

AN UNUSUAL SERIES OF AUTONOMOUS DISCRETE INTEGRABLE EQUATIONS ON A SQUARE LATTICE

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We present an infinite series of autonomous discrete equations on a square lattice with hierarchies of autonomous generalized symmetries and conservation laws in both directions. Their orders in both directions are equal to κN , where κ is an arbitrary natural number and N is the equation number in the series. Such a structure of hierarchies is new for discrete equations in the case $N > 2$. The symmetries and conservation laws are constructed using the master symmetries, which are found directly together with generalized symmetries. Such a construction scheme is apparently new in the case of conservation laws. Another new point is that in one of the directions, we introduce the master symmetry time into the coefficients of the discrete equations. In the most interesting case $N = 2$, we show that a second-order generalized symmetry is closely related to a relativistic Toda-type integrable equation. As far as we know, this property is very rare in the case of autonomous discrete equations.

Keywords: integrable system, quad equation, generalized symmetry, conservation law, L - A pair

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1. Introduction

The discrete integrable equation

$$(u_{n,m+1} + 1)(u_{n,m} - 1) = (u_{n+1,m+1} - 1)(u_{n+1,m} + 1) \quad (1)$$

is well known [1], [2]. Several integrable generalizations of this equation were recently found [3]–[5]. All of them are nonautonomous, and we write the two most interesting here. One of them is

$$(u_{n,m+1} + \chi_{n+m+1})(u_{n,m} - \chi_{n+m}) = (u_{n+1,m+1} - \chi_{n+m})(u_{n+1,m} + \chi_{n+m+1}), \quad (2)$$

$$\chi_k = \frac{1}{2}(1 + (-1)^k),$$

which is Eq. (77) in [4] up to the involution $n \leftrightarrow m$. The second example in fact represents a series of discrete equations corresponding to some periods of an n -dependent coefficient. For any fixed $N \geq 1$, the equation is

$$\alpha_n(u_{n,m+1} + 1)(u_{n,m} - 1) = \alpha_{n+1}(u_{n+1,m+1} - 1)(u_{n+1,m} + 1), \quad (3)$$

$$\alpha_{n+N} = \alpha_n \neq 0 \quad \text{for all } n \in \mathbb{Z}.$$

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It was studied in [5]. In both cases, these generalizations have hierarchies of generalized symmetries and conservation laws in both directions and also L - A pairs, but all these objects are nonautonomous, i.e., they depend explicitly on the discrete variable n or m .

Here, we construct a series of autonomous integrable generalizations of (1). We show that all equations of that series have autonomous L - A pairs, generalized symmetries, and conservation laws. In particular, that series provides examples of autonomous discrete equations such that the minimum possible orders of their autonomous generalized symmetries in any direction can be arbitrarily high. A series of equations constructed here is a particular case of (3), but our results here are not a direct consequence of the results in [5].

In Sec. 2, we consider an autonomous generalization of (1) with an arbitrary constant coefficient. This generalization includes the whole series under consideration, and we construct hierarchies of generalized symmetries and of conservation laws in the m direction for it. These results are needed in the subsequent sections. In Sec. 3, we construct and study a series of autonomous integrable generalizations of (1), which is our aim here. We construct autonomous generalized symmetries and conservation laws in the m direction in Sec. 3.1 and discuss symmetries and conservation laws in the n direction in Secs. 3.2 and 3.4. We consider the most interesting case $N = 2$ in more detail in Sec. 3.3 and discuss its relation to a relativistic Toda-type equation. In Sec. 3.5, we construct autonomous L - A pairs for equations of the series. In Sec. 4 based on our results, we formulate and discuss an important conjecture on the symmetry structure of equations of the series and also briefly discuss all the new results obtained.

2. Autonomous generalization of (1) with an arbitrary constant coefficient

The broadest generalization of Eq. (1) that we know is

$$\begin{aligned} (u_{n,m+1} + a_{n,m+1})(u_{n,m} - a_{n,m}) &= (u_{n+1,m+1} - b_{n+1,m+1})(u_{n+1,m} + b_{n+1,m}), \\ a_{n,m+2} = a_{n,m}, \quad b_{n,m+2} = b_{n,m}, \quad a_{n,m}^2 &= b_{n,m}^2. \end{aligned} \tag{4}$$

This is Eq. (40) in [4] up to the transformations $n \leftrightarrow m$ and $b_{n,m} \rightarrow -b_{n,m}$. Equations (1) and (2) are particular cases of it. In the case $b_{n,m} = a_{n,m} \neq 0$ for all n and m , after the rescaling $u_{n,m} = \hat{u}_{n,m} a_{n,m}$, we obtain the equation for $\hat{u}_{n,m}$:

$$\alpha_n(u_{n,m+1} + 1)(u_{n,m} - 1) = \alpha_{n+1}(u_{n+1,m+1} - 1)(u_{n+1,m} + 1), \quad \alpha_n \neq 0, \tag{5}$$

where $\alpha_n = a_{n,m+1}a_{n,m}$. This equation was introduced in Sec. 3 in [3] in a slightly different form.

There is an obvious autonomous subcase of (5) with an arbitrary constant coefficient β :

$$(u_{n,m+1} + 1)(u_{n,m} - 1) = \beta(u_{n+1,m+1} - 1)(u_{n+1,m} + 1), \quad \beta \neq 0. \tag{6}$$

It corresponds to the restriction $\alpha_{n+1}/\alpha_n = \beta$ for all n , i.e., we obtain $\alpha_n = \beta^n$ up to a factor. Equation (6) has an L - A pair and hierarchies of generalized symmetries and conservation laws in the m direction, but all these objects are nonautonomous. Equation (6) includes the whole series of equations that is our aim here. The results we present here are needed in the subsequent sections.

An L - A pair for Eq. (6) is given by

$$\Psi_{n+1,m} = L_{n,m}^{(1)} \Psi_{n,m}, \quad \Psi_{n,m+1} = L_{n,m}^{(2)} \Psi_{n,m}, \tag{7}$$

where $\Psi_{n,m}$ is a vector function,

$$\begin{aligned} L_{n,m}^{(1)} &= \begin{pmatrix} 1 & 2\lambda\beta^n(u_{n,m} + 1) \\ -\frac{2}{u_{n,m} - 1} & \frac{u_{n,m} + 1}{u_{n,m} - 1} \end{pmatrix}, \\ L_{n,m}^{(2)} &= \begin{pmatrix} 1 & -\lambda\beta^n(u_{n,m} + 1)(u_{n,m+1} - 1) \\ 1 & 0 \end{pmatrix}, \end{aligned} \quad (8)$$

and λ is the spectral parameter. The L - A pair corresponding to (5) was presented in a more general form in [5] and was first constructed in [3].

2.1. Generalized symmetries in the m direction. A differential–difference equation of the form

$$\partial_t u_{n,m} = h_{n,m}(u_{n,m+\mu}, u_{n,m+\mu-1}, \dots, u_{n,m-\mu}), \quad \mu > 0, \quad (9)$$

is called a generalized symmetry in the m direction of the discrete equation

$$\Phi_{n,m}(u_{n,m}, u_{n+1,m}, u_{n,m+1}, u_{n+1,m+1}) = 0 \quad (10)$$

if Eqs. (9) and (10) are compatible. The compatibility condition is obtained by differentiating (10) with respect to the time t by virtue of (9),

$$\sum_{i,j \in \{0,1\}} h_{n+i,m+j} \frac{\partial \Phi_{n,m}}{\partial u_{n+i,m+j}} = 0, \quad (11)$$

and must be satisfied identically on solutions of (10).

We suppose that there exist numbers n_1 , m_1 , n_2 , and m_2 , such that

$$\frac{\partial h_{n_1,m_1}}{\partial u_{n_1,m_1+\mu}} \neq 0, \quad \frac{\partial h_{n_2,m_2}}{\partial u_{n_2,m_2-\mu}} \neq 0. \quad (12)$$

The number μ is called the order of generalized symmetry (9). The form of Eq. (9) is symmetric in a sense. An explanation why such a form is natural for integrable differential–difference equations can be found in [6].

We first discuss the particular case of (6) where $\beta = 1$, which is known. It is important because we construct generalized symmetries for general case (3) in terms of symmetries of this particular case. Its simplest generalized symmetry in the m direction is

$$\partial_{t_1} u_{n,m} = (u_{n,m}^2 - 1)(u_{n,m+1} - u_{n,m-1}) = f_{n,m}^{(1)} \quad (13)$$

and is just the modified Volterra equation. The known master symmetry of (13) (see [7]) can be written in the form

$$\partial_{\tau'} u_{n,m} = (u_{n,m}^2 - 1)((m+1)u_{n,m+1} - (m-1)u_{n,m-1}) = g_{n,m}. \quad (14)$$

The hierarchy of Eq. (13) can be constructed as

$$\partial_{t_k'} u_{n,m} = f_{n,m}^{(k)}(u_{n,m+k}, u_{n,m+k-1}, \dots, u_{n,m-k}), \quad k \geq 1, \quad (15)$$

$$\begin{aligned} f_{n,m}^{(k+1)} &= \text{ad}_{g_{n,m}} f_{n,m}^{(k)} = [g_{n,m}, f_{n,m}^{(k)}] = D_{\tau'} f_{n,m}^{(k)} - D_{t_k'} g_{n,m} = \\ &= \sum_{j=-k}^k g_{n,m+j} \frac{\partial f_{n,m}^{(k)}}{\partial u_{n,m+j}} - \sum_{j=-1}^1 f_{n,m+j}^{(k)} \frac{\partial g_{n,m}}{\partial u_{n,m+j}}. \end{aligned} \quad (16)$$

Here, $D_{\tau'}$ and $D_{t'_k}$ are the operators of total derivatives by virtue of the respective Eqs. (14) and (15) with the definition shown in (16).

We thus obtain the standard, known symmetries of the modified Volterra equation. Because $[f_{n,m}^{(1)}, f_{n,m}^{(k)}] = 0$ for the thus constructed functions and

$$g_{n,m} = mf_{n,m}^{(1)} + (u_{n,m}^2 - 1)(u_{n,m+1} + u_{n,m-1}), \quad (17)$$

it is easy to prove by induction that all the functions $f_{n,m}^{(k)}$ do not depend explicitly on m , for example,

$$f_{n,m}^{(2)} = (u_{n,m}^2 - 1)[(u_{n,m+1}^2 - 1)(u_{n,m+2} + u_{n,m}) - (u_{n,m-1}^2 - 1)(u_{n,m} + u_{n,m-2})]. \quad (18)$$

It can be shown (see an explanation below) that (15) are also generalized symmetries of discrete equation (6) with $\beta = 1$. We also note that both (13) and its master symmetry (14) are generalized symmetries of discrete equation (6) with $\beta = 1$.

In general case (6), the simplest generalized symmetry in the m direction is

$$\partial_{t_1} u_{n,m} = \beta^n f_{n,m}^{(1)}, \quad (19)$$

and its master symmetry is given by

$$\partial_{\tau''} u_{n,m} = \beta^n g_{n,m}, \quad (20)$$

but it is not a generalized symmetry of (6) and therefore allows constructing generalized symmetries for (19) but not for (6). To solve this problem, we must introduce a special dependence on the master symmetry time in discrete equation (6) and both Eqs. (19) and (20). Such a scheme with the time of the master symmetry introduced into a discrete equation is probably used for the first time.

We consider a special generalization of (6):

$$A_n(\tau)(u_{n,m+1} + 1)(u_{n,m} - 1) = A_{n+1}(\tau)(u_{n+1,m+1} - 1)(u_{n+1,m} + 1), \quad (21)$$

where

$$A_n(\tau) = (\beta^{-n} + 4\tau)^{-1}, \quad A'_n(\tau) = -4A_n^2(\tau), \quad A_n(0) = \beta^n, \quad (22)$$

and τ is an external parameter. Here, τ is the time of a master symmetry. It can be verified that both the equations

$$\partial_{t_1} u_{n,m} = F_{n,m}^{(1)} = A_n(\tau) f_{n,m}^{(1)}, \quad (23)$$

$$\partial_{\tau} u_{n,m} = G_{n,m} = A_n(\tau) g_{n,m} \quad (24)$$

are generalized symmetries of (21). In particular, the important relation $A'_n = -4A_n^2$ in (22) is a consequence of the compatibility of (21) and (24). Because (24) does not commute with (23), it is reasonable to expect that for any $k \geq 1$, the functions

$$\begin{aligned} F_{n,m}^{(k+1)} &= \text{ad}_{G_{n,m}} F_{n,m}^{(k)} = [G_{n,m}, F_{n,m}^{(k)}] = D_{\tau} F_{n,m}^{(k)} - D_{t_k} G_{n,m} = \\ &= \frac{\partial F_{n,m}^{(k)}}{\partial \tau} + \sum_{j=-k}^k G_{n,m+j} \frac{\partial F_{n,m}^{(k)}}{\partial u_{n,m+j}} - \sum_{j=-1}^1 F_{n,m+j}^{(k)} \frac{\partial G_{n,m}}{\partial u_{n,m+j}} \end{aligned} \quad (25)$$

define nontrivial generalized symmetries of (21). We see that (24) allows constructing a hierarchy of generalized symmetries of (21). It also generates a hierarchy of conservation laws (see Sec. 2.2). Therefore, Eq. (24) plays the role of the master symmetry not only for (23) but also for discrete equation (21).

We now study the structure of these generalized symmetries to later extract the autonomous ones among them. By induction, we can prove the formula

$$F_{n,m}^{(k)} = A_n^k(\tau) \sum_{j=0}^{k-1} 4^j c_{k,j} f_{n,m}^{(k-j)}, \quad (26)$$

where $c_{k,j}$ are some constants, for example,

$$c_{1,0} = 1, \quad c_{2,0} = 1, \quad c_{2,1} = -1, \quad c_{3,0} = 1, \quad c_{3,1} = -3, \quad c_{3,2} = 2. \quad (27)$$

Substituting (26) in (25), we obtain

$$F_{n,m}^{(k+1)} = \frac{\partial A_n^k(\tau)}{\partial \tau} \sum_{j=0}^{k-1} 4^j c_{k,j} f_{n,m}^{(k-j)} + A_n^{k+1}(\tau) \sum_{j=0}^{k-1} 4^j c_{k,j} \text{ad}_{g_{n,m}} f_{n,m}^{(k-j)}. \quad (28)$$

Taking (16) and (22) into account, we obtain

$$F_{n,m}^{(k+1)} = -k A_n^{k+1}(\tau) \sum_{j=1}^k 4^j c_{k,j-1} f_{n,m}^{(k+1-j)} + A_n^{k+1}(\tau) \sum_{j=0}^{k-1} 4^j c_{k,j} f_{n,m}^{(k+1-j)}. \quad (29)$$

Comparing (26) and (29), we derive the recurrence relations

$$c_{k+1,j} = c_{k,j} - k c_{k,j-1}, \quad c_{k,-1} = c_{k,k} = 0, \quad c_{1,0} = 1, \quad 0 \leq j \leq k, \quad k \geq 1. \quad (30)$$

We see that generalized symmetries of (21) have the form

$$\partial_{t_k} u_{n,m} = F_{n,m}^{(k)}, \quad k \geq 1, \quad (31)$$

where the functions $F_{n,m}^{(k)}$ have form (26) and $f_{n,m}^{(k)}$, $A_n(\tau)$, and $c_{k,j}$ are given by (16), (22), and (30). The order of such a symmetry is k . An explicit dependence on n and τ is defined by the factor $A_n^k(\tau)$, and there is here no explicit dependence on m . If $\tau = 0$, then Eq. (21) becomes (6), and symmetries (31) become generalized symmetries of (6).

Theorem 1. *Discrete equation (6) has generalized symmetries of the form*

$$\partial_{t_k} u_{n,m} = \beta^{nk} \sum_{j=0}^{k-1} 4^j c_{k,j} f_{n,m}^{(k-j)}, \quad k \geq 1, \quad (32)$$

where $f_{n,m}^{(k)}$ and $c_{k,j}$ are defined by (16) and (30). These symmetries do not depend explicitly on m , and a dependence on n is given by the factor β^{nk} .

In the case $\beta = 1$, we can see that not only the special linear combination of $f_{n,m}^{(k)}$ given by (32) but also any of the functions $f_{n,m}^{(k)}$ define generalized symmetries of (6).

2.2. Conservation laws in the m direction. We consider the relation

$$(T_n - 1)p_{n,m} = (T_m - 1)q_{n,m}, \quad (33)$$

where the functions $p_{n,m}$ and $q_{n,m}$ depend on n , m , and $u_{n+i,m+j}$ and where T_n and T_m are the shift operators in the n and m directions: $T_n h_{n,m} = h_{n+1,m}$, $T_m h_{n,m} = h_{n,m+1}$. This relation is called the conservation law of discrete equation (10) if (33) is satisfied identically on solutions of (10). Using (10), we can rewrite $p_{n,m}$ and $q_{n,m}$ in terms of only n , m , and the functions $u_{n+i,m}$ and $u_{n,m+j}$, and we represent them in such a form. In the m -direction case, $p_{n,m}$ has the form $p_{n,m} = p_{n,m}(u_{n,m+k_1}, u_{n,m+k_1-1}, \dots, u_{n,m+k_2})$, where $k_1 \geq k_2$. This function $p_{n,m}$ can be called a conserved density by analogy with the discrete–differential case.

For $k_1 > k_2$, we obtain conserved densities $p_{n,m}$ such that

$$\frac{\partial^2 p_{n,m}}{\partial u_{n,m+k_1} \partial u_{n,m+k_2}} \neq 0 \quad \text{for all } n, m.$$

The number $k_1 - k_2$ can then be called the order of this conservation law (see, e.g., [6]). If $k_1 = k_2$ and $p_{n,m}$ is not constant, then the conservation law is not trivial, and its order is equal to zero. Conservation laws of different orders are essentially different.

Conservation laws for (6) were constructed in [8] using L – A pair (8). But we have only one way to construct conservation laws and a few laws. In this one way, it is difficult to follow the structure of the conservation laws and extract the autonomous ones among them. Here, we solve the problem using master symmetry (24). Such a strategy for constructing conservation laws is apparently new.

It is known in the discrete–differential case that differentiating a conservation law by virtue of the master symmetry, we obtain new conservation laws [6]. Here, we show that the same is true for discrete conservation laws (33). We demonstrate this in detail in the example of discrete equation (21) and its master symmetry (24). As in the preceding section, we then pass to Eq. (6) by choosing $\tau = 0$.

It is easy to verify that the functions

$$p_{n,m}^{(1)} = A_n(\tau)(u_{n,m+1} - 1)(u_{n,m} + 1), \quad q_{n,m}^{(1)} = -2A_n(\tau)u_{n,m} \quad (34)$$

define a conservation law for (21) in the m direction. Using it and master symmetry (24), we can construct a hierarchy of conservation laws for Eq. (21):

$$(T_n - 1)p_{n,m}^{(k)} = (T_m - 1)q_{n,m}^{(k)}, \quad k \geq 1, \quad (35)$$

all of which do not depend explicitly on m . We do this by induction, using the property that

$$(T_n - 1)D_\tau p_{n,m}^{(k)} = (T_m - 1)D_\tau q_{n,m}^{(k)} \quad (36)$$

is also a conservation law, where D_τ is the total derivative by virtue of master symmetry (24), which is one of generalized symmetries of (21) in the m direction. As a result, the operator D_τ automatically commutes with T_m and commutes with T_n on solutions of discrete equation (21).

New conservation law (36) depends on m explicitly. To eliminate m , we use the fact that we can add a function of the form $(T_n - 1)(T_m - 1)h_{n,m}$ to both sides of conservation law (33) and obtain a new conservation law defined by

$$\tilde{p}_{n,m} = p_{n,m} + (T_m - 1)h_{n,m}, \quad \tilde{q}_{n,m} = q_{n,m} + (T_n - 1)h_{n,m}. \quad (37)$$

In addition, we use the fact that $p_{n,m}^{(k)}$ are also conserved densities for differential–difference equation (23),

$$D_{t_1} p_{n,m}^{(k)} = (T_m - 1)r_{n,m}^{(k)}, \quad (38)$$

because (38) is satisfied for $k = 1$ with

$$r_{n,m}^{(1)} = A_n^2(\tau)(u_{n,m+1} - 1)(u_{n,m}^2 - 1)(u_{n,m-1} + 1)$$

and it is known from the differential–difference case that if $p_{n,m}^{(k)}$ is a conserved density of (23), then the function $D_\tau p_{n,m}^{(k)}$ is also a conserved density of it.

We suppose that the functions $p_{n,m}^{(k)}$, $q_{n,m}^{(k)}$, and $r_{n,m}^{(k)}$ do not depend explicitly on m for some $k \geq 1$. The total derivative $D_\tau p_{n,m}^{(k)}$ then depends on m linearly:

$$D_\tau p_{n,m}^{(k)} = (m - 1)D_{t_1} p_{n,m}^{(k)} + \dots \quad (39)$$

Because of (38), we can use transformation (37) with $h_{n,m} = -(m - 1)r_{n,m}^{(k)}$ and as a result obtain

$$p_{n,m}^{(k+1)} = D_\tau p_{n,m}^{(k)} - (T_m - 1)[(m - 1)r_{n,m}^{(k)}], \quad (40)$$

$$q_{n,m}^{(k+1)} = D_\tau q_{n,m}^{(k)} - (T_n - 1)[(m - 1)r_{n,m}^{(k)}]. \quad (41)$$

The function $p_{n,m}^{(k+1)}$ is a new conserved density for discrete equation (21) and for its symmetry (23) and does not depend explicitly on m .

We now explain how to construct the function $r_{n,m}^{(k+1)}$ and why it has no explicit dependence on m . We also give a simpler construction scheme for the functions $p_{n,m}^{(k)}$, which provides important information about their structure and also a second, more rigorous justification that these functions are conserved densities of (23).

The function

$$v_{n,m} = A_n(\tau)(u_{n,m+1} - 1)(u_{n,m} + 1) \quad (42)$$

satisfies the equations

$$\partial_{t_1} v_{n,m} = v_{n,m}(v_{n,m+1} - v_{n,m-1}), \quad (43)$$

$$\partial_\tau v_{n,m} = v_{n,m}((m + 2)v_{n,m+1} + v_{n,m} - (m - 1)v_{n,m-1}), \quad (44)$$

which is just the Volterra equation and its master symmetry [9]. Relation (42) is a slight nonautonomous generalization of the well-known discrete Miura transformation. It transforms the problem of constructing $p_{n,m}^{(k)}$ and $r_{n,m}^{(k)}$ into the well-known problem for the Volterra equation. In particular, the initial conserved density $p_{n,m}^{(1)}$ becomes $p_{n,m}^{(1)} = v_{n,m}$, and it is a common density for all generalized symmetries of Volterra equation (43). Therefore, it can be rigorously proved that the functions $D_\tau^k p_{n,m}^{(1)}$ for all k are conserved densities for (43) (see Theorem 20 in [6]).

The function $r_{n,m}^{(1)}$ becomes $r_{n,m}^{(1)} = v_{n,m}v_{n,m-1}$. All the functions $p_{n,m}^{(k)}$ and $r_{n,m}^{(k)}$ can also be expressed in terms of $v_{n,m+j}$, i.e., relations (38) become conservation laws of Volterra equation (43). The structure of these conservation laws is described by the following lemma.

Lemma 1. *For any $k \geq 1$, the function $p_{n,m}^{(k)}$ is an autonomous and homogeneous polynomial of degree k and has the form*

$$p_{n,m}^{(k)} = P^{(k)}(v_{n,m}, v_{n,m+1}, \dots, v_{n,m+k-1}), \quad \frac{\partial^2 P^{(k)}}{\partial v_{n,m} \partial v_{n,m+k-1}} \neq 0. \quad (45)$$

The function $r_{n,m}^{(k)}$ is also an autonomous and homogeneous polynomial of degree $k + 1$ and has the form

$$r_{n,m}^{(k)} = R^{(k)}(v_{n,m-1}, v_{n,m}, v_{n,m+1}, \dots, v_{n,m+k-1}). \quad (46)$$

We recall that for any $k \geq 1$, two such functions define a conservation law of order $k - 1$ for differential-difference equation (43) (see, e.g., [6]). The functions $p_{n,m}^{(k)}$ and $r_{n,m}^{(k)}$ are autonomous in the sense that they depend explicitly on neither n nor m .

Outline of proof. The lemma holds for $k = 1$. We suppose that it holds for some $k \geq 1$ and prove that it holds for $k + 1$. We use the same formula (40) to construct $p_{n,m}^{(k+1)}$. In this case, we can easily verify that this function satisfies the conditions of the lemma. The function $p_{n,m}^{(k+1)}$ is the next conserved density of (43). There hence exists a function $r_{n,m}^{(k+1)}$ satisfying relation (38) and depending on $v_{n,m+j}$. It can be easily constructed directly from (38) (see, e.g., [6]). Moreover, the left-hand side of (38) is an autonomous homogeneous polynomial of $v_{n,m+j}$ of degree $k + 2$. If we seek $r_{n,m}^{(k+1)}$ as a homogeneous polynomial, then it exists and is unique and autonomous. The resulting function satisfies the assertion of the lemma.

If we replace $v_{n,m+j}$ with $u_{n,m+j}$ in both $p_{n,m}^{(k)}$ and $r_{n,m}^{(k)}$ using (42), then we obtain a conservation law for symmetry (23), and its order is k (see Theorem 18 in [6]). It is clear that the thus constructed functions $p_{n,m}^{(k)}$ and $r_{n,m}^{(k)}$ do not depend explicitly on m . Because $p_{n,m}^{(k)}$ in (45) is a homogeneous polynomial of degree k , its structure in terms of $u_{n,m+j}$ is

$$p_{n,m}^{(k)} = A_n^k(\tau) \widehat{P}^{(k)}(u_{n,m}, u_{n,m+1}, \dots, u_{n,m+k}), \quad (47)$$

where $\widehat{P}^{(k)}$ is an autonomous polynomial. The dependence on n and τ here is determined by only the factor $A_n^k(\tau)$.

We can now show that $q_{n,m}^{(k+1)}$ does not depend explicitly on m and the structure of $q_{n,m}^{(k)}$ is similar to (47).

Lemma 2. For any $k \geq 1$, the function $q_{n,m}^{(k)}$ has the form

$$q_{n,m}^{(k)} = A_n^k(\tau) \widehat{Q}^{(k)}(u_{n,m}, u_{n,m+1}, \dots, u_{n,m+k-1}), \quad (48)$$

where $\widehat{Q}^{(k)}$ is an autonomous polynomial. The function $q_{n,m}^{(k+1)}$ can be constructed using the recurrence relation

$$q_{n,m}^{(k+1)} = \frac{\partial q_{n,m}^{(k)}}{\partial \tau} + A_n(\tau) \sum_{j=0}^{k-1} (u_{n,m+j}^2 - 1) ((j+2)u_{n,m+j+1} - ju_{n,m+j-1}) \frac{\partial q_{n,m}^{(k)}}{\partial u_{n,m+j}}. \quad (49)$$

Proof. It follows from relations (39) and (41) that $q_{n,m}^{(k+1)}$ depends on m linearly:

$$q_{n,m}^{(k+1)} = (m-1)W_{n,m}^{(k)} + Z_{n,m}^{(k)}, \quad W_{n,m}^{(k)} = D_{t_1} q_{n,m}^{(k)} - (T_n - 1)r_{n,m}^{(k)}.$$

Relation (35) where k is replaced with $k + 1$ and the fact that $p_{n,m}^{(k+1)}$ is independent of m imply that $(T_m - 1)W_{n,m}^{(k)} = 0$ on solutions of (21).

The function $W_{n,m}^{(k)}$ can be expressed in terms of only n , τ , and $u_{n,m+j}$. This is obvious for $D_{t_1} q_{n,m}^{(k)}$ and holds for $r_{n,m}^{(k)}$ by virtue of (42) and (46). Definition (42) of $v_{n,m}$ and discrete equation (21) imply that

$$v_{n+1,m} = A_n(\tau)(u_{n,m+1} + 1)(u_{n,m} - 1). \quad (50)$$

Therefore, $T_n r_{n,m}^{(k)} = R^{(k)}(v_{n+1,m-1}, v_{n+1,m}, \dots, v_{n+1,m+k-1})$ can also be expressed in this way. It is important that the dependence on $u_{n,m+j}$ in $W_{n,m}^{(k)}$ is polynomial. Such a function $W_{n,m}^{(k)}$ satisfies $(T_m - 1)W_{n,m}^{(k)} = 0$ if and only if it is independent of $u_{n,m+j}$, i.e., $W_{n,m}^{(k)} = \eta_n^{(k)}(\tau)$. This function vanishes if $u_{n,m+j} = 1$ for all j , and therefore $W_{n,m}^{(k)} \equiv 0$.

For $q_{n,m}^{(k+1)}$, we now obtain the formula $q_{n,m}^{(k+1)} = (D_\tau - (m-1)D_{t_1})q_{n,m}^{(k)}$, which can be rewritten as (49). Structure (48) for $q_{n,m}^{(k+1)}$ follows from (22), (34), and recurrence relation (49).

We thus obtain the explicit formulas

$$p_{n,m}^{(1)} = v_{n,m}, \quad q_{n,m}^{(1)} = -2A_n(\tau)u_{n,m}, \quad r_{n,m}^{(1)} = v_{n,m}v_{n,m-1}, \quad (51)$$

$$p_{n,m}^{(2)} = v_{n,m}(2v_{n,m+1} + v_{n,m}),$$

$$q_{n,m}^{(2)} = -4A_n^2(\tau)(u_{n,m+1}u_{n,m}^2 - u_{n,m+1} - 2u_{n,m}), \quad (52)$$

$$r_{n,m}^{(2)} = 2v_{n,m}v_{n,m-1}(v_{n,m+1} + v_{n,m}),$$

$$p_{n,m}^{(3)} = 2v_{n,m}(3v_{n,m+2}v_{n,m+1} + 3v_{n,m+1}v_{n,m} + 3v_{n,m+1}^2 + v_{n,m}^2),$$

$$q_{n,m}^{(3)} = -4A_n^3(\tau)[3(u_{n,m}^2 - 1)(u_{n,m+2}u_{n,m+1}^2 + u_{n,m+1}^2u_{n,m} -$$

$$- u_{n,m+2} - 4u_{n,m+1} - 5u_{n,m}) + 16u_{n,m}^3], \quad (53)$$

$$r_{n,m}^{(3)} = 6v_{n,m}v_{n,m-1}(v_{n,m+2}v_{n,m+1} + 2v_{n,m+1}v_{n,m} + v_{n,m+1}^2 + v_{n,m}^2),$$

where $v_{n,m}$ is given by (42). This illustrates the construction scheme described above.

If $\tau = 0$, then discrete equation (21) becomes (6), and its conservation laws become conservation laws of (6). Because $A_n(0) = \beta^n$, we obtain the following result for the conservation laws of (6).

Theorem 2. For any $k \geq 1$, discrete equation (6) has conservation law (35) of the order k defined by functions of the forms

$$p_{n,m}^{(k)} = \beta^{nk} \widehat{P}^{(k)}(u_{n,m}, u_{n,m+1}, \dots, u_{n,m+k}),$$

$$q_{n,m}^{(k)} = \beta^{nk} \widehat{Q}^{(k)}(u_{n,m}, u_{n,m+1}, \dots, u_{n,m+k-1}), \quad (54)$$

where the polynomials $\widehat{P}^{(k)}$ and $\widehat{Q}^{(k)}$ depend explicitly on neither n nor m .

3. A series of autonomous integrable generalizations

In Sec. 2, we considered autonomous discrete equation (6) with an L - A pair and hierarchies of generalized symmetries and conservation laws in the m direction. But all those objects are essentially nonautonomous. The symmetries, conservation laws, and L - A pairs of autonomous discrete equations that we consider here are autonomous, and these equations have hierarchies of generalized symmetries and conservation laws in both n and m directions. It was shown in [5] that discrete equation (3) that has a periodic coefficient α_n , should have hierarchies of generalized symmetries and conservation laws in both the n and m directions. In the case of conservation laws, this was shown by using an L - A pair. In the case of symmetries, we studied some particular cases.

Because we are interested in autonomous equations, we consider the intersection of Eqs. (3) and (6). It follows from $\alpha_n = \beta^n$ that $\beta^N = 1$. We therefore consider the equations

$$(u_{n,m+1} + 1)(u_{n,m} - 1) = \beta_N(u_{n+1,m+1} - 1)(u_{n+1,m} + 1), \quad \beta_N^N = 1, \quad N \geq 1. \quad (55)$$

To separate equations with different N , we here consider primitive roots of unity. It is clear that $\beta_1 = 1$, and this case is well-known (see (1)). If $N > 1$, then

$$\beta_N^N = 1, \quad \beta_N^j \neq 1 \quad \text{for all } 1 \leq j < N. \quad (56)$$

In particular,

$$\beta_1 = 1, \quad \beta_2 = -1, \quad \beta_3 = -\frac{1}{2} \pm i\frac{\sqrt{3}}{2}, \quad \beta_4 = \pm i, \quad (57)$$

i.e., in the last two cases, we have two primitive roots corresponding to the signs $+$ and $-$. For any $N > 4$, at least two primitive roots exist, which are given by $\beta_N = e^{\pm 2i\pi/N}$. Hence, we below consider the series of Eqs. (55) such that β_N are primitive roots of unity.

Currently, we know only one similar series of integrable discrete equations [10]. Those equations are Darboux integrable and of the Burgers type, and the minimum order of their first integrals can be arbitrarily high. Equations of series (55) are integrable by the inverse scattering method.

For Eq. (55) with (56) and $N = 2$, we have $\beta_2 = -1$, i.e., the equation is written as

$$(u_{n,m+1} + 1)(u_{n,m} - 1) = -(u_{n+1,m+1} - 1)(u_{n+1,m} + 1). \quad (58)$$

This is the most interesting example in the series because it has real coefficients. It was found in [11], where the authors sought discrete equations on a square lattice using five-point differential–difference equations obtained in the recent symmetry classification [12], [13] as a generalized symmetry.

3.1. Autonomous generalized symmetries and conservation laws in the m direction. Here, we construct autonomous generalized symmetries and conservation laws in the m direction for Eq. (55) with (56) using the results in Sec. 2.

In Theorem 1, we constructed symmetries (32), where an explicit dependence on n was given by the factor β^{nk} . It follows from this theorem that Eqs. (55) with (56) have infinitely many autonomous generalized symmetries in the m direction, which are given by (32) with $k = N, 2N, 3N, \dots$

Corollary 1. *For any $N \geq 2$, discrete equation (55) with (56) has autonomous generalized symmetries in the m direction given by (32), (16), and (30) with $k = \kappa N$, $\kappa \in \mathbb{N}$.*

For Eq. (58), the simplest autonomous generalized symmetry in the m direction is given by

$$\partial_{t_2} u_{n,m} = c_{2,0} f_{n,m}^{(2)} + 4c_{2,1} f_{n,m}^{(1)}. \quad (59)$$

From (30), we find that $c_{2,0} = 1$ and $c_{2,1} = -1$, and using (13) and (18) for the functions $f_{n,m}^{(1)}$ and $f_{n,m}^{(2)}$, we obtain the explicit formulas

$$\begin{aligned} \partial_{t_2} u_{n,m} = & (u_{n,m}^2 - 1)[(u_{n,m+1}^2 - 1)(u_{n,m+2} + u_{n,m}) - \\ & - (u_{n,m-1}^2 - 1)(u_{n,m} + u_{n,m-2}) - 4(u_{n,m+1} - u_{n,m-1})]. \end{aligned} \quad (60)$$

This symmetry was first found in [11].

In Theorem 2, we constructed conservation laws for Eq. (6) given by (54) where an explicit dependence on n was given by the factor β^{nk} . It follows from this theorem that Eq. (55) with (56) have infinitely many autonomous conservation laws in the m direction, and they are given by (54) with $k = N, 2N, 3N, \dots$

Corollary 2. *For any $N \geq 2$, discrete equation (55) with (56) has infinitely many autonomous conservation laws, and their orders are multiples of N .*

In the case of Eq. (58), the simplest autonomous conservation law, taken from (52), has the order 2 and is given by

$$\begin{aligned} p_{n,m}^{(2)} &= v_{n,m}(2v_{n,m+1} + v_{n,m}), & v_{n,m} &= (u_{n,m+1} - 1)(u_{n,m} + 1), \\ q_{n,m}^{(2)} &= -4(u_{n,m+1}u_{n,m}^2 - u_{n,m+1} - 2u_{n,m}). \end{aligned}$$

3.2. Generalized symmetries in the n direction. We consider generalized symmetries in the n direction. Discrete equation (10) has a generalized symmetry in the n direction

$$\partial_\theta u_{n,m} = \zeta_{n,m}(u_{n+\nu,m}, u_{n+\nu-1,m}, \dots, u_{n-\nu,m}), \quad \nu > 0, \quad (61)$$

if (10) and (61) are compatible, i.e., the equation

$$D_\theta \Phi_{n,m} = \sum_{i,j \in \{0,1\}} \zeta_{n+i,m+j} \frac{\partial \Phi_{n,m}}{\partial u_{n+i,m+j}} = 0 \quad (62)$$

is satisfied identically on solutions of (10). It is natural to suppose that there exist numbers n_1, m_1, n_2 , and m_2 such that

$$\frac{\partial \zeta_{n_1, m_1}}{\partial u_{n_1+\nu, m_1}} \neq 0, \quad \frac{\partial \zeta_{n_2, m_2}}{\partial u_{n_2-\nu, m_2}} \neq 0. \quad (63)$$

The number ν is called the order of symmetry (61). The form of Eq. (61) is symmetric as in Sec. 2.1 for the same reason as (9).

Two theorems for Eq. (5) and its “nondegenerate” symmetries of orders 1 and 2 were proved in [5]. Here, we prove analogous theorems for Eq. (6) and its symmetries of orders 1, 2 and 3 without using any nondegeneracy conditions.

Theorem 3. *The following two statements hold:*

1. *If Eq. (6) has a generalized symmetry (61) in the n direction of order N such that $N = 1, 2, 3$, then $\beta^N = 1$, i.e., Eq. (6) has form (55).*
2. *Equation (55) where $N = 1, 2, 3$ and β_N is a primitive root of unity has a generalized symmetry of order N and does not have generalized symmetries of lower orders.*

Outline of proof. To construct generalized symmetries for the discrete equations, we use a method developed in [3], [14] (see [15] for its most advanced version). Compatibility condition (62) is a functional equation for the unknown function $\zeta_{n,m}$. The method allows obtaining results from (62) in the form of partial differential equations for $\zeta_{n,m}$ using the so-called annihilation operators introduced in [16].

1. If Eq. (6) has a generalized symmetry (61) of order $N = 1, 2, 3$, then the simplest differential consequences of (62) have the forms

$$(\beta^N - 1) \frac{\partial \zeta_{n,m}}{\partial u_{n+N,m}} = 0, \quad (\beta^N - 1) \frac{\partial \zeta_{n,m}}{\partial u_{n-N,m}} = 0, \quad (64)$$

and these relations must be satisfied for all n and m . Conditions (63) and (64) imply that $\beta^N = 1$.

2. For Eq. (55) where $N = 1, 2, 3$ and β_N is a primitive root of unity, we seek symmetries of form (61) with $\nu = N$ and we use no restriction like (63). We find the following generalized symmetries.

In the case $N = 1$, it has the form

$$\partial_{\theta_1} u_{n,m} = (u_{n,m}^2 - 1) \left(\frac{a_{n+1}}{u_{n+1,m} + u_{n,m}} - \frac{a_n}{u_{n,m} + u_{n-1,m}} \right), \quad (65)$$

where $a_n = b + cn$ with arbitrary constants b and c .

In the case $N = 2$, it has the form

$$\partial_{\theta_2} u_{n,m} = (u_{n,m}^2 - 1)(T_n - 1) \left(\frac{a_{n+1}(u_{n+1,m} + u_{n,m})}{U_{n,m}} + \frac{a_n(u_{n-1,m} + u_{n-2,m})}{U_{n-1,m}} \right), \quad (66)$$

where

$$U_{n,m} = (u_{n+1,m} + u_{n,m})(u_{n,m} + u_{n-1,m}) - 2(u_{n,m}^2 - 1). \quad (67)$$

The function a_n is given by $a_n = b_n + cn$, where c is a constant and $b_{n+2} \equiv b_n$ is an arbitrary two-periodic function on n . It can be represented as $b_n = b^{(1)} + (-1)^n b^{(2)}$ with two arbitrary constants $b^{(1)}$ and $b^{(2)}$.

In the case $N = 3$, it has the form

$$\partial_{\theta_3} u_{n,m} = (u_{n,m}^2 - 1)(T_n - 1) \left(\frac{a_{n+2} V_{n,m}}{U_{n,m}} + \frac{a_n W_{n,m}}{U_{n-2,m}} + (T_n + 1) \frac{a_{n+1} Z_{n,m}}{U_{n-1,m}} \right), \quad (68)$$

where

$$\begin{aligned} V_{n,m} &= \beta_3^2 (u_{n+1,m}^2 - 1) + u_{n+1,m} (u_{n+2,m} - u_{n-1,m}) - u_{n+2,m} u_{n-1,m} + 1, \\ W_{n,m} &= \beta_3 (u_{n-2,m}^2 - 1) + u_{n-2,m} (u_{n-1,m} + u_{n-3,m}) + u_{n-1,m} u_{n-3,m} + 1, \\ Z_{n,m} &= (u_{n+1,m} + u_{n,m})(u_{n-1,m} + u_{n-2,m}), \\ U_{n,m} &= \beta_3^2 (u_{n+1,m}^2 - 1)(u_{n,m} + u_{n-1,m}) + \beta_3 (u_{n,m}^2 - 1)(u_{n+2,m} + u_{n+1,m}) + \\ &\quad + (u_{n+1,m} u_{n,m} + 1)(u_{n+2,m} + u_{n-1,m}) + (u_{n+1,m} + u_{n,m})(u_{n+2,m} u_{n-1,m} + 1). \end{aligned}$$

Here, β_3 is either of the two primitive roots in (57). The function a_n is given by $a_n = b_n + cn$, where c is a constant and $b_{n+3} \equiv b_n$ is an arbitrary three-periodic function. It can be represented as $b_n = b^{(1)} + b^{(2)} \beta_3^n + b^{(3)} \beta_3^{2n}$, where $b^{(1)}$, $b^{(2)}$, and $b^{(3)}$ are arbitrary constants.

We see that such an Eq. (55) has a generalized symmetry of order N in all three cases. We can also see that generalized symmetries of lower orders do not exist in the cases $N = 2, 3$ because the requirements $\partial \zeta_{n,m} / \partial u_{n+N,m} \equiv 0$ or $\partial \zeta_{n,m} / \partial u_{n-N,m} \equiv 0$ imply that $\zeta_{n,m} \equiv 0$.

In the case $N = 2$, we have only the primitive root $\beta_2 = -1$, and the formulas for b_n in the cases $N = 2$ and $N = 3$ are analogous. We have the following important corollary of Theorem 3 for autonomous equations (55).

Corollary 3. *Any of Eqs. (55) where $N = 1, 2, 3$ and β_N is a primitive root of unity has an autonomous generalized symmetry of order N given by (65)–(68) with $a_n \equiv 1$, and does not have autonomous generalized symmetries of lower orders.*

These autonomous symmetries exemplify integrable differential–difference equations with one continuous variable θ_N and one discrete variable n , while the parameter m is not essential. Symmetry (65) with $a_n \equiv 1$ was first found in [3] and corresponds to a well-known Volterra-type integrable equation [6], [17]. Symmetry (66) is a particular case of a nonautonomous symmetry of discrete equation (5) and was found in [5]. Nevertheless, Eqs. (66) and (67) with $a_n \equiv 1$ provide new examples of autonomous integrable differential–difference equations of orders 2 and 3.

If $N = 2$, then the subcases $a_n \equiv 1$ and $a_n \equiv (-1)^n$ of (66) are compatible, i.e., we here have two commuting generalized symmetries of order 2. If $N = 3$, then the subcases $a_n \equiv 1$, $a_n \equiv \beta_3^n$, and $a_n \equiv \beta_3^{2n}$ of (67) are compatible, i.e., we have three commuting generalized symmetries of order 3.

Equation (65) with $a_n \equiv n$ is a known master symmetry for differential–difference equation (65) with $a_n \equiv 1$ [18]. It is important for us that in all the three cases $N = 1, 2, 3$, the symmetry corresponding to $a_n \equiv n$ plays the role of the master symmetry for discrete equation (55) where β_N is a primitive root of unity. Compared with Sec. 2.1, these master symmetries are more convenient to use because they do not depend explicitly on the time of the master symmetry.

We let

$$\partial_{\hat{\theta}_N} u_{n,m} = \Xi_{n,m}^{(N)} \quad (69)$$

denote the generalized symmetry of discrete equation (55) with $N = 2$ or $N = 3$ corresponding to $a_n \equiv n$ in (66) or (67), which plays the role of the master symmetry. We show how to construct generalized symmetries of higher orders

$$\partial_{\hat{\theta}_{k,N}} u_{n,m} = \Upsilon_{n,m}^{(k,N)}, \quad k \in \mathbb{N}, \quad (70)$$

starting from symmetries (66) or (67) with a periodic coefficient $a_n \equiv b_n$, which correspond to (70) with $k = 1$. The order of such a symmetry is equal to kN . The right-hand sides of these symmetries are constructed using the recurrence relation

$$\begin{aligned} \Upsilon_{n,m}^{(k+1,N)} &= \text{ad}_{\Xi_{n,m}^{(N)}} \Upsilon_{n,m}^{(k,N)} = D_{\hat{\theta}_N} \Upsilon_{n,m}^{(k,N)} - D_{\hat{\theta}_{k,N}} \Xi_{n,m}^{(N)} = \\ &= \sum_{j=-kN}^{kN} \Xi_{n+j,m}^{(N)} \frac{\partial \Upsilon_{n,m}^{(k,N)}}{\partial u_{n+j,m}} - \sum_{j=-N}^N \Upsilon_{n+j,m}^{(k,N)} \frac{\partial \Xi_{n,m}^{(N)}}{\partial u_{n+j,m}}, \end{aligned} \quad (71)$$

where $D_{\hat{\theta}_N}$ and $D_{\hat{\theta}_{k,N}}$ are the total derivatives by virtue of (69) and (70).

3.3. Comparison of the case $N = 2$ with a known example: Relation to relativistic Toda-type equations. We consider discrete equation (58) in more detail. We know the only autonomous example analogous to (58) from the standpoint of a generalized symmetry structure. It was found in [19] and then studied in [20]. This example is

$$u_{n+1,m+1}(u_{n,m} - u_{n,m+1}) - u_{n+1,m}(u_{n,m} + u_{n,m+1}) + 2 = 0. \quad (72)$$

Its generalized symmetries of orders 1 and 2 in the m direction are

$$\partial_{t_1} u_{n,m} = (-1)^n \frac{u_{n,m+1}u_{n,m-1} + u_{n,m}^2}{u_{n,m+1} + u_{n,m-1}}, \quad (73)$$

$$\partial_{t_2} u_{n,m} = \frac{(u_{n,m+2} - u_{n,m-2})(u_{n,m+1}^2 - u_{n,m}^2)(u_{n,m}^2 - u_{n,m-1}^2)}{(u_{n,m} + u_{n,m-2})(u_{n,m+1} + u_{n,m-1})^2(u_{n,m+2} + u_{n,m})}. \quad (74)$$

The simplest symmetry in the n direction has the order 2:

$$\partial_{\hat{\theta}_2} u_{n,m} = (u_{n+1,m}u_{n,m} - 1)(u_{n,m}u_{n-1,m} - 1)(b_{n+1}u_{n+2,m} - b_nu_{n-2,m}), \quad (75)$$

where $b_{n+2} \equiv b_n$ is an arbitrary two-periodic function, i.e., $b_n = b^{(1)} + b^{(2)}(-1)^n$ with arbitrary constant coefficients $b^{(1)}$ and $b^{(2)}$. In the case of discrete equation (58), a symmetry analogous to (73) is

$$\partial_{t_1} u_{n,m} = (-1)^n (u_{n,m}^2 - 1)(u_{n,m+1} - u_{n,m-1}). \quad (76)$$

Generalized symmetries of Eq. (58) similar to (74) and (75) are (60) and (66) with $a_n \equiv b_n$.

Autonomous discrete equations (58) and (72) have hierarchies of autonomous generalized symmetries in both directions. The orders of those autonomous symmetries are even, and as can be seen from the examples above, the simplest autonomous generalized symmetries in both directions have order 2.

In [20], we showed that differential–difference equation (75) is equivalent to a Tsuchida system [21] (see details below). But in the class of five-point differential–difference equations, this in itself is an interesting

integrable example. Equation (66) with $a_n \equiv b_n$ seems a new integrable example of a five-point differential–difference equation. In [20], we briefly noted that Eq. (75) is similar to relativistic Toda-type equations (see Secs. 4.2 and 4.3 in [22] and Sec. 3.3 in [6]) according to its generalized symmetry properties. In [5], we demonstrated such a relation to relativistic Toda-type equations more explicitly for a nonautonomous equation. Here, following [5], we demonstrate such an explicit relation for Eqs. (75) and (66) with $a_n \equiv b_n$.

We first consider generalized symmetry (75). For any fixed m , we introduce $v_k = u_{2k,m}$, $w_k = u_{2k-1,m}$, $\varsigma = b_{2k}$, and $\eta = b_{2k-1}$ and rewrite (75) as a system:

$$\begin{aligned}\partial_{\tilde{\theta}_2} v_k &= (\eta v_{k+1} - \varsigma v_{k-1})(w_{k+1} v_k - 1)(v_k w_k - 1), \\ \partial_{\tilde{\theta}_2} w_k &= (\varsigma w_{k+1} - \eta w_{k-1})(v_k w_k - 1)(w_k v_{k-1} - 1).\end{aligned}\tag{77}$$

This is just Tsuchida system (3.13) in [21]. In either of the two cases $\varsigma = 1$, $\eta = 0$ or $\varsigma = 0$, $\eta = 1$, we introduce $U_k = \log v_k$ or $U_k = -\log w_k$ and in any of these four cases obtain the relativistic Toda-type equation

$$\ddot{U}_k = \dot{U}_k(\dot{U}_{k+1} - \dot{U}_{k-1} - e^{U_{k+1}-U_k} + e^{U_k-U_{k-1}}),\tag{78}$$

where we set $\dot{U}_k = \partial_{\tilde{\theta}_2} U_k$. This is the known Eq. (Ld3) in [6] with $\mu = 0$ and $\nu = 1$.

We now consider symmetry (66) with $a_n \equiv b_n$. For any fixed m , we introduce \tilde{u}_n :

$$u_{n,m} = \frac{\tilde{u}_n + \tilde{u}_{n+1}}{\tilde{u}_n - \tilde{u}_{n+1}}.$$

This transformation is not invertible but is linearizable, i.e., it is not of the Miura type in the terminology in [23]. As a result, we obtain the integrable modification of (66) with $a_n \equiv b_n$

$$\begin{aligned}\partial_{\tilde{\theta}_2} \tilde{u}_n &= \frac{(\tilde{u}_{n+2} - \tilde{u}_n)(\tilde{u}_{n+1} - \tilde{u}_n)(\tilde{u}_n - \tilde{u}_{n-1})}{2(\tilde{u}_{n+2}\tilde{u}_n + \tilde{u}_{n+1}\tilde{u}_{n-1}) - (\tilde{u}_{n+2} + \tilde{u}_n)(\tilde{u}_{n+1} + \tilde{u}_{n-1})} b_{n+1} + \\ &+ \frac{(\tilde{u}_{n+1} - \tilde{u}_n)(\tilde{u}_n - \tilde{u}_{n-1})(\tilde{u}_n - \tilde{u}_{n-2})}{2(\tilde{u}_{n+1}\tilde{u}_{n-1} + \tilde{u}_n\tilde{u}_{n-2}) - (\tilde{u}_{n+1} + \tilde{u}_{n-1})(\tilde{u}_n + \tilde{u}_{n-2})} b_n.\end{aligned}\tag{79}$$

We now pass to the notation $v_k = \tilde{u}_{2k}$, $w_k = \tilde{u}_{2k-1}$, $\varsigma = b_{2k}$, and $\eta = b_{2k-1}$ and obtain the system

$$\begin{aligned}\partial_{\tilde{\theta}_2} v_k &= \frac{(v_{k+1} - v_k)(w_{k+1} - v_k)(v_k - w_k)}{2(v_{k+1}v_k + w_{k+1}w_k) - (v_{k+1} + v_k)(w_{k+1} + w_k)} \eta + \\ &+ \frac{(w_{k+1} - v_k)(v_k - w_k)(v_k - v_{k-1})}{2(w_{k+1}w_k + v_kv_{k-1}) - (w_{k+1} + w_k)(v_k + v_{k-1})} \varsigma, \\ \partial_{\tilde{\theta}_2} w_k &= \frac{(w_{k+1} - w_k)(v_k - w_k)(w_k - v_{k-1})}{2(w_{k+1}w_k + v_kv_{k-1}) - (w_{k+1} + w_k)(v_k + v_{k-1})} \varsigma + \\ &+ \frac{(v_k - w_k)(w_k - v_{k-1})(w_k - w_{k-1})}{2(v_kv_{k-1} + w_kw_{k-1}) - (v_k + v_{k-1})(w_k + w_{k-1})} \eta.\end{aligned}\tag{80}$$

In either of the two subcases $\varsigma = 2$, $\eta = 0$ or $\varsigma = 0$, $\eta = 2$, we introduce $U_k = v_k$ or $U_k = w_k$ and in any of these four cases obtain the relativistic Toda-type equation

$$\ddot{U}_k = \dot{U}_k^2 \left(\frac{\dot{U}_{k-1}}{(U_k - U_{k-1})^2} - \frac{\dot{U}_{k+1}}{(U_k - U_{k+1})^2} + \frac{1}{U_k - U_{k-1}} + \frac{1}{U_k - U_{k+1}} \right),$$

where we set $\dot{U}_k = \partial_{\tilde{\theta}_2} U_k$. This is the known Eq. (L2) in [6] with $r(x, y) = (x - y)^2/2$.

New examples of five-point differential–difference equations similar to (66) with $a_n \equiv b_n$ and (75) recently appeared in [24].

3.4. Conservation laws in the n direction. Because Eq. (55) with $N = 1$ is well known, we here consider the cases $N = 2$ and $N = 3$ and conservation laws in the n direction, which are autonomous.

The relation

$$(T_n - 1)\check{p}_{n,m} = (T_m - 1)\check{q}_{n,m}, \quad (81)$$

where $\check{p}_{n,m}$ and $\check{q}_{n,m}$ depend on n, m , and $u_{n+i,m+j}$, is called the conservation law of discrete equation (55) with (56) if it is satisfied identically on solutions of this equation. Using (55), we can rewrite $\check{p}_{n,m}$ and $\check{q}_{n,m}$ in terms of only n, m , and the functions $u_{n+i,m}$ and $u_{n,m+j}$, and we write them in namely this representation. In n -direction case, $\check{q}_{n,m}$ becomes

$$\check{q}_{n,m} = \check{q}_{n,m}(u_{n+k_1,m}, u_{n+k_1-1,m}, \dots, u_{n+k_2,m}), \quad k_1 \geq k_2.$$

The function $\check{q}_{n,m}$ can be called the conserved density by analogy with the discrete-differential case.

For $k_1 > k_2$, we obtain the densities $\check{q}_{n,m}$, such that

$$\frac{\partial^2 \check{q}_{n,m}}{\partial u_{n+k_1,m} \partial u_{n+k_2,m}} \neq 0 \quad \text{for all } n, m.$$

We then call $k_1 - k_2$ the order of such a conservation law. If $k_1 = k_2$ and the function $\check{q}_{n,m}$ is not constant, then (81) is the nontrivial zeroth-order conservation law. Conservation laws of different orders are essentially different.

We use master symmetries (69) to construct conservation laws in the cases $N = 2, 3$. They are simpler than in Sec. 2.1 in the sense that they do not depend explicitly on the time $\hat{\theta}_N$ of the master symmetry. But constructing conservation laws in such a way is new even in the $\hat{\theta}_N$ -independent case.

We construct a hierarchy of n -independent conservation laws for Eq. (55) with (56) with $N = 2, 3$:

$$(T_n - 1)\check{p}_{n,m}^{(k)} = (T_m - 1)\check{q}_{n,m}^{(k)}, \quad k \geq 1. \quad (82)$$

Similarly to Sec. 2.2, we can construct the conservation laws by induction using the property that

$$(T_n - 1)D_{\hat{\theta}_N} \check{p}_{n,m}^{(k)} = (T_m - 1)D_{\hat{\theta}_N} \check{q}_{n,m}^{(k)} \quad (83)$$

is also a conservation law. Here, $D_{\hat{\theta}_N}$ is the total derivative by virtue of (69).

In both the $N=2$ and $N=3$ cases, the starting conservation law is constructed using (33) in [3]. We let

$$\partial_{\hat{\theta}_N} u_{n,m} = \Omega_{n,m}^{(N)} \quad (84)$$

denote autonomous symmetries (66) and (67) with $a_n \equiv 1$ and rewrite the corresponding discrete equation (55) with (56) in the form

$$u_{n+1,m+1} = \varphi^{(N)}(u_{n,m}, u_{n+1,m}, u_{n,m+1}).$$

Then the starting conservation laws are

$$(1 - T_n^N)T_n^{-1} \log \frac{\partial \varphi^{(N)}}{\partial u_{n+1,m}} = (T_m - 1) \log \frac{\partial \Omega_{n,m}^{(N)}}{\partial u_{n+N,m}}. \quad (85)$$

Such a conservation law is autonomous with the conserved density $\check{q}_{n,m}^{(1)} = \log(\partial \Omega_{n,m}^{(N)} / \partial u_{n+N,m})$, but conservation law (83) depends explicitly on the variable n . How to eliminate this dependence on n from

such a conservation law is explained in Sec. 2.2. We can do this because master symmetry (69) depends on n linearly, the function $\check{q}_{n,m}^{(1)}$ is also a conserved density of Eq. (84), and we can add a function of form (37) to both sides of conservation law (83).

If $N = 2$, then we can rewrite conservation law (85) in the form

$$(T_m - 1)\check{q}_{n,m}^{(k)} = (T_n^2 - 1)\check{p}_{n,m}^{(k)}, \quad (86)$$

where $k = 1$,

$$\check{q}_{n,m}^{(1)} = \log \frac{(u_{n+1,m}^2 - 1)(u_{n,m}^2 - 1)}{U_{n+1,m}^2}, \quad \check{p}_{n,m}^{(1)} = \log \frac{u_{n,m} + 1}{u_{n,m+1} - 1}, \quad (87)$$

and $U_{n,m}$ is given by (68). The form of this conservation law is specific, but it is a particular case of (82) with $\check{p}_{n,m}^{(k)} = (T_n + 1)\check{p}_{n,m}^{(k)}$. Using master symmetry (69), we obtain the next conservation law, which can be made autonomous and rewritten in the same specific form (86). It is given by

$$\check{q}_{n,m}^{(2)} = -\frac{(u_{n+4,m} + u_{n+3,m})(u_{n+2,m}^2 - 1)(u_{n+1,m} + u_{n,m})}{U_{n+3,m}U_{n+1,m}} + \frac{u_{n+1,m}^2 - 1}{U_{n+1,m}},$$

$$\check{p}_{n,m}^{(2)} = \frac{(u_{n+2,m} + u_{n+1,m})(u_{n,m} - 1)}{U_{n+1,m}}.$$

The orders of these conservation laws are 2 and 4. These conservation laws were constructed in [5] in a slightly different form using an L - A pair.

If $N = 3$, then we can rewrite conservation law (85) in the form

$$(T_m - 1)\check{q}_{n,m}^{(k)} = (T_n^3 - 1)\check{p}_{n,m}^{(k)}, \quad (88)$$

where $k = 1$, $\check{p}_{n,m}^{(1)}$ is given by (87) as before,

$$\check{q}_{n,m}^{(1)} = \log \frac{(u_{n+2,m}^2 - 1)(u_{n+1,m}^2 - 1)(u_{n,m}^2 - 1)}{U_{n+1,m}^2},$$

and $U_{n,m}$ is given by (68). Specific form (88) is also a particular case of (82). This conservation law is autonomous and has the order 3. Using master symmetry (69), we can obtain the next conservation law, which can be made autonomous and rewritten in the same specific form (88) with $k = 2$. It has the order 6. But it is too cumbersome to show here.

We note that using nonautonomous generalized symmetries (66) and (67) with $a_n \equiv b_n$ and the same formula (85) for starting conservation laws, we can try to obtain nonautonomous conservation laws, but nothing new arises because the operator $T_m - 1$ annihilates the explicit dependence on n .

3.5. Autonomous L - A pairs. Here, we construct autonomous L - A pairs for the discrete equations of series (55) with (56) using nonautonomous L - A pair (7), (8) for Eq. (6). In the $N=1$ case, we have $\beta_1 = 1$, and this L - A pair is obviously autonomous.

Applying the operator T_n^{N-1} to the first equation in (7), we obtain the result

$$\Psi_{n+N,m} = L_{n,m}^{(1,N)} \Psi_{n,m}, \quad \Psi_{n,m+1} = L_{n,m}^{(2)} \Psi_{n,m}, \quad (89)$$

where $N \geq 2$, $L_{n,m}^{(1,N)} = L_{n+N-1,m}^{(1)} L_{n+N-2,m}^{(1)} \cdots L_{n+1,m}^{(1)} L_{n,m}^{(1)}$, and β is replaced with β_N in the matrices $L_{n,m}^{(1)}$ and $L_{n,m}^{(2)}$. The compatibility condition for (89) is

$$L_{n,m+1}^{(1,N)} L_{n,m}^{(2)} = L_{n+N,m}^{(2)} L_{n,m}^{(1,N)}. \quad (90)$$

Because $\beta_N^N = 1$, we can see that the factor β_N^n is unchanged not only in the matrix $L_{n,m+1}^{(1,N)}$ but also in $L_{n+N,m}^{(2)}$. It therefore plays no role in relation (90), and we can replace β_N^n with a constant. It can be eliminated by scaling the spectral parameter λ , and we obtain the relation

$$\Lambda_{n,m+1}^{(1,N)} \Lambda_{n,m}^{(2)} = \Lambda_{n+N,m}^{(2)} \Lambda_{n,m}^{(1,N)}, \quad (91)$$

defined by the matrices

$$\Lambda_{n,m}^{(1,N)} = \Theta_{n,m}^{(N-1)} \Theta_{n,m}^{(N-2)} \dots \Theta_{n,m}^{(1)} \Theta_{n,m}^{(0)}. \quad (92)$$

Here,

$$\Theta_{n,m}^{(k)} = \begin{pmatrix} 1 & 2\lambda\beta_N^k(u_{n+k,m} + 1) \\ -\frac{1}{u_{n+k,m} - 1} & \frac{u_{n+k,m} + 1}{u_{n+k,m} - 1} \end{pmatrix},$$

$$\Lambda_{n,m}^{(2)} = \begin{pmatrix} 1 & -\lambda(u_{n,m} + 1)(u_{n,m+1} - 1) \\ 1 & 0 \end{pmatrix}.$$

For any $N \geq 2$, matrix relation (91) is a consequence of discrete equation (55) where (56) is satisfied. By direct calculation, we verified that for $N = 2, 3, 4$, relation (91) is equivalent to (55) with condition (56). It is quite likely that the same holds for any $N \geq 2$, and for Eq. (55) with (56), we obtain the autonomous L - A pair

$$\Psi_{n+N,m} = \Lambda_{n,m}^{(1,N)} \Psi_{n,m}, \quad \Psi_{n,m+1} = \Lambda_{n,m}^{(2)} \Psi_{n,m}. \quad (93)$$

4. Conclusions

We have constructed a series of autonomous integrable discrete equations (55) where β_N is a primitive root of unity. Equation (55) with $N = 1$ is well known. Equations (55) with (56) have hierarchies of autonomous generalized symmetries and conservation laws in both directions and also autonomous L - A pairs.

We constructed symmetries and conservation laws of (55) with (56) using the master symmetries. Those master symmetries arise as generalized symmetries of these discrete equations and depend linearly on one of two discrete variables (see Secs. 2.1 and 3.2). One of them also has an explicit dependence on its time. In the case of conservation laws, such a construction scheme seems new. Introducing the time of the master symmetry into the corresponding discrete equation also seems a new feature in the method.

It seems to us that the following conjecture on the generalized symmetry structure should hold.

Conjecture. *Each autonomous equation (55) with condition (56) has an infinite hierarchy of autonomous generalized symmetries in both directions of orders κN , $\kappa \geq 1$. The minimum possible order of an autonomous generalized symmetry in any direction is equal to N .*

We do not know any examples of this kind in case of hyperbolic partial differential equations analogous to discrete equations of the form (10). The results presented in Secs. 3.1 and 3.2 support this conjecture. Corollary 1 states that there exist autonomous generalized symmetries of orders κN in the m direction. In Theorem 3, in particular, we proved that Eqs. (55) with condition (56) for $N = 2, 3$ have autonomous generalized symmetries of the order N in the n direction and do not have autonomous symmetries of lower orders. We can prove a similar result for the m direction.

Theorem 4. *Equations (55) with condition (56) for $N = 2, 3$ have autonomous generalized symmetries of the order N in the m direction and do not have autonomous symmetries of lower orders.*

Proof. The proof is similar to the proof of Theorem 3 but is very cumbersome, and we do not give it here.

This conjecture is important from the standpoint of the generalized symmetry method for discrete equations when classifying discrete equations using the existence of generalized symmetries of a fixed order [14], [19]. Because the minimum order of an autonomous generalized symmetry can be arbitrarily high, we cannot classify all integrable discrete equations (10) in this way in the autonomous case.

As our results show, the hierarchies of autonomous conservation laws should have a similar structure. Each autonomous equation (55) with condition (56) should have an infinite hierarchy of autonomous conservation laws of the orders κN , $\kappa \geq 1$, in both directions. This is true in the m direction (see Corollary 2).

The case $N = 2$ is most interesting because discrete equation (58) has no complex coefficients. We considered it in more detail in Sec. 3.3. For this Eq. (58) and its known analogue (72), we showed that their second-order generalized symmetries in the n direction are closely related to integrable differential-difference equations of the relativistic Toda-type. We do not know any autonomous discrete example, except (58), (72), with generalized symmetries of this kind.

Finally, we note that although autonomous equations (55) with condition (56) are more or less obvious particular cases of nonautonomous equations (5), many results related to these autonomous equations are new.

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