpha_{k+1} - \alpha_k}, \tag{79}$$

from which we obtain:

$$\left(1 + \varepsilon^2\alpha_{k+1}^2\right) v_{n,k+1} + \varepsilon^2\alpha_{k+1} = \left(1 + \varepsilon^2\alpha_k^2\right) v_{n,k} + \varepsilon^2\alpha_k. \tag{80}$$

This equation is equivalent to a total difference and therefore its solution is given by:

$$v_{n,k} = \frac{\theta_n - \varepsilon^2 \alpha_k}{1 + \varepsilon^2 \alpha_k^2}, \tag{81}$$

with an arbitrary function θ_n. Putting $\alpha_k = \kappa_{2k-1}$, we obtain the solution for $u_{n,2k-1}$:

$$u_{n,2k-1} = \frac{1 + \kappa_{2k-1}\theta_n}{\theta_n - \varepsilon^2 \kappa_{2k-1}}. \tag{82}$$

The second equation is just a linear ordinary difference equation which can be written as a total difference, performing the substitution $\rho_{2k+1} = \kappa_{2k+2} - \kappa_{2k}$, and we get:

$$u_{n,2k} = \omega_n + \kappa_{2k}. \tag{83}$$

The resulting solution reads:

$$u_{n,m} = F_m^{(+)} (\omega_n + \kappa_m) + F_m^{(-)} \frac{1 + \kappa_m \theta_n}{\theta_n - \varepsilon^2 \kappa_m}. \tag{84}$$

This solution depends on *three* arbitrary functions, as we started from a second order first integral, which is a consequence of the quad-equation. This means that there must exist a relation between θ_n and ω_n. This relation can be retrieved by inserting (84) into (11*a*). As a result we obtain the following definition for ω_n:

$$\omega_n - \omega_{n+1} = \alpha_2 \frac{\varepsilon^2 + \theta_n \theta_{n+1}}{\theta_{n+1} - \theta_n}, \tag{85}$$

which gives us the final expression for the solution of (11*a*) up to the discrete integration given by (85). □

Remark 4.1. The general solution (77) obtained from the first integral (32*a*) is the same as the general solution (84) obtained from the first integral (32*b*) in the sense that one of them can easily be transformed into the other one.

As a final remark we note that it has been proved in [51, 57] the following theorem:

Theorem 4.2. *Given a Darboux integrable quad-equation* (17) *then it possesses generalized symmetries depending on arbitrary function. In particular if the generalized symmetry generator is given by the vector field:*

$$\hat{X} = g_{n,m}\left(\mathbf{u}_{n,m}^D\right) \partial_{u_{n,m}}, \quad \mathbf{u}_{n,m}^D = \{u_{n+i,m+j}\}_{i=l_1,\ldots,k_1, j=l_2,\ldots,k_2}, \tag{86}$$

then the characteristic of the generalized symmetry $g_{n,m}\left(\mathbf{u}_{n,m}^D\right)$ *has the following form:*

$$g_{n,m}\left(\mathbf{u}_{n,m}^D\right) = g_{n,m}^{(1)}\left(u_{n+l_i,m}, \ldots, u_{n+k_1,m}\right) + g_{n,m}^{(2)}\left(u_{n,m+l_2}, \ldots, u_{n,m+k_2}\right), \tag{87}$$

and the functions $g_{n,m}^{(1)}$ *and* $g_{n,m}^{(2)}$ *are given by:*

$$g_{n,m}^{(1)} = R^{(1)}\left(F_n\left(T_n^{p_1} W_1, \ldots, T_n^{q_1} W_1\right)\right), \tag{88a}$$

$$g_{n,m}^{(2)} = R^{(2)}\left(G_m\left(T_m^{p_2} W_2, \ldots, T_m^{q_2} W_2\right)\right), \tag{88b}$$

with $p_i < q_i$ *and* $R^{(i)}$ *are difference operators called Laplace operators and have the form:*

$$R^{(1)} = \sum_{r=j_1}^{h_1} \lambda_r \left(u_{n+l_1',m}, \ldots, u_{n+k_1',m} \right) T_n^r, \tag{89a}$$

$$R^{(2)} = \sum_{r=j_2}^{h_2} \mu_r \left(u_{n,m+l_2'}, \ldots, u_{n,m+k_2'} \right) T_m^r, \tag{89b}$$

with $j_i < h_i$ *and* $l_i' < k_i'$.

The three-point generalized symmetries, i.e. the generalized symmetries of the form (87) with $k_i = -l_i = 1$, of the trapezoidal H^4 equation (11) and of the H^6 equation (12) were derived in [13, 23]. The ${}_tH_1^\varepsilon$ equation (11*a*) possesses three-points generalized symmetries depending on arbitrary functions. This is due to fact that the first integrals of the ${}_tH_1^\varepsilon$ equation (11*a*), displayed in (32) are of first and second order respectively. Theorem 1.1 alongside with theorem 4.2 implies that there should exists an order at which all the other trapezoidal H^4 equation (11) and the H^6 equation (12) possess generalized symmetries depending on arbitrary functions. Therefore the discovery of the Darboux integrability of the trapezoidal H^4 equation (11) and the H^6 equation (12) poses the challenging problem of finding the explicit form of such generalized symmetries. These symmetries will be highly nontrivial, especially in the case of the ${}_tH_2^\varepsilon$ equation (11*b*) and of the ${}_tH_3^\varepsilon$ equation (11*c*), where the order of the first integrals is particularly high.

Another important open problem is the continuum limits of the trapezoidal H^4 equation (11) and of the H^6 equation (12), which as far as we know are still unknown. The continuum limits for the trapezoidal H^4 equation (11) and of the H^6 equation (12) which are two-periodic quad-equations must be understood as the continuum limits of the systems arising from these equations getting rid of the two-periodic factors. This can be done by considering the cases when n and m are even or odd and then defining new unknown functions as follows [22, 53]:

$$u_{2k,2l} = v_{k,l}, \qquad u_{2k+1,2l} = w_{k,l}, \tag{90a}$$

$$u_{2k,2l+1} = y_{k,l}, \quad u_{2k+1,2l+1} = z_{k,l}. \tag{90b}$$

An interesting result would be to show that they can arise as discretization of continuous systems of hyperbolic Darboux integrable PDEs. This might shed light on the preservation of integrability properties upon discretization which is another important topic in mathematical physics.

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ORCID

G Gubbiotti https://orcid.org/0000-0003-0423-3736

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