ISSN 2304-0122 Ufa Mathematical Journal. Vol. 9. No 3 (2017). P. 158-164.

doi:10.13108/2017-9-3-158 UDC 517.9

ON INTEGRABILITY OF A DISCRETE ANALOGUE OF KAUP-KUPERSHMIDT EQUATION

R.N. GARIFULLIN, R.I. YAMILOV

Abstract. We study a new example of the equation obtained as a result of a recent generalized symmetry classification of differential-difference equations defined on five points of an one-dimensional lattice. We establish that in the continuous limit this new equation turns into the well-known Kaup-Kupershmidt equation. We also prove its integrability by constructing an $L - A$ pair and conservation laws. Moreover, we present a possibly new scheme for constructing conservation laws from $L - A$ pairs.

We show that this new differential-difference equation is similar by its properties to the discrete Sawada-Kotera equation studied earlier. Their continuous limits, namely the Kaup–Kupershmidt and Sawada-Kotera equations, play the main role in the classification of fifth order evolutionary equations made by V.G. Drinfel'd, S.I. Svinolupov and V.V. Sokolov.

Keywords: differential-difference equation, integrability, Lax pair, conservation law.

Mathematics Subject Classification: 37K10, 35G50, 39A10

1. Introduction

We consider the differential-difference equation

$$u_{n,t} = (u_n^2 - 1)\left(u_{n+2}\sqrt{u_{n+1}^2 - 1} - u_{n-2}\sqrt{u_{n-1}^2 - 1}\right), \tag{1}$$

where $n \in \mathbb{Z}$ and $u_n(t)$ is the unknown function of one discrete variable n and one continuous variable t, and the subscript t denotes the time derivative. Equation (1) is obtained as a result of generalized symmetry classification of five-point differential-difference equations

$$u_{n,t} = F(u_{n+2}, u_{n+1}, u_n, u_{n-1}, u_{n-2}) \tag{2}$$

made in [8]. Equation (1) coincides with the equation [8, (E17)] up to a scaling of u_n and t .

Equations (2) play an important role in the study of four-point discrete equations on the square lattice, which are very relevant for today, see e.g. [1, 5, 6, 15]. No relation between (1) and any other known equation of the form (2) is known. More precisely, here we mean the relations in the form of the transformations

$$\hat{u}_n = \varphi(u_{n+k}, u_{n+k-1}, \ldots, u_{n+m}), \quad k > m, \tag{3}$$

and their compositions, see a detailed discussion of such transformations in [7]. The only information we have at the moment on (1) is that it possesses a nine-point generalized symmetry of the form:

$$u_{n,\theta} = G(u_{n+4}, u_{n+3}, \ldots, u_{n-4}).$$

Р.Н. Гарифуллин, Р.И. Ямилов, Об интегрируемости дискретного аналога уравнения Каупа-Купершмидта.

The research is supported by the Russian Science Foundation (project no. 15-11-20007).

Поступила 12 декабря 2016 г.

In this article we study equation (1) in details. In Section 2 we find its continuous limit, which is the well-known Kaup-Kupershmidt equation [4,10]:

$$U_\tau = U_{xxxxx} + 5UU_{xxx} + \frac{25}{2}U_xU_{xx} + 5U^2U_x, \tag{4}$$

where the subscripts τ and x denote τ and x partial derivatives. In order to justify the integrability of (1), we construct an $L - A$ pair in Section 3 and in Section 4, we show that it provides an infinity hierarchy of conservation laws. In Section 5 we discuss possible generalizations of a scheme for constructing the conservation laws, which is formulated in Section 4 for equation (1).

2. Continuous limit

In the continuous limit, most of the equations of form (2) presented in [8] turns into the Korteweg-de Vries equation. The exceptions are (1) and the following two equations:

$$u_{n,t} = u_n^2(u_{n+2}u_{n+1} - u_{n-1}u_{n-2}) - u_n(u_{n+1} - u_{n-1}), \tag{5}$$

$$u_{n,t} = (u_n + 1)\left(\frac{u_{n+2}u_n(u_{n+1}+1)^2}{u_{n+1}} - \frac{u_{n-2}u_n(u_{n-1}+1)^2}{u_{n-1}} + (1+2u_n)(u_{n+1} - u_{n-1})\right), \tag{6}$$

which correspond to equations (E15) and (E16) in [8]. Equation (5) is known for a long time [17]. Equation (6) was found recently in [2] and it is related to (5) by a composition of transformations of the form (3). In the continuous limit, these three equations correspond to the fifth order equations of the form:

$$U_\tau = U_{xxxxx} + F(U_{xxxx}, U_{xxx}, U_{xx}, U_x, U). \tag{7}$$

There is a complete list of integrable equations (7), see [3, 11, 14]. Two equations play the main role there, namely, (4) and the Sawada-Kotera equation [16]:

$$U_\tau = U_{xxxxx} + 5UU_{xxx} + 5U_xU_{xx} + 5U^2U_x. \tag{8}$$

All the other are transformed into these two by transformations of the form:

$$\hat{U} = \Phi(U, U_x, U_{xx}, \dots, U_{x\dots x}).$$

It is known [1] that in the continuous limit equation (5) becomes the Sawada-Kotera equation (8). The other results below are new.

Using the substitution

$$u_n(t) = \frac{2\sqrt{2}}{3} + \frac{\sqrt{2}}{16}\varepsilon^2 U\left(\tau - \frac{9}{80}\varepsilon^5 t, x + \frac{2}{3}\varepsilon t\right), \quad x = \varepsilon n, \tag{9}$$

in equation (1), as $\varepsilon \to 0$ we get the Kaup-Kupershmidt equation (4).

It is interesting that equation (6) has two different continuous limits. The substitution

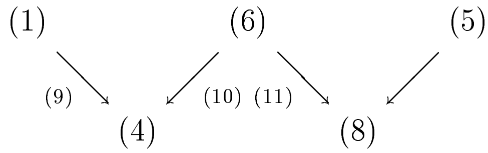
$$u_n(t) = -\frac{4}{3} - \varepsilon^2 U\left(\tau - \frac{18}{5}\varepsilon^5 t, x + \frac{4}{3}\varepsilon t\right), \quad x = \varepsilon n, \tag{10}$$

in (6) leads us to equation (4), while the substitution

$$u_n(t) = -\frac{2}{3} + \varepsilon^2 U\left(\tau - \frac{18}{5}\varepsilon^5 t, x + \frac{4}{3}\varepsilon t\right), \quad x = \varepsilon n, \tag{11}$$

gives rise to equation (8). As well as (1), equation (6) deserves further study.

In conclusion, let us present a picture that shows the link between discrete and continuous equations:



3. $L-A$ pair

As the continuous limit shows, the integrability properties of equation (1) should be close to those of equation (5). Following the $L-A$ pair [1, (15,17)], we look for an $L-A$ pair of the form:

$$L_n\psi_n = 0, \quad \psi_{n,t} = A_n\psi_n \tag{12}$$

with the operator L_n of the form:

$$L_n = l_n^{(2)}T^2 + l_n^{(1)}T + l_n^{(0)} + l_n^{(-1)}T^{-1},$$

where $l_n^{(k)},\ k = -1, 0, 1, 2$, depend on finitely many functions u_{n+j}. Here T is the shift operator: $Th_n = h_{n+1}$. In this case the operator A_n can be chosen as

$$A_n = a_n^{(1)}T + a_n^{(0)} + a_n^{(-1)}T^{-1}.$$

The compatibility condition for the system (12) is

$$\frac{d(L_n\psi_n)}{dt} = (L_{n,t} + L_nA_n)\psi_n = 0 \tag{13}$$

and it must be satisfied on virtue of equations (1) and $L_n\psi_n = 0$.

If we suppose that the coefficients $l_n^{(k)}$ depend on u_n only, as in [1], we can see that $a_n^{(k)}$ depend on u_{n-1}, u_n only. However, in this case the problem has no solution. This is why we proceed to the case when the functions $l_n^{(k)}$ depend on u_n, u_{n+1}. Then the coefficients $a_n^{(k)}$ must depend on u_{n-1}, u_n, u_{n+1}. In this case we succeeded to find the operators L_n and A_n with one irremovable arbitrary constant λ playing the role of a spectral parameter:

$$L_n = u_n\sqrt{u_{n+1}^2 - 1}T^2 + u_{n+1}T + \lambda\left(u_n - u_{n+1}\sqrt{u_n^2 - 1}T^{-1}\right), \tag{14}$$

$$A_n = \frac{\sqrt{u_n^2 - 1}}{u_n}\left(\sqrt{u_n^2 - 1}(u_{n+1}T + u_{n-1}T^{-1}) - \lambda^{-1}u_{n-1}T + \lambda u_{n+1}T^{-1}\right). \tag{15}$$

The $L-A$ pair (12,14,15) can be rewritten in the standard matrix form with 3×3 matrices $\tilde{L}_n, \tilde{A}_n$:

$$\Psi_{n+1} = \tilde{L}_n\Psi_n, \quad \Psi_{n,t} = \tilde{A}_n\Psi_n.$$

Here a new spectral function is given by

$$\Psi_n = 2^{-n}\begin{pmatrix} \frac{\sqrt{u_n^2-1}}{u_n}\psi_{n+1} \\ \psi_n \\ \psi_{n-1} \end{pmatrix},$$

and the matrices $\tilde{L}_n, \tilde{A}_n$ read:

$$\tilde{L}_n = \begin{pmatrix} -\frac{1}{\sqrt{u_n^2-1}} & -\frac{\lambda}{u_{n+1}} & \frac{\lambda\sqrt{u_n^2-1}}{u_n} \\ \frac{u_n}{\sqrt{u_n^2-1}} & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}, \tag{16}$$

$$\tilde{A}_n = \begin{pmatrix} \lambda^{-1} - \frac{u_{n-2}}{u_n}\sqrt{u_{n-1}^2 - 1} & u_{n+1}\sqrt{u_n^2 - 1} & \frac{(u_n^2-1)(\lambda u_{n+2}\sqrt{u_{n+1}^2-1}-u_n)}{u_n^2} \\ u_{n+1}\sqrt{u_n^2 - 1} - \lambda^{-1}u_{n-1} & 0 & \frac{\lambda u_{n+1}\sqrt{u_n^2-1}+u_{n-1}(u_n^2-1)}{u_n} \\ u_n + \lambda^{-1}u_{n-2}\sqrt{u_{n-1}^2 - 1} & u_n u_{n-1} & \lambda + \frac{u_{n-2}}{u_n}\sqrt{u_{n-1}^2 - 1} \end{pmatrix}. \tag{17}$$

In this case, unlike (13), the compatibility condition can be represented in matrix form:

$$\tilde{L}_{n,t} = \tilde{A}_{n+1}\tilde{L}_n - \tilde{L}_n\tilde{A}_n$$

without using the spectral function Ψ_n.

There are two methods for constructing the conservation laws by using such matrix $L - A$ pairs [5, 9, 12]. However, we do not see how to apply those methods in case of matrices (16) and (17). In the next section, we shall use a different scheme for constructing conservation laws from the $L - A$ pair (12), and this scheme seems to be new.

4. Conservation laws

The structure of operators (14,15) allows us to rewrite $L - A$ pair (12) in form of the Lax pair. The operator L_n depends linearly on λ:

$$L_n = P_n - \lambda Q_n, \tag{18}$$

where

$$P_n = u_n\sqrt{u_{n+1}^2 - 1}T^2 + u_{n+1}T, \quad Q_n = u_{n+1}\sqrt{u_n^2 - 1}T^{-1} - u_n.$$

Introducing $\hat{L}_n = Q_n^{-1}P_n$, we get an equation of the form:

$$\hat{L}_n\psi_n = \lambda\psi_n. \tag{19}$$

The functions $\lambda\psi_n$ and $\lambda^{-1}\psi_n$ in the second equation of (12) can be expressed in terms of $\hat{L}_n$ and ψ_n by using (19) and its consequence $\lambda^{-1}\psi_n = \hat{L}_n^{-1}\psi_n$. As a result we have:

$$\psi_{n,t} = \hat{A}_n\psi_n, \tag{20}$$

where

$$\hat{A}_n = \frac{\sqrt{u_n^2 - 1}}{u_n}\left(\sqrt{u_n^2 - 1}(u_{n+1}T + u_{n-1}T^{-1}) - u_{n-1}TP_n^{-1}Q_n + u_{n+1}T^{-1}Q_n^{-1}P_n\right).$$

It is important that new operators $\hat{L}_n$ and $\hat{A}_n$ in the $L - A$ pair (19,20) are independent of the spectral parameter λ. For this reason, the compatibility condition can be written in the operator form without using ψ-function:

$$\hat{L}_{n,t} = \hat{A}_n\hat{L}_n - \hat{L}_n\hat{A}_n = [\hat{A}_n, \hat{L}_n], \tag{21}$$

i.e., now it is of the form of the Lax equation. The difference between this $L - A$ pair and well-known Lax pairs for the Toda and Volterra equations is that now the operators $\hat{L}_n$ and $\hat{A}_n$ are nonlocal. Nevertheless, using the definition of inverse operators being linear:

$$P_nP_n^{-1} = P_n^{-1}P_n = 1, \quad Q_nQ_n^{-1} = Q_n^{-1}Q_n = 1, \tag{22}$$

by straightforward calculations we can check that (21) holds true.

The conservation laws of equation (1), which are expressions of the form

$$\rho_{n,t}^{(k)} = (T - 1)\sigma_n^{(k)}, \ k \geqslant 0,$$

can be derived from the Lax equation (21), notwithstanding nonlocal structure of the operators $\hat{L}_n, \hat{A}_n$, see [18]. For this we must, first of all, represent the operators $\hat{L}_n,\ \hat{A}_n$ as formal series in powers of T^{-1}:

$$H_n = \sum_{k\leqslant N} h_n^{(k)} T^k. \tag{23}$$

Formal series of this kind can be multiplied according the rule:

$$(a_n T^k)(b_n T^j) = a_n b_{n+k} T^{k+j}.$$

The inverse series can be obtained by definition (22), for instance:

$$Q_n^{-1} = -(1 + q_n T^{-1} + (q_n T^{-1})^2 + \ldots + (q_n T^{-1})^k + \ldots)\frac{1}{u_n}, \quad q_n = \frac{u_{n+1}}{u_n}\sqrt{u_n^2 - 1}.$$

The series $\hat{L}_n$ has the second order:

$$\hat{L}_n = \sum_{k\leqslant 2} l_n^{(k)} T^k = -(\sqrt{u_{n+1}^2 - 1}T^2 + u_{n+1}u_n T + u_{n+1}u_{n-1}\sqrt{u_n^2 - 1} + \ldots).$$

The conserved densities $\rho_n^{(k)}$ of equation (1) can be found as:

$$\rho_n^{(0)} = \log l_n^{(2)}, \qquad \rho_n^{(k)} = \operatorname{res} \hat{L}_n^k, \quad k \geqslant 1, \tag{24}$$

where the residue of formal series (23) is defined by the rule: $\operatorname{res} H_n = h_n^{(0)}$, see [18]. The corresponding functions $\sigma_n^{(k)}$ can easily be found by direct calculations.

In this way below we find the conserved densities $\hat{\rho}_n^{(k)}$ and then we simplify in accordance with the rule:

$$\hat{\rho}_n^{(k)} = c_k \rho_n^{(k)} + (T - 1) g_n^{(k)},$$

where c_k are constant. First three densities of equation (1) read:

$$\begin{aligned}
\hat{\rho}_n^{(0)} &= \log(u_n^2 - 1),\\
\hat{\rho}_n^{(1)} &= u_{n+1}u_{n-1}\sqrt{u_n^2 - 1},\\
\hat{\rho}_n^{(2)} &= (u_n^2 - 1)(2u_{n+2}u_{n-2}\sqrt{u_{n+1}^2 - 1}\sqrt{u_{n-1}^2 - 1} + u_{n+1}^2 u_{n-1}^2)\\
&\quad + u_{n+1}u_{n-1}u_n\sqrt{u_n^2 - 1}(u_{n+2}\sqrt{u_{n+1}^2 - 1} + u_{n-2}\sqrt{u_{n-1}^2 - 1}).
\end{aligned}$$

5. Discussion of the construction scheme

In the previous section we have outlined the scheme for constructing the conservation laws by example of equation (1). It can easily be generalized for the equations of an arbitrarily high order:

$$u_{n,t} = F(u_{n+M}, u_{n+M-1}, \ldots, u_{n-M}).$$

Assume that such equation has an $L - A$ pair of the form (12) with a linear in λ operator L_n, and let the operators P_n, Q_n of (18) have the form:

$$R_n = \sum_{k=k_1}^{k_2} r_n^{(k)} T^k, \quad k_1 \leqslant k_2 \in \mathbb{Z}, \tag{25}$$

with the coefficients $r_n^{(k)}$ depending on finitely many functions u_{n+j}. We suppose that

$$\hat{L}_n = Q_n^{-1} P_n = \sum_{k\leqslant N} l_n^{(k)} T^k$$

has a positive order $N \geqslant 1$. If $N \leqslant -1$, then we change $\lambda \to \lambda^{-1}$ and introduce $\tilde{L}_n = P_n^{-1} Q_n$ of a positive order. In the case $N = 0$, the scheme does not work.

As $\lambda^k \psi_n = \hat{L}_n^k \psi_n$ for any integer k, we can consider operators A_n of the form:

$$A_n = \sum_{k=m_1}^{m_2} a_n^{(k)}[T] \lambda^k, \quad m_1 \leqslant m_2 \in \mathbb{Z},$$

where $a_n^{(k)}[T]$ are operators of the form (25). Then we can rewrite A_n as

$$\hat{A}_n = \sum_{k=m_1}^{m_2} a_n^{(k)}[T] L_n^k = \sum_{k \leqslant \hat{N}} \hat{a}_n^{(k)} T^k.$$

We are led to Lax equation (21) with $\hat{L}_n, \hat{A}_n$ of form (23) and, therefore, we can construct the conserved densities as written above, namely, according (24) with the only difference $\rho_n^{(0)} = \log l_n^{(N)}$.

It should be remarked that the scheme can easily be applied to equation (5) with the $L - A$ pair [1, (15,17)].

In a quite similar way this scheme can also be applied in the continuous case, namely, to PDEs of the form

$$u_t = F(u, u_x, u_{xx}, \dots, u_{x \dots x}).$$

We consider the operators (25) with D_x instead of T, which become the differential operators, where D_x is the operator of total x-derivative. Besides, $k_2 \geqslant k_1 \geqslant 0$ and the coefficients $r_n^{(k)}$ depend on finitely many functions $u, u_x, u_{xx}, \dots$. Instead of (23) we consider the formal series in powers of D_x^{-1}. A theory of such formal series and, in particular, the definition of the residue were discussed in [13].

REFERENCES

1. V.E. Adler. *On a discrete analog of the Tzitzeica equation* // arXiv:1103.5139.
2. V.E. Adler, *Integrable Möbius invariant evolutionary lattices of second order* // Funkts. Anal. Pril. **50**:4, 13–25 (2016). [Funct. Anal. Appl. **50**:4 268–280, (2016).]
3. V.G. Drinfel'd, S.I. Svinolupov, V.V. Sokolov. *Classification of fifth order evolution equations possessing infinite series of conservation laws* // Dokl. AN Ukr. SSR. Ser. A. 10, 8–10 (1985) (in Urkainin).
4. A.P. Fordy and J. Gibbons. *Factorization of operators I. Miura transformations* // J. Math. Phys. **21**:10, 2508–2510 (1980).
5. R.N. Garifullin, A.V. Mikhailov and R.I. Yamilov. *Discrete equation on a square lattice with a nonstandard structure of generalized symmetries* // Teor. Matem. Fiz. **180**:1, 17–34 (2014) [Theor. Math. Phys. **180**:1, 765–780 (2014)].
6. R.N. Garifullin and R.I. Yamilov. *Generalized symmetry classification of discrete equations of a class depending on twelve parameters* // J. Phys. A: Math. Theor. **45**:34, 345205 (2012).
7. R.N. Garifullin, R.I. Yamilov and D. Levi. *Non-invertible transformations of differential-difference equations* // J. Phys. A: Math. Theor. **49**:37, 37LT01 (2016).
8. R.N. Garifullin, R.I. Yamilov, D. Levi. *Classification of five-point differential-difference equations* // J. Phys. A: Math. Theor. **50**:12, 125201 (2017).
9. I.T. Habibullin, M.V. Yangubaeva. *Formal diagonalization of a discrete Lax operator and conservation laws and symmetries of dynamical systems* // Teor. Matem. Fiz. **177**:3, 441–467 (2013) [Theor. Math. Phys. **177**:3, 1655–1679 (2013).]
10. D.J. Kaup. *On the inverse scattering problem for cubic eigenvalue problems of the class $\psi_{xxx} + 6Q\psi_x + 6R\psi = \lambda\psi$* // Stud. Appl. Math. **62**:3, 189–216 (1980).

11. A.G. Meshkov, V.V. Sokolov. *Integrable evolution equations with a constant separant* // Ufimsk. Matem. Zhurn. **4**:3, 104–154 (2012) [Ufa Math. Journal **4**:3, 104–152 (2012).]
12. A.V. Mikhailov. *Formal diagonalisation of Lax-Darboux schemes* // Model. Anal. Inform. Sist. **22**:6, 795–817 (2015).
13. A.V. Mikhailov, A.B. Shabat and R.I. Yamilov. *The symmetry approach to the classification of nonlinear equations. Complete lists of integrable systems* // Uspekhi Matem. Nauk. **42**:4, 3–53 (1987) [Russ. Math. Surv. **42**:4, 1–63 (1987).]
14. A.V. Mikhailov, V.V. Sokolov and A.B. Shabat. *The symmetry approach to classification of integrable equations* // What is Integrability? Ed. V.E. Zakharov. Springer series in Nonlinear Dynamics. 115–184 (1991).
15. A.V. Mikhailov and P. Xenitidis. *Second order integrability conditions for difference equations: an integrable equation* // Lett. Math. Phys. **104**:4, 431–450 (2014).
16. K. Sawada and T. Kotera. *A method for finding N-soliton solutions of the K.d.V. equation and K.d.V.-like equation* // Progr. Theoret. Phys. **51**:5, 1355–1367 (1974).
17. S. Tsujimoto and R. Hirota. *Pfaffian representation of solutions to the discrete BKP hierarchy in bilinear form* // J. Phys. Soc. Jpn. **65**, 2797–2806 (1996).
18. R. Yamilov. *Symmetries as integrability criteria for differential difference equations* // J. Phys. A: Math. Gen. **39**:45, R541–R623 (2006).

Garifullin Rustem Nailevich,
Institute of Mathematics, Ufa Scientific Center, RAS,
Chenryshevsky str. 112,
450008, Ufa, Russia
E-mail: `rustem@matem.anrb.ru`

Yamilov Ravil Islamovich,
Institute of Mathematics, Ufa Scientific Center, RAS,
Chenryshevsky str. 112,
450008, Ufa, Russia
E-mail: `RvlYamilov@matem.anrb.ru`