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2017 J. Phys. A: Math. Theor. 50 125201

(http://iopscience.iop.org/1751-8121/50/12/125201)

IOP Publishing Journal of Physics A: Mathematical and Theoretical
J. Phys. A: Math. Theor. **50** (2017) 125201 (27pp) doi:10.1088/1751-8121/aa5cc3

Classification of five-point differential-difference equations

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Received 24 October 2016, revised 12 January 2017
Accepted for publication 30 January 2017
Published 20 February 2017

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Abstract
Using the generalized symmetry method, we carry out, up to autonomous point transformations, the classification of integrable equations of a subclass of the autonomous five-point differential-difference equations. This subclass includes such well-known examples as the Itoh–Narita–Bogoyavlensky and the discrete Sawada–Kotera equations. The resulting list contains 17 equations, some of which seem to be new. We have found non-point transformations relating most of the resulting equations among themselves and their generalized symmetries.

Keywords: integrability, generalized symmetry, classification, non-invertible transformation

1. Introduction

The generalized symmetry method uses the existence of generalized symmetries as an integrability criterion and allows one to classify integrable equations of a certain class. Using this method, the classification problem has been solved for some important classes of partial differential equations (PDEs) [29, 30], of differential-difference equations (DΔEs) [8, 40], and of partial difference equations (PΔEs) [14, 28].

This is not the only integrability criterion introduced to produce integrable PΔEs. Using the compatibility around the cube (CAC) technique introduced in [9, 33, 34], Adler, Bobenko and Suris (ABS) [5] obtained a class of integrable equations on a quad graph. More recent results on this line of research can be found in [6, 12, 17]. All equations obtained by ABS and their extensions have generalized symmetries which are integrable DΔEs, belonging to the classification presented in [25, 40] and given, in general, by DΔEs defined on three-point lattices [18, 19, 24].

Recently one can find many results in which P∆Es defined on the square but not determined in the ABS classification or in its extensions have generalized symmetries defined on more than three-point lattices [1, 14, 31, 35]. An extension of the classification of the integrable P∆Es defined on a square is very difficult to perform. An alternative that seems more easy to perform is to classify integrable five-point D∆Es

$$\dot{u}_n = \Psi(u_{n+2}, u_{n+1}, u_n, u_{n-1}, u_{n-2}). \tag{1.1}$$

Here $\dot{u}_n$ is derivative of u_n with respect to a continuous variable t. Few results in this line of research are already known, see e.g. [2–4, 16]. The integrable P∆Es are then obtained as Bäcklund transformations of these D∆Es [15, 22, 23, 26]. The best known integrable example in this class is the Ito–Narita–Bogoyavlensky (INB) equation [10, 21, 32]:

$$\dot{u}_n = u_n(u_{n+2} + u_{n+1} - u_{n-1} - u_{n-2}). \tag{1.2}$$

Volterra type equations

$$\dot{u}_n = \Phi(u_{n+1}, u_n, u_{n-1}) \tag{1.3}$$

have been completely classified [39], and the resulting list of equations is quite big, see the details in the review article [40]. The classification of five-point lattice equations of the form (1.1) will contain equations coming from the classification of Volterra equations (1.3). For example, they appear if we consider equations of the form

$$\dot{u}_n = \Phi(u_{n+2}, u_n, u_{n-2}). \tag{1.4}$$

It is clear that if u_n is a solution of (1.4), then the functions $\tilde{u}_k = u_{2k}$ and $\hat{u}_k = u_{2k+1}$ satisfy (1.3) with k instead of n. Equation (1.4) is just a three-point lattice equation equivalent to (1.3). A second case is when we consider generalized symmetries of (1.3). Any integrable Volterra type equation has a five-point symmetry of the form (1.1). See the explicit results for Volterra type equations presented for example in [20, 36, 37, 40].

To avoid those two cases, which are included in the classification of Volterra type equations and to simplify the problem, we consider here equations of the form

$$\begin{aligned}\dot{u}_n = {} & A(u_{n+1}, u_n, u_{n-1})u_{n+2} + B(u_{n+1}, u_n, u_{n-1})u_{n-2} \\ & + C(u_{n+1}, u_n, u_{n-1}),\end{aligned} \tag{1.5}$$

where the form of A and B will be defined later, see (1.7).

Few equations of the Volterra classification (1.3) are also included in the five-point classification (1.5). They are those equations which are linearly dependent on u_{n+1} and u_{n-1}. All of them are polynomial [40]. Such equations, rewritten in the form (1.4), belong to the class (1.5). Also their five-point symmetries are of the form (1.5). Moreover the majority of the examples of D∆Es of the form (1.1) known up to now belong to the class (1.5) [4, 7, 10, 11, 14, 21, 31, 32, 38]. So the class (1.5) is not void.

The theory of the generalized symmetry method is well-developed in case of Volterra type equations [40] and it has been modified for the case of equations depending on 5 and more lattice points in [2, 3]. The classification problem of the class (1.5) seems to be technically quite complicate. For this reason we use a simpler version of the method compared with the one presented in [2, 3].

For equations analogous to (1.2), which are the first members of their hierarchies, the simplest generalized symmetry has the form [1, 14, 31, 41]:

$$u_{n,\tau} = G(u_{n+4}, u_{n+3}, u_{n+2}, u_{n+1}, u_n, u_{n-1}, u_{n-2}, u_{n-3}, u_{n-4}), \tag{1.6}$$

where $u_{n,\tau}$ denotes τ-derivative of u_n. We will use the existence of such symmetry as an integrability criterion.

The problem naturally splits into cases depending on the form of the functions A and B. In this article we study the case when the autonomous DΔEs (1.5) is such that A and B satisfy to the following conditions:

$$A \neq \alpha(u_{n+1}, u_n)\alpha(u_n, u_{n-1}), \quad B \neq \beta(u_{n+1}, u_n)\beta(u_n, u_{n-1}) \tag{1.7}$$

for any functions α and β of their arguments. We call this class the *Class I*. Class I includes such well-known examples as the INB equation and the discrete Sawada–Kotera equation, see [38] and (E15) in section 3. The following simple criterion for checking conditions (1.7) takes place:

$$\frac{\partial}{\partial u_n}\frac{a_{n+1}a_{n-1}}{a_n} \neq 0, \qquad \frac{\partial}{\partial u_n}\frac{b_{n+1}b_{n-1}}{b_n} \neq 0, \tag{1.8}$$

where

$$a_n = A(u_{n+1}, u_n, u_{n-1}), \quad b_n = B(u_{n+1}, u_n, u_{n-1}),$$

as we will show in theorem 1 in section 2.3. In the proof of this criterion an essential role is played by the fact that (1.5) is autonomous, i.e. has no explicit dependence on n.

In this article we present a complete list of equations of the Class I possessing a generalized symmetry of the form (1.6). Among them there are a few apparently new integrable examples. Then we show the non-point transformations relating most of resulting equations among themselves.

In section 2 we discuss a theory of the generalized symmetry method suitable to solve our specific problem. In particular, in section 2.2 some integrability conditions are derived and criteria for checking those conditions are proved in section 2.3. In section 3 we present the obtained list of integrable equations and the relations between those equations expressed in the form of the non-point transformations which are presented in appendix A. In section 4 the generalized symmetries of the key equations of the resulting list are given. Section 5 is devoted to some concluding remarks. A scheme for costructing integrable partial difference equations, using the differential-difference equations obtained in this paper, is briefly discussed in appendix B.

2. Theory

To simplify the notation let us represent (1.5) as:

$$\dot{u}_n = a_n u_{n+2} + b_n u_{n-2} + c_n \equiv f_n, \tag{2.1}$$

where

$$\begin{gathered} a_n = A(u_{n+1}, u_n, u_{n-1}), \quad b_n = B(u_{n+1}, u_n, u_{n-1}), \\ c_n = C(u_{n+1}, u_n, u_{n-1}). \end{gathered} \tag{2.2}$$

In (2.1) we require

$$a_n \neq 0, \quad b_n \neq 0. \tag{2.3}$$

For convenience we represent the symmetry (1.6) as

$$u_{n,\tau} = g_n, \tag{2.4}$$

with the restriction:

$$\frac{\partial g_n}{\partial u_{n+4}} \neq 0, \qquad \frac{\partial g_n}{\partial u_{n-4}} \neq 0. \tag{2.5}$$

The compatibility condition for (2.1) and (2.4) is

$$u_{n,\tau,t} - u_{n,t,\tau} \equiv D_t g_n - D_\tau f_n = 0. \tag{2.6}$$

As (2.1) and (2.4) are autonomous, their form, as well as the form of the compatibility condition (2.6), do not depend on the point n. For this reason, we write them down at the point $n = 0$:

$$D_t g_0 = D_\tau f_0. \tag{2.7}$$

Here D_t and D_τ are the operators of total differentiation with respect to t and τ given respectively by:

$$D_t = \sum_{k\in\mathbb{Z}} f_k \frac{\partial}{\partial u_k}, \quad D_\tau = \sum_{k\in\mathbb{Z}} g_k \frac{\partial}{\partial u_k}. \tag{2.8}$$

We assume as independent variables the functions

$$u_0, u_1, u_{-1}, u_2, u_{-2}, u_3, u_{-3}. \dots \tag{2.9}$$

Thus (2.7) must be satisfied identically for all values of the independant variables (2.9). Equation (2.7) depends on variables $u_{-6}, u_{-5}, \dots, u_5, u_6$ and it is an overdetermined equation for the unknown function g_0, with given f_0. Using a standard technique of the generalized symmetry method [40], we can calculate g_0 step by step, obtaining conditions for the function f_0.

2.1. General case

The first steps for the calculation of g_0 can be carried out with no restriction on the form of the equation (2.1) given by the function f_0.

In fact, differentiating (2.7) with respect to u_6, we obtain:

$$a_4 \frac{\partial g_0}{\partial u_4} = a_0 \frac{\partial g_2}{\partial u_6}. \tag{2.10}$$

Introducing the shift operator T: $Th_n = h_{n+1}$, we can rewrite (2.10) as:

$$(T^2 - 1)\left(\frac{1}{a_0 a_2}\frac{\partial g_0}{\partial u_4}\right) = 0.$$

The kernel of the operator $T^2 - 1$ in the autonomous case consists just of constants [40]. Then, up to a τ-scaling in (2.4), we can write

$$\frac{\partial g_0}{\partial u_4} = a_0 a_2. \tag{2.11}$$

By differentiating (2.7) with respect to u_5 and taking into account (2.11), we obtain:

$$a_3 \frac{\partial g_0}{\partial u_3} - a_0 \frac{\partial g_2}{\partial u_5} - a_1 a_3 \frac{\partial f_0}{\partial u_1} + a_0 a_2 \frac{\partial f_4}{\partial u_5} = 0. \tag{2.12}$$

If we define

$$h_0^+ = \frac{\partial g_0}{\partial u_3} - a_1 \frac{\partial f_0}{\partial u_1} - a_0 \frac{\partial f_2}{\partial u_3}, \tag{2.13}$$

then (2.12) is equivalent to

$$a_3 h_0^+ = a_0 h_2^+, \tag{2.14}$$

where $h_2^+ = T^2 h_0^+$.

Then we can state the following lemma:

Lemma 1. *If $h_0^+ \neq 0$, then there exists $\hat{\alpha}_n = \alpha(u_n, u_{n-1})$, such that $a_0 = \hat{\alpha}_1 \hat{\alpha}_0$, i.e. the equation is not of Class I.*

Proof. When we multiply (2.14) by $\frac{h_1^+}{a_0 a_1 a_2 a_3}$ we obtain

$$(T-1)\frac{h_0^+ h_1^+}{a_0 a_1 a_2} = 0.$$

As the kernel of the operator $T-1$ consists of constants, we have

$$\frac{h_0^+ h_1^+}{a_0 a_1 a_2} = \eta^2 \neq 0. \tag{2.15}$$

Equation (2.15) is equivalent to $a_0 = \hat{\alpha}_1 \hat{\alpha}_0$, where $\hat{\alpha}_0 = \frac{h_0^+}{\eta a_1}$. ■

There are two possibilities:

- Case 1. $a_0 \neq \hat{\alpha}_1 \hat{\alpha}_0$ for any $\hat{\alpha}_n = \alpha(u_n, u_{n-1})$, see (1.8). Then $h_0^+ = 0$ due to lemma 1.
- Case 2. $a_0 = \hat{\alpha}_0 \hat{\alpha}_1$ for some $\hat{\alpha}_n = \alpha(u_n, u_{n-1})$. Then we can find from (2.14) that $h_0^+ = \mu^+ \hat{\alpha}_0 \hat{\alpha}_1 \hat{\alpha}_2$ with a constant μ^+.

In both cases (2.13) gives us $\frac{\partial g_0}{\partial u_3}$.

In quite similar way, differentiating (2.7) with respect to u_{-6} and u_{-5}, we get a set of relations analogous to (2.11) and (2.14). Namely,

$$\frac{\partial g_0}{\partial u_{-4}} = \nu b_0 b_{-2}, \tag{2.16}$$

$$b_{-3} h_0^- = b_0 h_{-2}^-, \tag{2.17}$$

where $\nu \neq 0$ is a constant, and

$$h_0^- = \frac{\partial g_0}{\partial u_{-3}} - \nu b_{-1} \frac{\partial f_0}{\partial u_{-1}} - \nu b_0 \frac{\partial f_{-2}}{\partial u_{-3}}. \tag{2.18}$$

As a consequence of a lemma similar to lemma 1 we get two cases:

1. $b_0 \neq \hat{\beta}_0 \hat{\beta}_{-1}$ for any $\hat{\beta}_n = \beta(u_{n+1}, u_n)$, then $h_0^- = 0$.
2. $b_0 = \hat{\beta}_0 \hat{\beta}_{-1}$, then we can find from (2.17) that $h_0^- = \mu^- \hat{\beta}_0 \hat{\beta}_{-1} \hat{\beta}_{-2}$ with a constant μ^-.

In both cases (2.18) provides us $\frac{\partial g_0}{\partial u_{-3}}$.

So the results presented in this section provide a natural frame for splitting further calculation of g_0 into several different cases. In the following in this paper we consider Class I in which condition (1.7) is satisfied and therefore $h_0^{\pm} = 0$.

2.2. Class I

When condition (1.7) is satisfied and therefore $h_0^+ = h_0^- = 0$, then due to (2.13) and (2.18)

$$\frac{\partial g_0}{\partial u_3} = a_1 \frac{\partial f_0}{\partial u_1} + a_0 \frac{\partial f_2}{\partial u_3}, \tag{2.19}$$

$$\frac{\partial g_0}{\partial u_{-3}} = \nu b_{-1} \frac{\partial f_0}{\partial u_{-1}} + \nu b_0 \frac{\partial f_{-2}}{\partial u_{-3}}. \tag{2.20}$$

Partial derivatives $\frac{\partial g_0}{\partial u_4}, \frac{\partial g_0}{\partial u_{-4}}$ are always given by (2.11), (2.16).

Differentiating (2.7) with respect to u_4 and u_{-4} and introducing the functions:

$$q_0^+ = \frac{1}{a_0}\frac{\partial g_0}{\partial u_2} - D_t \log a_0 - \frac{\partial f_0}{\partial u_0} - \frac{\partial f_2}{\partial u_2} - \frac{1}{a_0}\frac{\partial f_0}{\partial u_1}\frac{\partial f_1}{\partial u_2}, \tag{2.21}$$

$$q_0^- = \frac{1}{\nu b_0}\frac{\partial g_0}{\partial u_{-2}} - D_t \log b_0 - \frac{\partial f_0}{\partial u_0} - \frac{\partial f_{-2}}{\partial u_{-2}} - \frac{1}{b_0}\frac{\partial f_0}{\partial u_{-1}}\frac{\partial f_{-1}}{\partial u_{-2}}, \tag{2.22}$$

we obtain, up to a common factor $a_0\, a_2$, the relation:

$$2D_t \log a_0 = q_2^+ - q_0^+, \tag{2.23}$$

and up to a common factor $\nu b_0\, b_{-2}$, the relation:

$$2D_t \log b_0 = q_{-2}^- - q_0^-. \tag{2.24}$$

It is evident that

$$q_0^+ = q_0^+(u_1, u_0, u_{-1}, u_{-2}, u_{-3}), \qquad q_0^- = q_0^-(u_3, u_2, u_1, u_0, u_{-1}). \tag{2.25}$$

The relations (2.23) and (2.24) have the form of conservation laws and are necessary conditions for the integrability. If a symmetry (2.4) exists for (2.1), then there must exist some functions q_n^+, q_n^- of the form (2.25) such that relations (2.23) and (2.24) are satisfied.

The integrability conditions are formulated in terms of equation (2.1) only, i.e. for an integrable equation (2.1), there must exist functions $q_n^{\pm}$ of the form (2.25) satisfying the relations (2.23) and (2.24).

If, for a given equation (2.1), conditions (2.23) and (2.24) are satisfied and the functions $q_n^{\pm}$ are known, then partial derivatives $\frac{\partial g_0}{\partial u_2}, \frac{\partial g_0}{\partial u_{-2}}$ can be found from (2.21), (2.22). In this case the right hand side of symmetry (2.4) is defined up to one unknown function of 3 variables:

$$\psi(u_{n+1}, u_n, u_{n-1}). \tag{2.26}$$

This function can be found directly from the compatibility condition (2.7).

In this way we can carry out the classification of the equations of Class I. On the first stage we use the integrability conditions (2.23), (2.24). Then we define the symmetry up to function (2.26) and try to find it from the compatibility condition.

2.3. Criteria for checking the integrability conditions (2.23), (2.24)

Let us now explain how to use the integrability conditions (2.23), (2.24). More precisely, we present some criteria for checking those conditions. We will also prove the criterion (1.8).

Let us introduce for any function

$$\varphi = \varphi(u_{m_1}, u_{m_1-1}, \ldots, u_{m_2}), \qquad m_1 \geqslant m_2, \tag{2.27}$$

the formal variational derivative:

$$\frac{\delta\varphi}{\delta u_0} = \sum_{k=m_2}^{m_1} T^{-k}\frac{\partial\varphi}{\partial u_k} = \frac{\partial}{\partial u_0}\sum_{k=m_2}^{m_1} T^{-k}\varphi, \tag{2.28}$$

see e.g. [40], as well as its adjoint version:

$$\frac{\bar\delta\varphi}{\bar\delta u_0} = \sum_{k=m_2}^{m_1}(-1)^k T^{-k}\frac{\partial\varphi}{\partial u_k} = \frac{\partial}{\partial u_0}\sum_{k=m_2}^{m_1}(-1)^k T^{-k}\varphi, \tag{2.29}$$

Then we can state the following lemma:

Lemma 2. *The following statements are true:*

$$\frac{\delta\varphi}{\delta u_0} = 0 \quad \text{iff} \quad \varphi = \kappa + (T-1)\omega, \tag{2.30}$$

$$\frac{\bar\delta\varphi}{\bar\delta u_0} = 0 \quad \text{iff} \quad \varphi = (T+1)\omega, \tag{2.31}$$

where κ is a constant, while ω is a function of a finite number of independent variables (2.9).

This lemma implies that if the function φ is such that its variational derivative with respect to u_0 is zero, then it can be represented in terms of some constant κ and a function ω depending on a finite number of independent variables.

Proof. The proof of (2.30) and (2.31) are similar. The proof of (2.30) is given in [40]. So we present here only the proof of (2.31).

If the function φ can be expressed as $(T+1)\omega$ and $m_1 > m_2$, then $\omega = \omega(u_{m_1-1}, \ldots, u_{m_2})$. For such functions φ we have:

$$\begin{aligned}\frac{\bar\delta\varphi}{\bar\delta u_0} &= \frac{\partial}{\partial u_0}\sum_{k=m_2}^{m_1}(-1)^k T^{-k}(T+1)\omega \\ &= \frac{\partial}{\partial u_0}((-1)^{m_2}T^{-m_2+1}\omega + (-1)^{m_1}T^{-m_1}\omega) = 0,\end{aligned}$$

where the last equality is identically satisfied, as both terms in the last sum do not depend on u_0.

In the case when $m_1 = m_2$ the assumption $\varphi = (T+1)\omega$ implies that the function ω and therefore φ are constant functions, and hence $\frac{\bar\delta\varphi}{\bar\delta u_0} = 0$.

Let us now assume that φ satisfies the equation $\frac{\bar\delta\varphi}{\bar\delta u_0} = 0$ with $m_1 > m_2$. Then it follows that the following equation is also satisfied

$$\frac{\partial}{\partial u_{m_1}}(-1)^{m_2}T^{m_2}\frac{\bar{\delta}\varphi}{\bar{\delta}u_0}=0.$$

Explicitating the variational derivative we have

$$\frac{\partial^2\varphi}{\partial u_{m_1}\partial u_{m_2}}=0.$$

So we can represent φ as

$$\begin{aligned}\varphi&=\theta(u_{m_1},u_{m_1-1},\ldots,u_{m_2+1})+\rho(u_{m_1-1},u_{m_1-2},\ldots,u_{m_2})\\&=(T+1)\rho+\hat{\varphi}(u_{m_1},u_{m_1-1},\ldots,u_{m_2+1}),\end{aligned}$$

with $\hat{\varphi}=\theta-T\rho$.

As $\frac{\bar{\delta}\varphi}{\bar{\delta}u_0}=\frac{\bar{\delta}\hat{\varphi}}{\bar{\delta}u_0}=0$, we have reduced the problem to a lower number of variables. Repeating this procedure, we finally obtain

$$\varphi=\tilde{\varphi}(u_{m_1})+(T+1)\tilde{\omega}.$$

Then it is easy to see that

$$\frac{\partial\tilde{\varphi}}{\partial u_{m_1}}=(-1)^{m_1}T^{m_1}\frac{\bar{\delta}\tilde{\varphi}}{\bar{\delta}u_0}=0,$$

i.e. $\tilde{\varphi}$ is a constant, and $\varphi=(T+1)(\tilde{\omega}+\tilde{\varphi}/2)$. ■

This lemma allows us to formulate and prove the following theorems:

Theorem 1. *The request that conditions (1.7) are satisfied is equivalent to (1.8).*

Proof. The condition $A\neq\alpha_n\alpha_{n+1}$ can be rewritten in terms of the function $\log a_n=\log A$ as

$$\log A\neq(T+1)\log\alpha(u_n,u_{n-1})$$

or, in equivalent form,

$$\frac{\bar{\delta}}{\bar{\delta}u_0}\log a_n=-\frac{\partial}{\partial u_n}\log\frac{a_{n+1}a_{n-1}}{a_n}\neq 0.$$

Their equivalence follows from lemma 2. In an analogous way we get the result for b_n. ■

Theorem 2. *The conditions*

$$\frac{\delta}{\delta u_0}D_t\log a_0=0,\quad \frac{\bar{\delta}}{\bar{\delta}u_0}D_t\log a_0=0 \tag{2.32}$$

imply

$$2D_t\log a_0=\kappa^++(T^2-1)q_0^+,\quad \kappa^+\in\mathbb{C}, \tag{2.33}$$

and vice versa.

Proof. As (2.33) can be rewritten in the form

$$2D_t \log a_0 = \kappa^+ + (T-1)(T+1)q_0^+ = (T+1)(\kappa^+/2 + (T-1)q_0^+),$$

we see that (2.33) implies (2.32) due to lemma 2.

Let us start from (2.32). From the first relation of (2.32) and statement (2.30), we obtain

$$2D_t \log a_0 = \kappa^+ + (T-1)Q_0^+ = \kappa^+ + (T+1)Q_0^+ - 2Q_0^+,$$

therefore $\frac{\bar\delta}{\bar\delta u_0} Q_0^+ = 0$ due to the second part of (2.32). Writing $Q_0^+ = (T+1)q_0^+$, we get (2.33). ∎

Theorem 3. *The conditions*

$$\frac{\delta}{\delta u_0} D_t \log b_0 = 0, \quad \frac{\bar\delta}{\bar\delta u_0} D_t \log b_0 = 0 \tag{2.34}$$

imply

$$2D_t \log b_0 = \kappa^- + (T^{-2}-1)q_0^-, \quad \kappa^- \in \mathbb{C}, \tag{2.35}$$

and vice versa.

As $(T^{-2}-1)q_0^- = (T^2-1)(-q_{-2}^-)$, we see that conditions (2.33) and (2.35) are of the same type and so the proof of theorem 3 repeats that of theorem A2.

Let us simplify the integrability conditions (2.32) and (2.34). To do so we use a new representation for the variational derivatives (2.28) and (2.29):

$$\frac{\delta \varphi}{\delta u_0} = \sum_{j=-m_1}^{-m_2} \frac{\partial}{\partial u_0} T^j \varphi, \quad \frac{\bar\delta \varphi}{\bar\delta u_0} = \sum_{j=-m_1}^{-m_2} (-1)^j \frac{\partial}{\partial u_0} T^j \varphi. \tag{2.36}$$

Moreover in place of conditions (2.32) and (2.34) we use their linear combinations:

$$\frac{1}{2}\left(\frac{\delta}{\delta u_0} + \frac{\bar\delta}{\bar\delta u_0}\right) D_t \log a_0 = 0, \quad \frac{1}{2}\left(\frac{\delta}{\delta u_0} - \frac{\bar\delta}{\bar\delta u_0}\right) D_t \log a_0 = 0,$$

$$\frac{1}{2}\left(\frac{\delta}{\delta u_0} + \frac{\bar\delta}{\bar\delta u_0}\right) D_t \log b_0 = 0, \quad \frac{1}{2}\left(\frac{\delta}{\delta u_0} - \frac{\bar\delta}{\bar\delta u_0}\right) D_t \log b_0 = 0.$$

As

$$D_t \log a_0 = \frac{f_1}{a_0}\frac{\partial a_0}{\partial u_1} + \frac{f_0}{a_0}\frac{\partial a_0}{\partial u_0} + \frac{f_{-1}}{a_0}\frac{\partial a_0}{\partial u_{-1}},$$

then we use (2.36) with $m_1 = -m_2 = 3$ and we see that the conditions (2.32) are equivalent to

$$\frac{\partial}{\partial u_0} D_t \log(a_{-2}a_0a_2) = 0, \tag{2.37a}$$

$$\frac{\partial}{\partial u_0} D_t \log(a_{-3}a_{-1}a_1a_3) = 0. \tag{2.37b}$$

Similarly, the conditions (2.34) are equivalent to

$$\frac{\partial}{\partial u_0} D_t \log(b_{-2}b_0b_2) = 0, \tag{2.38a}$$

$$\frac{\partial}{\partial u_0} D_t \log(b_{-3}b_{-1}b_1b_3) = 0. \tag{2.38b}$$

These are explicit and simple conditions for checking the integrability conditions (2.23) and (2.24). When these conditions are satisfied for a given equation (2.1), one has to write down the representations (2.33) and (2.35) and check that they are satisfied, i.e. $\kappa^{\pm} = 0$.

2.4. The beginning of classification

We have explicit formulae (2.11), (2.16), (2.19), (2.20) for the partial derivatives

$$\frac{\partial g_0}{\partial u_4}, \quad \frac{\partial g_0}{\partial u_{-4}}, \quad \frac{\partial g_0}{\partial u_3}, \quad \frac{\partial g_0}{\partial u_{-3}} \tag{2.39}$$

and implicit definitions for

$$\frac{\partial g_0}{\partial u_2}, \ \frac{\partial g_0}{\partial u_{-2}} \tag{2.40}$$

given by the relations (2.21)–(2.24). The compatibility of the partial derivatives (2.39) gives no conditions. The compatibility of (2.40) with each other and with (2.39) provides three restrictions. In fact, differentiating, for example, (2.23) with respect to u_{-3} and using (2.21), we get $\frac{\partial^2 g_0}{\partial u_{-3}\partial u_2}$ in explicit form. This has to be equal to the differentiation with respect to u_2 of $\frac{\partial g_0}{\partial u_{-3}}$ given by (2.20). In this way, using the compatibility of $\frac{\partial g_0}{\partial u_{-3}}$ and $\frac{\partial g_0}{\partial u_2}$, $\frac{\partial g_0}{\partial u_3}$ and $\frac{\partial g_0}{\partial u_{-2}}$, $\frac{\partial g_0}{\partial u_{-2}}$ and $\frac{\partial g_0}{\partial u_2}$, we are led to

$$(\nu+1)\frac{\partial b_0}{\partial u_1} = 0, \quad (\nu+1)\frac{\partial a_0}{\partial u_{-1}} = 0, \quad (\nu+1)\frac{\partial}{\partial u_0}\frac{a_0}{b_0} = 0, \tag{2.41}$$

where a_0, b_0 and ν are defined in (2.1) and by (2.16).

Conditions (2.41) allow one to split the classification problem into two cases:

1. The case $\nu \neq -1$ which is simpler and leads to (E8).
2. The case when $\nu = -1$.

Using the explicit integrability conditions (2.37) and (2.38) for $\nu = -1$, we can get two results given in the following theorems 4 and 5 which are formulated in terms of equation (2.1) only.

Theorem 4. *If an equation* (2.1) *satisfies conditions* (2.37) *and* (2.38)*, then its coefficients must have the form:*

$$a_0 = a^{(1)}(u_1, u_0)a^{(2)}(u_0, u_{-1}), \qquad b_0 = b^{(1)}(u_1, u_0)b^{(2)}(u_0, u_{-1}). \tag{2.42}$$

Proof. This theorem is divided in two parts. One related to a_0 and the second one to b_0. We will prove in the following the first part only as the second can be derived in the same way.

The first relation of (2.42) is equivalent to require that

$$a^{(3)} = \frac{\partial^2}{\partial u_{-1}\partial u_1} \log a_0 \equiv 0. \tag{2.43}$$

Let us define $a^{(4)}$ and $a^{(5)}$ to be the functions on the left hand side of (2.37*a*) and (2.37*b*). At first we see, that

$$\frac{\partial a^{(3)}}{\partial u_1} = \frac{1}{b_{-1}} T^{-3} \frac{\partial^2}{\partial u_4^2} a^{(5)} = 0, \qquad \frac{\partial a^{(3)}}{\partial u_{-1}} = \frac{1}{a_1} T^{-1} \frac{\partial^2}{\partial u_4 \partial u_0} a^{(5)} = 0,$$

i.e. $a^{(3)} = a^{(3)}(u_0)$.

Let us suppose, reductio ad absurdum, that $a^{(3)} \neq 0$. As a consequence we have

$$\frac{\partial^2 a_0}{\partial u_{-1}^2} = \frac{1}{2Ta^{(3)}} T^{-1} \frac{\partial^3 a^{(4)}}{\partial u_3^2 \partial u_0} = 0,$$

and this implies the identity

$$\frac{\partial}{\partial u_{-1}} (a_0^2 a^{(3)}) = 0.$$

As $a^{(3)} = a^{(3)}(u_0) \neq 0$ by assumption, we find that $\frac{\partial a_0}{\partial u_{-1}} = 0$. Now, due to the definition of $a^{(3)}$ given in (2.43), we get $a^{(3)} = 0$. This gives a contradiction which proves the theorem. ∎

Taking into account the results just obtained in theorem 4, we can prove the following theorem:

Theorem 5. *If an equation* (2.1) *belongs to the Class I and satisfies conditions* (2.37), (2.38) *and* (2.42)*, then for the functions $a^{(2)}$ and $b^{(1)}$ one has:*

$$\frac{\partial^2 a^{(2)}}{\partial u_{-1}^2} = 0, \qquad \frac{\partial^2 b^{(1)}}{\partial u_1^2} = 0. \tag{2.44}$$

Proof. We prove just the first equality in (2.44), as the second one can be proved in exactly the same way.

By direct calculation we derive from (2.37*b*) and (2.42) that

$$\frac{\partial^2 a^{(2)}}{\partial u_{-1}^2} \cdot \frac{\partial^2 \log a^{(1)}}{\partial u_0 u_1} = \frac{\partial}{\partial u_1} \left(\frac{1}{a^{(1)}} T^{-1} \frac{\partial^2 a^{(5)}}{\partial u_0 \partial u_3} \right) = 0.$$

In this proof $a^{(5)}$ and $a^{(4)}$ are the same as in previous one. So, either $\frac{\partial^2 a^{(2)}}{\partial u_{-1}^2} = 0$ or $\frac{\partial^2 \log a^{(1)}}{\partial u_0 u_1} = 0$. In the first instance the theorem has been proved.

Let us then assume that $\frac{\partial^2 a^{(2)}}{\partial u_{-1}^2} \neq 0$. Changing notations for the functions $a^{(1)}$ and $a^{(2)}$, we get $a^{(1)} = a^{(1)}(u_1)$. By a straightforward calculation we can show that

$$\frac{\partial (a^{(2)} T^{-2} a^{(1)})}{\partial u_{-1}} = \frac{a^{(2)} T^{-2} a^{(1)}}{T^{-1} a^{(1)}} \frac{\partial^2 T^{-2} a^{(4)}}{\partial u_{-2} \partial u_1} / T^{-1} \frac{\partial^2 a^{(2)}}{\partial u_{-1}^2} = 0,$$

i.e. $a^{(2)} = \frac{\theta(u_0)}{a^{(1)}(u_{-1})}$. The function θ is constant, as

$$\frac{d\theta}{du_0} = \frac{\theta}{a^{(1)}} \frac{\partial^2 T^{-1} a^{(5)}}{\partial u_2 \partial u_{-1}} / \frac{\partial^2 a^{(2)}}{\partial u_{-1}^2} = 0.$$

The condition (1.8) is not satisfied for $a_n = \theta a^{(1)}(u_{n+1})/a^{(1)}(u_{n-1})$, and the resulting equation would not be of Class I. This is in contradiction with one of the hypothesis of this theorem. So this instance is not possible. □

3. Complete list of integrable equations

In this section we present the complete list of integrable equations of Class I together with the non-point relations between them. These equations are referred by special numbers (E1)–(E17). Some of the obtained equations seem to be new.

The classification is usually carried out in two steps: at first one finds all integrable equations of a certain class up to point transformations, then one searches for non-point transformations which link the different resulting equations. In this paper we use autonomous point transformations which, because of the specific form (1.5) of the equations, are linear transformations with constant coefficients:

$$\hat{u}_0 = c_1 u_0 + c_2, \quad \hat{t} = c_3 t, \qquad c_1 c_3 \neq 0. \tag{3.1}$$

Non-point transformations we use here are transformations of the form

$$\hat{u}_0 = \varphi(u_k, u_{k-1}, \ldots, u_m), \quad k > m, \tag{3.2}$$

and their compositions.

The equation (1.1) is transformed into

$$\hat{u}_{0,t} = \hat{\Psi}(\hat{u}_2, \hat{u}_1, \hat{u}_0, \hat{u}_{-1}, \hat{u}_{-2}) \tag{3.3}$$

by (3.2), if for any solution u_n of (1.1), formula (3.2) provides a solution $\hat{u}_n$ of (3.3). To check this, we substitute (3.2) into (3.3), differentiate with respect to t in virtue of equation (1.1) and check that the resulting relation is satisfied identically for all values of independent variables (2.9).

We see that transformation (3.2) is explicit in one direction. If an equation A is transformed into B by a transformation (3.2), then this transformation has the *direction* from A to B and we will write in diagrams below $A \to B$, so indicating the direction where it is explicit.

The complete list has been obtained up to autonomous linear point transformations (3.1). The precise result is formulated as follows:

Theorem 6. *If a nonlinear equation of the form* (2.1)–(2.3) *belongs to Class I, given by restrictions* (1.8)*, and has a generalized symmetry* (1.6), (2.4), (2.5)*, then up to point transformation* (3.1) *it is equivalent to one of the equations (E1)–(E17). Any of the equations (E1)–(E17) has a generalized symmetry of the form* (1.6), (2.4), (2.5).

For easier understanding of the results, we split the complete list into smaller Lists 1-5. In each List the equations are related to each other by non-point transformations which are discussed in appendix A. For each of these lists we show relations between equations in diagrams, where the transformations (3.2) are shown by arrows. Transformations used in those diagrams have special numbers (T1), (T2), ... and are listed in appendix A.4. Generalized symmetries for key equations of the complete list are presented in the following section.

List 1. Equations related to the Volterra equation

$$\dot{u}_0 = u_0(u_2 - u_{-2}) \tag{E1}$$

$$\dot{u}_0 = u_0^2(u_2 - u_{-2}) \tag{E2}$$

$$\dot{u}_0 = (u_0^2 + u_0)(u_2 - u_{-2}) \tag{E3}$$

$$\dot{u}_0 = (u_2 + u_1)(u_0 + u_{-1}) - (u_1 + u_0)(u_{-1} + u_{-2}) \tag{E4}$$

$$\dot{u}_0 = (u_2 - u_1 + a)(u_0 - u_{-1} + a) + (u_1 - u_0 + a)(u_{-1} - u_{-2} + a) + b \tag{E5}$$

$$\dot{u}_0 = u_2u_1u_0(u_0u_{-1} + 1) - (u_1u_0 + 1)u_0u_{-1}u_{-2} + u_0^2(u_{-1} - u_1) \tag{E6}$$

All equations of List 1 are transformed into (E1) as shown in diagram (3.4).

$$\begin{tikzcd}
(E4) \arrow[dr, "(T1)"'] & (E5) \arrow[d, "(T2)"'] & (E6) \arrow[d, "(T3)"'] \\
(E2) \arrow[r, "(T4)"'] & (E1) & (E3) \arrow[l, "(T5)"]
\end{tikzcd} \tag{3.4}$$

All transformations here are linearizable except for (T5) which is of Miura type. The notions of linearizable and of Miura type transformations are discussed in appendix A.1. Transformations $\tilde{u}_k = u_{2k}$ or $\tilde{u}_k = u_{2k+1}$ turn equations (E1)–(E3) into the well-known Volterra equation and its modifications in their standard form. Transformations (T4) and (T5) also turn into the standard ones, see e.g. [40].

List 2. Linearizable equations

$$\dot{u}_0 = (T - a)\left(\frac{(u_1 + au_0 + b)(u_{-1} + au_{-2} + b)}{u_0 + au_{-1} + b} + u_0 + au_{-1} + b\right) + cu_0 + d \tag{E7}$$

$$\dot{u}_0 = \frac{u_2u_0}{u_1} + u_1 - a^2\left(u_{-1} + \frac{u_0u_{-2}}{u_{-1}}\right) + cu_0 \tag{E8}$$

In both equations $a \neq 0,\ (a+1)d = bc$.

Both equations of List 2 are related to the linear one:

$$\dot{u}_0 = u_2 - a^2u_{-2} + cu_0/2 \tag{3.5}$$

as it is shown in diagram (3.6).

$$\begin{tikzcd}
(3.5) \arrow[r, "(T3)"] & (E8) & (E7) \arrow[l, "(T6)"']
\end{tikzcd} \tag{3.6}$$

However (E7) is related to the linear equation (3.5) by a linearizable transformation which is implicit in both directions.

List 3. Equations of the relativistic Toda type

$$\dot{u}_0 = (u_0 - 1)\left(\frac{u_2(u_1 - 1)u_0}{u_1} - \frac{u_0(u_{-1} - 1)u_{-2}}{u_{-1}} - u_1 + u_{-1}\right) \tag{E9}$$

$$\dot{u}_0 = \frac{u_2 u_1^2 u_0^2(u_0 u_{-1} + 1)}{u_1 u_0 + 1} - \frac{(u_1 u_0 + 1)u_0^2 u_{-1}^2 u_{-2}}{u_0 u_{-1} + 1} - \frac{(u_1 - u_{-1})(2u_1 u_0 u_{-1} + u_1 + u_{-1})u_0^3}{(u_1 u_0 + 1)(u_0 u_{-1} + 1)} \tag{E10}$$

The equations of List 3 are related to the following one:

$$\dot{u}_0 = (u_1 u_0 - 1)(u_0 u_{-1} - 1)(u_2 - u_{-2}) \tag{3.7}$$

as it is shown in diagram (3.8).

$$(3.7) \xrightarrow{(T3)} (E9) \xleftarrow{(T7)} (E10) \tag{3.8}$$

Equation (3.7), which is out of Class I, is known and well-studied [13, 14]. It is shown in [13] that it is of the relativistic Toda type. It is natural that equations (E9) and (E10), of Class I, being related to (3.7) are of the same type. Equation (E9) obtained by transformation (T3) from (3.7) is also known, see [4]. The relation between (E10) and (3.7) is linearizable, but completely implicit.

List 4. Equations related to the INB

$$\dot{u}_0 = u_0(u_2 + u_1 - u_{-1} - u_{-2}) \tag{E11}$$

$$\begin{aligned}\dot{u}_0 = {} & (u_2 - u_1 + a)(u_0 - u_{-1} + a) + (u_1 - u_0 + a)(u_{-1} - u_{-2} + a) \\ & + (u_1 - u_0 + a)(u_0 - u_{-1} + a) + b\end{aligned} \tag{E12}$$

$$\dot{u}_0 = (u_0^2 + a u_0)(u_2 u_1 - u_{-1} u_{-2}) \tag{E13}$$

$$\dot{u}_0 = (u_1 - u_0)(u_0 - u_{-1})\left(\frac{u_2}{u_1} - \frac{u_{-2}}{u_{-1}}\right) \tag{E14}$$

All equations of List 4 are transformed into the INB equation (E11) as it is shown in diagram (3.9).

$$\begin{tikzcd} & (E13) \arrow[d, "(T8)"] & \\ (E12) \arrow[r, "(T2)"] & (E11) & (E14) \arrow[l, "(T9)"'] \end{tikzcd} \tag{3.9}$$

Equation (E12) with $a = 0$ and (E13) with $a = 0$ are simple modifications of the INB. The corresponding transformations are linearizable, and these equations are presented in [31] and [11], respectively. Equation (E13) with $a = 1$ has been found in [7, 31] together with corresponding transformations (T8). As it is shown in [7, 16], these transformations are of Miura type.

Equation (E14) is discussed in [16] as a preliminary result of the present classification. It is shown there that the transformation (T9) is linearizable, as it can be decomposed as a superposition of simple linearizable transformations, see diagram (3.10).

$$\begin{tikzcd}
(3.12) \arrow[d, "\eta_1:(T3)"'] & (3.11) \arrow[l, "\eta_2:(T10)"'] \arrow[d, "\eta_1:(T3)"'] \\
(E11) & (E14) \arrow[l, "\eta:(T9)"']
\end{tikzcd} \tag{3.10}$$

Equations (3.11) and (3.12) used in this diagram have the form:

$$\dot{u}_0 = (u_2 - u_0)(u_1 - u_{-1})(u_0 - u_{-2}), \tag{3.11}$$

$$\dot{u}_0 = u_0(u_1u_2 - u_{-1}u_{-2}). \tag{3.12}$$

Equation (3.12) is well-known [11]. It is interesting that this superposition consists of linearizable transformations in different directions, but it can be rewritten in the explicit form (T9):

$$\eta = \eta_1 \circ \eta_2 \circ \eta_1^{-1}.$$

List 5. Other equations

$$\dot{u}_0 = u_0^2(u_2u_1 - u_{-1}u_{-2}) - u_0(u_1 - u_{-1}) \tag{E15}$$

$$\dot{u}_0 = (u_0 + 1)\left(\frac{u_2u_0(u_1 + 1)^2}{u_1} - \frac{u_{-2}u_0(u_{-1} + 1)^2}{u_{-1}} + (1 + 2u_0)(u_1 - u_{-1})\right) \tag{E16}$$

$$\dot{u}_0 = (u_0^2 + 1)\left(u_2\sqrt{u_1^2 + 1} - u_{-2}\sqrt{u_{-1}^2 + 1}\right) \tag{E17}$$

Equation (E15) has been found in [38] and can be called the discrete Sawada–Kotera equation [1, 38]. Equation (E17) has been found as a result of the present classification and seems to be new. We know no relations between it and the other known equations. Equation (E16) has been found in [4] and is related to (E15) as it is shown in the following diagram [4]:

$$(E15) \xleftarrow{(T11)} (3.14) \xrightarrow{(T12)} (E16) \tag{3.13}$$

Here

$$\dot{u}_0 = \frac{(u_1 - u_0)(u_0 - u_{-1})(u_2 - u_{-2})}{(u_2 - u_{-1})(u_1 - u_{-2})}. \tag{3.14}$$

Introducing

$$v_0 = u_1 - u_0, \qquad w_0 = v_0/v_{-1}, \tag{3.15}$$

we can rewrite transformations (T11) and (T12) in form of the Riccati equations for unknown function w_n:

$$w_1 w_0 + \frac{1+\hat{u}_0}{\hat{u}_0} w_0 + 1 = 0, \quad w_1 w_0 + \frac{1}{1+\tilde{u}_1} w_1 + w_0 + 1 = 0. \tag{3.16}$$

This means, as shown in [16], that, up to linearizable transformations, both transformations (T11) and (T12) are of Miura type. It is very difficult to use these relations (3.16) for the construction of solutions, as, if we know one of the solutions $\hat{u}_n$ or $\tilde{u}_n$ and need to find the other one, we have to solve a Riccati equation.

Let us explain now why (E15)–E17) cannot be transformed into the INB by an explicit non-point transformation. Transformation $\tilde{u}_0 = au_0$, $\tilde{t} = a^{-3}t$ introduces into equations (E15)–(E17) the parameter a, e.g. in case of (E15) it gives:

$$\dot{u}_0 = u_0^2(u_2 u_1 - u_{-1}u_{-2}) - a^2 u_0(u_1 - u_{-1}).$$

All these three equations with the parameter a are analogues of (E13) as they generalize the same equation. More precisely, if we substitute $a = 0$, then in all cases we obtain (E13) with $a = 0$. This is a well-known modification of the INB equation, see List 4. Equation (E13) is transformed into the INB by (T8), and it is natural to search analogoues transformations for (E15)–(E17).

Let us consider a more general equation:

$$\dot{u}_0 = a(u_0)b(u_1)u_2 - a(u_0)b(u_{-1})u_{-2} + c(u_1, u_0, u_{-1}), \tag{3.17}$$

where the function $a(x)/b(x)$ is not constant. This equation generalizes (E13), (E15)–(E17). Let us look for a transformation of the form:

$$\hat{u}_0 = \phi(u_k, u_{k-1}, \dots, u_0), \quad k > 0, \quad \frac{\partial\phi}{\partial u_k} \neq 0, \quad \frac{\partial\phi}{\partial u_0} \neq 0, \tag{3.18}$$

which can be obtained with no loss of generality from (3.2) by a shift. If we consider a transformation (3.18) which transforms (3.17) into the INB (E11), then the function ϕ must satisfy the differential-functional equation

$$\dot{\phi} \equiv \sum_{j=0}^{k} \frac{\partial\phi}{\partial u_j}\dot{u}_j = \phi(T^2 + T - T^{-1} - T^{-2})\phi, \tag{3.19}$$

where $\dot{u}_j$ are defined by shifting j times (3.17).

For all $k \neq 2$ it is sufficient to use two consequences of (3.19), which are obtained by differentiation with respect to u_{k+2} and u_{-2}:

$$\frac{\partial\phi}{\partial u_k} a(u_k)b(u_{k+1}) = \phi T^2 \frac{\partial\phi}{\partial u_k},$$

$$\frac{\partial\phi}{\partial u_0} a(u_0)b(u_{-1}) = \phi T^{-2} \frac{\partial\phi}{\partial u_0}.$$

By a short calculation it is easy to show that the transformation (3.18) does not exist for all $k \neq 2$. The case $k = 2$ requires a more detailed investigation, as a solution of the problem exists in case of (E13). We are led to the following result:

Theorem 7. *Equations (E15)–(E17) cannot be transformed into the equation INB (E11) by a transformation of the form* (3.18).

In conclusion, let us briefly discuss the existence of a possible link between the various Lists 1-5. In List 5 we exclude from the consideration (E17), as at the moment we have no information about it.

The $L - A$ pair for the Volterra equation (E1), presented in List 1, is given by 2×2 matrices as well as for (3.7), related to List 3, see [13]. The $L - A$ pair of the INB equation (E11) and of the discrete Sawada–Kotera equation (E15), contained in List 4 and 5, is given by 3×3 matrices, see [10] and [1] respectively. List 2 consists of linearizable equations. For this reason, three groups of equations, namely, List 2, Lists 1,3 and Lists 4,5 should not be related by transformations (3.2) and their compositions.

Volterra type equations of List 1 essentially differ from relativistic Toda type equations of List 3 in their algebraic properties [8, 40]. Then such equations cannot be related by the transformations mentioned above. As for equations of Lists 4 and 5, except for (E17), at the moment we see no essential difference between them. We have a negative result for the explicit transformations (3.2) formulated in theorem 7. However, there might be some compositions of transformations (3.2), analogues to the one shown in diagram (3.13), relating these Lists.

4. Generalized symmetries of key equations

In this section we present the generalized symmetries for the key equations of the Lists 1–5. The symmetries for other equations can be easily obtained by using the simple transformations shown in diagrams (3.4), (3.6), (3.8), (3.9) and contained in appendix A.4. Those transformations either are contained in table A2 or are point equivalent to transformations of table A2. The way how to construct the symmetries, using such transformations, is explained in section A.3. There is one exception which will be commented separately.

List 1. Generalized symmetries for the Volterra equation and its modifications are well-known, see e.g. [40], and we just replace u_{n+j} by u_{n+2j}.

The simplest generalized symmetry for (E1) reads:

$$u_{0,\tau} = u_0(u_2(u_4 + u_2 + u_0) - u_{-2}(u_0 + u_{-2} + u_{-4})).$$

The symmetry

$$u_{0,\tau} = (u_0^2 + cu_0)((u_2^2 + cu_2)(u_4 + u_0 + c) - (u_{-2}^2 + cu_{-2})(u_0 + u_{-4} + c))$$

corresponds to (E2), if $c = 0$, and to (E3), if $c = 1$. Generalized symmetries of (E4)–(E6) are constructed by using the transformations (T1)–(T3).

List 2. A generalized symmetry for (E8) could be obtained from the linear equation (3.5) by applying the transformation (T3). However, we write it down here in explicit form:

$$u_{0,\tau} = (T^4 - a^4)\left(\frac{u_{-1}u_{-3}}{u_{-2}} + \frac{u_0u_{-2}u_{-4}}{u_{-1}u_{-3}}\right). \tag{4.1}$$

A symmetry for (E7) can be constructed by the transformation (T6). Equation (4.1) can be represented as

$$u_{0,\tau} = (T + a)h, \tag{4.2}$$

and the symmetry for (E7) is obtained from (4.2) as follows:

$$u_{0,\tau} = h|_{u_k \to u_{k+1}+au_k+b}.$$

List 3. One could get a generalized symmetry of (E9), using the known symmetry of (3.7) [13, 14] and the transformation (T3). However, we write it down explicitly:

$$u_{0,\tau} = (u_0-1)(T-T^{-1})((T+T^{-1})s_1-(1+T^{-1})s_2+s_3),$$

$$s_1 = \frac{u_2u_0u_{-2}(u_1-1)(u_0-1)(u_{-1}-1)}{u_1u_{-1}}, \qquad s_2 = \frac{u_1u_{-1}(u_1-1)(u_0-1)}{u_0},$$

$$s_3 = \frac{u_1(u_0-1)(u_1u_0-u_1-u_0)u_{-1}^2}{u_0^2} - \frac{(u_2u_1-u_2-u_1)u_0(u_0-1)}{u_1}.$$

A symmetry for (E10) can be obtained from this one by the transformation (T7).

List 4. Generalized symmetries for almost all equations of this List are known; for the INB equation (E11) and for (E13) with $a=0$ see [41], and for (E13) with $a=1$ and for (E12) with $a=0$ see [31]. Nevertheless, for completeness, we present here the generalized symmetries for the most interesting equations of this List. For (E11) the symmetry has the form

$$u_{0,\tau} = u_0(T^2+T-T^{-1}-T^{-2})((T+T^{-1})u_1u_{-1}+(T+1)u_0u_{-1}+u_0^2),$$

for (E13) the form

$$u_{0,\tau} = (u_0^2+au_0)(T^2-T^{-1})(u_0u_{-1}(1+T^{-1})((u_2u_1+u_{-1}u_{-2})(u_0+a))+au_0^2u_{-1}^2),$$

and for (E14) the form

$$u_{0,\tau} = \frac{(u_1-u_0)(u_0-u_{-1})}{u_0}\Bigg((T^2-T^{-2})\frac{u_2(u_1-u_0)(u_0-u_{-1})u_{-2}}{u_1u_{-1}} \\ +(T-T^{-1})\frac{u_1(u_1-u_0)(u_0-u_{-1})u_{-1}}{u_0^2} - p_1+p_2\Bigg),$$

$$p_1 = \frac{u_3(u_2-u_1)u_0u_{-2}}{u_2u_{-1}} + \frac{u_2u_0(u_{-1}-u_{-2})u_{-3}}{u_1u_{-2}},$$

$$p_2 = \frac{(u_2u_{-1}-u_1u_{-2})(u_1u_0+u_0u_{-1}-u_1u_{-1})}{u_1u_{-1}}.$$

The symmetry for (E12) is constructed by using the transformation (T2).

List 5. The generalized symmetry for the discrete Sawada–Kotera equation (E15) has been found in [1] and it has the form:

$$u_{0,\tau} = u_0(w_1(w_3+w_2+w_1+w_0)-w_{-1}(w_0+w_{-1}+w_{-2}+w_{-3}) \\ -u_1(w_3+w_{-1})+u_{-1}(w_1+w_{-3})), \qquad w_0 = u_0(1+u_1u_{-1}).$$

For (E16) one has:

$$
\begin{aligned}
u_{0,\tau} &= (u_0+1)((T^2-T^{-2})A+(T-T^{-2})B+(T-T^{-1})C+(1-T^{-1})D),\\
A &= U_1U_{-1}u_2u_{-2}u_0(1+u_0)+U_0u_1u_{-1},\\
B &= u_2u_{-1}((u_0U_1+1)(u_1U_0+1)+u_0^2U_1+u_1^2U_0-1)\\
&\quad +\frac{u_2u_0^2}{u_1}+\frac{u_1^2u_{-1}}{u_0}+u_1u_0,\\
C &= (U_0u_1u_{-1}+1)^2+U_0u_0(1+u_1u_{-1}(2+3u_1+3u_{-1}))\\
&\quad +u_1u_0u_{-1}(2(u_0+1)-3(u_1+u_{-1})),\\
D &= U_1U_0u_2u_1u_0u_{-1}+U_2u_3u_1u_0(u_1+1)+U_{-1}u_1u_0u_{-2}(u_0+1)+U_1u_2u_0^2\\
&\quad +U_0u_1^2u_{-1}+u_1^2(u_3u_2+2u_3+u_2+2u_{-1})+u_0^2(u_{-1}u_{-2}+u_{-1}+2u_{-2}+2u_2)\\
&\quad +u_1^2u_0(4u_{-1}-2u_2^2+3)+u_1u_0^2(4u_2-2u_{-1}^2+3)+2u_1^2u_0^2+5u_1u_0,\\
U_0 &= \frac{(u_0+1)^2}{u_0}.
\end{aligned}
$$

For (E17) the symmetry reads:

$$
\begin{aligned}
u_{0,\tau} &= U_0\Big(u_4\sqrt{U_3}\,U_2\sqrt{U_1}-\sqrt{U_{-1}}\,U_{-2}\sqrt{U_{-3}}\,u_{n-4}\\
&\quad +u_2u_1\sqrt{U_1}\left(u_3\sqrt{U_2}+\sqrt{U_0}\,u_{-1}\right)-\sqrt{U_{-1}}\,u_{-1}u_{-2}\left(u_1\sqrt{U_0}+\sqrt{U_{-2}}\,u_{-3}\right)\\
&\quad +u_0(u_2^2U_1-U_{-1}u_{-2}^2)\Big),\qquad U_0=u_0^2+1.
\end{aligned}
$$

5. Conclusion

In this article we have done the classification of the differential-difference equations depending on five lattice points and belonging to Class I. This Class is a natural subclass of the differential-difference equations (1.5) from the point of view of the integrability conditions. In this Class we have found 17 equations, some of which seem new. We have found the non-point transformations which relate them to fewer key equations and presented for them the generalized symmetries.

This work is the starting point of a research which we plan to continue. From one side, using the results presented here, we can extend the classification outside Class I by considering the case when (1.7) is not satisfied. We can call this class of equations as a Class II. We already have an example belonging to this class, namely (3.7). From the other side, we can construct the Bäcklund transformations for differential-difference equations obtained in this paper. They possibly will provide autonomous integrable partial difference equations, defined on a square lattice, different from the known examples, see e.g. [1, 5, 14, 28, 31]. The construction of integrable partial difference equations is briefly discussed in appendix B.

Acknowledgments

The authors RNG and RIY gratefully acknowledge financial support from a Russian Science Foundation grant (project 15-11-20007). DL has been partly supported by the Italian Ministry of Education and Research, 2010 PRIN *Continuous and discrete nonlinear integrable evolutions: from water waves to symplectic maps* and by INFN IS-CSN4 *Mathematical Methods of Nonlinear Physics*.

Appendix A. Some remarks on non-point transformations of differential-difference equations

In this appendix we follow the paper [16] and briefly formulate some results of the transformation theory necessary for this paper. In particular, we introduce the notions of linearizable and Miura type transformations used here. We also explain how to find transformations relating equations obtained. Schemes of the construction of modified equations by using those transformations are presented in appendices A.2 and A.3 in a new way compared to [16]. Then we present the list of transformations necessary to reduce the list of equations to their key representatives.

A.1. Miura type and linearizable transformations

There are two essentially different classes of transformations (3.2). The first one consists of *Miura type transformations*. The inversion of such transformations is equivalent to solving the discrete Riccati equations and their generalizations [16]. For example the well-known discrete Miura transformation

$$\hat{u}_0 = (1 + u_0)(1 - u_1) \tag{A.1}$$

relates the Volterra equation to its modification. Equation (A.1) may be considered as a discrete Riccati equation for the unknown function u_n with $\hat{u}_n$ a given function.

The second class consists of *linearizable transformations* which are not of Miura type. We will use bellow *linear transformations* with *constant coefficients*:

$$\hat{u}_0 = \nu_k u_k + \nu_{k-1} u_{k-1} + \ldots + \nu_m u_m + \nu, \quad k > m. \tag{A.2}$$

We can introduce the following definition:

Definition A.1. A transformation of the form (3.2) is called *linearizable* if it can be represented as a superposition of linear transformations (A.2) and point transformations $\hat{u}_0 = \psi(u_0)$. A superposition of linearizable transformations in different directions will be also called a linearizable transformation.

For example, the transformations

$$\hat{u}_0^+ = u_1 u_0, \qquad \hat{u}_0^- = u_1/u_0, \qquad \tilde{u}_0 = u_2 u_1 u_0, \tag{A.3}$$

which are of the form (3.2), are linearizable since they can be expressed as

$$\hat{u}_0^\pm = (\exp \circ (T \pm 1) \circ \log)\, u_0, \qquad \tilde{u}_0 = (\exp \circ (T^2 + T + 1) \circ \log)\, u_0.$$

Transformations can be in different directions. As an example we have the composition

$$B \leftarrow D \leftarrow C \rightarrow A,$$

consisting of linearizable transformations, which can be rewritten as a transformation $A \rightarrow B$ of the form (3.2), see diagram (3.10). In diagrams (3.6) and (3.8) we have superpositions of the form $A \rightarrow B \leftarrow C$. In these cases, linearizable transformations relating A and C are implicit in both directions.

The inversion of any linearizable transformation is reduced to solving a number of linear equations with constant coefficients. For this reason the linearizable transformation is not of Miura type and the construction of solutions by such transformations is more easy.

Any linearizable transformation of the form (3.2) can be expressed as a composition of the transformation

Table A1. Examples of point symmetries (A.6) and corresponding autonomous linearizable transformations (A.5).

$\sigma_n(u_n)$	1	$(-1)^n$	u_n	$(-1)^n u_n$
$\phi(u_1, u_0)$	$u_1 - u_0$	$u_1 + u_0$	u_1/u_0	$u_1 u_0$

Table A2. Examples of conserved densities of (A.13) and corresponding autonomous linearizable transformations (A.11).

$\rho_n(u_n)$	u_n	$(-1)^n u_n$	$\log u_n$	$(-1)^n \log u_n$
$\psi(\tilde u_1, \tilde u_0)$	$\tilde u_1 - \tilde u_0$	$\tilde u_1 + \tilde u_0$	$\tilde u_1/\tilde u_0$	$\tilde u_1 \tilde u_0$

$$\hat u_0 = (T-1)u_0 \tag{A.4}$$

and non-autonomous point transformations $\hat u_n = \psi_n(u_n)$ [16]. So, in order to invert a linearizable transformation, we have to solve a few times the simplest discrete linear equation (A.4) for the unknown function u_n. The transformation (A.4) is solved by the discrete analogue of the integration.

Next, we will explain how to construct simple autonomous linearizable transformations, according to definition A.1,

$$\hat u_0 = \phi(u_1, u_0) \tag{A.5}$$

by using *point symmetries* and *conservation laws*. These techniques will be hereinafter used to find links between equations of the resulting list presented in section 3.

A.2. Point symmetries

Here we construct the transformations (A.5), starting from point symmetries.

For an equation of the form (1.1) we can describe all non-autonomous point symmetries of the form

$$\partial_\tau u_n = \sigma_n(u_n), \quad \sigma_n(u_n) \neq 0, \ \forall\, n, \tag{A.6}$$

by solving the determining equation:

$$\sigma_n'(u_n)\Psi = \sum_{j=-2}^{2} \frac{\partial \Psi}{\partial u_{n+j}} \sigma_{n+j}(u_{n+j}), \tag{A.7}$$

where by a σ_n' we mean the derivative of the function with respect to its argument.

Theorem A.1. *If (1.1) has a point symmetry (A.6), then it admits the following non-autonomous linearizable transformation:*

$$\hat u_n = (T-1)\eta_n(u_n), \quad \eta_n'(u_n) = \frac{1}{\sigma_n(u_n)}, \tag{A.8}$$

which allows us to construct a modification of (1.1).

We are interested here in autonomous linearizable transformations of the form (A.5), and therefore primarily in autonomous point symmetries (A.6). However, sometimes a non-autonomous point symmetry of the form (A.6) may also lead to the autonomous result. This is the case when there exists a non-autonomous point transformation

$$U_n = \psi_n(\hat{u}_n) \tag{A.9}$$

which turns the transformation (A.8) into an autonomous one. We guarantee that, in this case, the resulting equation for U_n will be also autonomous [16].

For any equation (1.1) and any transformation (A.5), we can get an equation of the form:

$$\hat{u}_{0,t} = \hat{\Psi}(\hat{u}_2, \hat{u}_1, \hat{u}_0, \hat{u}_{-1}, \hat{u}_{-2}, u_0). \tag{A.10}$$

To do so we differentiate (A.5) w.r.t. to t in virtue of (1.1) and express all variables u_k, $k \neq 0$, in terms of $\hat{u}_j, u_0$ by using (A.5). In the case of the autonomous transformations obtained with a help of theorem A.1 and the remark just after it, the dependence on u_0 in (A.10) disappears, and one gets in this way an autonomous modified equation (3.3).

Let us write down in table A1 the four most typical examples of point symmetries, two of which not autonomous, together with the corresponding autonomous transformations (A.5). Some of them have been simplified by applying an autonomous point transformation of $\hat{u}_0$.

A.3. Conservation laws

We can construct simple autonomous linearizable transformations

$$u_0 = \psi(\tilde{u}_1, \tilde{u}_0), \tag{A.11}$$

starting from conservation laws. Transformation (A.11) relates equation (1.1) and an equation of the form

$$\tilde{u}_{0,t} = \tilde{\Psi}(\tilde{u}_2, \tilde{u}_1, \tilde{u}_0, \tilde{u}_{-1}, \tilde{u}_{-2}). \tag{A.12}$$

For any equation (1.1) we can find all conservation laws of the form

$$\partial_t \rho_n(u_n) = (T-1)h_n, \quad \rho_n'(u_n) \neq 0, \ \forall\, n, \tag{A.13}$$

where $h_n = h_n(u_{n+1}, \ldots, u_{n-2})$. The conserved density ρ_n is found by using a criterion introduced in [25]. A function $\rho_n(u_n)$ is a conserved density of (1.1) iff

$$\frac{\delta(\partial_t \rho_n(u_n))}{\delta u_n} \equiv \sum_{j=-2}^{2} T^{-j} \frac{\partial(\rho_n'(u_n)\Psi)}{\partial u_{n+j}} = 0. \tag{A.14}$$

If ρ_n is known, then the function h_n can be easily constructed [25].

Theorem A.2. *If* (1.1) *has a conservation law (A.13), then it admits the following non-autonomous linearizable transformation:*

$$u_n = \rho_n^{-1}(\tilde{u}_{n+1} - \tilde{u}_n), \tag{A.15}$$

which allows us to construct a modification of (1.1)*:*

$$\partial_t \tilde{u}_n = h_n(\rho_{n+1}^{-1}(\tilde{u}_{n+2} - \tilde{u}_{n+1}), \ldots, \rho_{n-2}^{-1}(\tilde{u}_{n-1} - \tilde{u}_{n-2})). \tag{A.16}$$

In the case of autonomous conservation law (A.13), we get an autonomous modified equation (A.12). When a non-autonomous point transformation

$$U_n = \mu_n(\tilde{u}_n) \tag{A.17}$$

makes the transformation (A.15) autonomous, we often can get an autonomous modification too, see [16]. In the particular case when

$$\rho_n(u_n) = (-1)^n p(u_n), \tag{A.18}$$

the transformation (A.17) has the form $U_n = (-1)^{n+1}\tilde{u}_n$, hence

$$u_n = p^{-1}(U_{n+1} + U_n),$$

and we are guaranteed that there exists an autonomous modification [16].

Let us write down in table A2 the four most typical examples of conserved densities, two of which not autonomous and of the form (A.18), together with corresponding autonomous transformations (A.11). Some of them have been simplified by autonomous point transformations of $\tilde{u}_0$.

Let us explain how to find the modified equation for these examples. For the first and second ones we have to obtain the representations:

$$\dot{u}_0 = (T \mp 1)h,$$

and then we get the modification $\tilde{u}_{0,t} = h$, where the variables u_k in h are replaced by $\tilde{u}_{k+1} \mp \tilde{u}_k$. For the third and fourth examples we have to write down the representations

$$\partial_t \log u_0 = (T \mp 1)h.$$

A modification has the form $\tilde{u}_{0,t} = \tilde{u}_0 h$, where the variables u_k in h are replaced by $\tilde{u}_{k+1}\tilde{u}_k^{\mp 1}$.

In conclusion, we note that if for a given conserved density $\rho_n(u_n)$ we get the representation (A.13) with a different linear difference operator with constant coefficients, then we can get a different linearizable transformation and corresponding modified equation in a similar way. For instance, for the conserved density $\log u_n$ of the INB equation (1.2) we have:

$$\begin{aligned}\partial_t \log u_0 &= (T^4 + T^3 - T - 1)u_{-2}\\ &= (T-1)(T+1)(T-c_1)(T-c_2)u_{-2},\end{aligned}$$

where $c_{1,2} = -\frac{1}{2} \pm \frac{\sqrt{3}i}{2}$, and this provides a lot of possibilities to construct modified equations, see details and other examples in [16].

A.4. List of non-point transformations

Here we list all non-point transformations used to link the equations presented in section 3:

$$\hat{u}_0 = u_1 + u_0, \tag{T1}$$

$$\hat{u}_0 = u_1 - u_0 + a, \tag{T2}$$

$$\hat{u}_0 = u_1 u_0, \tag{T3}$$

$$\hat{u}_0 = u_2 u_0, \tag{T4}$$

$$\hat{u}_0 = u_2(u_0 + 1) \quad \text{or} \quad \hat{u}_0 = (u_2 + 1)u_0, \tag{T5}$$

$$\hat{u}_0 = u_1 + a u_0 + b, \tag{T6}$$

$$\hat{u}_0 = u_1 u_0 + 1, \tag{T7}$$

$$\hat{u}_0 = u_2 u_1 (u_0 + a) \quad \text{or} \quad \hat{u}_0 = (u_2 + a) u_1 u_0, \tag{T8}$$

$$\hat{u}_0 = \frac{(u_1 - u_0)(u_0 - u_{-1})}{u_0}, \tag{T9}$$

$$\hat{u}_0 = u_1 - u_{-1}, \tag{T10}$$

$$\hat{u}_0 = \frac{u_0 - u_1}{u_2 - u_{-1}}, \tag{T11}$$

$$\tilde{u}_0 = -\frac{(u_1 - u_{-1})(u_0 - u_{-2})}{(u_1 - u_{-2})(u_0 - u_{-1})}. \tag{T12}$$

Some of these transformations were known before, as for example, the Miura type transformations (T5), (T8), (T11) and (T12). The other transformations, which are linearizable, were constructed in this work by using the transformation theory presented above in appendix A. A more detailed discussion of this theory can be found in the recent article [16].

Appendix B. Construction of integrable partial difference equations

Here we briefly discuss the construction of integrable completely discrete equations of the form

$$F(u_{n+1,m+1}, u_{n+1,m}, u_{n,m+1}, u_{n,m}) = 0 \tag{B.1}$$

by using the differential-difference equations. We do it by example of the well-known modified Volterra equation:

$$\dot{u}_n = (u_n^2 - 1)(u_{n+1} - u_{n-1}). \tag{B.2}$$

A construction scheme and the same example (B.2) have been first considered in [27]. A different scheme by example of the INB equation (1.2) has been presented in [26, appendix], but with a negative result.

We follow here the article [15], where the three-point equations (1.3) are considered in the more general non-autonomous case. However, the method of [15] can also be applied to five-point equations (1.1). That method is suitable for equations with polynomial and rational right hand side. In the case of more complex equations like (E17), it is better to follow the example of [26, appendix].

We will look for equations (B.1) linear in all their arguments, i.e. such that $\frac{\partial^2 F}{\partial u_{n+i,m+j}^2} = 0$ for all $i, j \in \{0, 1\}$. In this case the function F is a fourth degree polynomial with 16 constants coefficients c_k, $1 \leqslant k \leqslant 16$. Besides, we look for nondegenerate equations. If we rewrite equation (B.1) in the form

$$u_{n+1,m+1} = f(u_{n+1,m}, u_{n,m+1}, u_{n,m}), \tag{B.3}$$

then the nondegeneracy conditions read:

$$\frac{\partial F}{\partial u_{n+1,m+1}} \cdot \frac{\partial f}{\partial u_{n+1,m}} \cdot \frac{\partial f}{\partial u_{n,m+1}} \cdot \frac{\partial f}{\partial u_{n,m}} \neq 0. \tag{B.4}$$

These conditions allow us to avoid degenerate examples like $(u_{n+1,m+1} + u_{n+1,m}) u_{n,m+1} u_{n,m} = 0$.

We rewrite (B.2) in the form

$$\dot{u}_{n,m} = (u_{n,m}^2 - 1)(u_{n+1,m} - u_{n-1,m}) \tag{B.5}$$

and require that the equations (B.1) and (B.5) would be compatible. The discrete equation (B.1) defines the Bäcklund transformation for (B.2), while (B.5) is the generalized symmetry of this dicrete equation.

To check the compatibility, we differentiate (B.1) with respect to the time in virtue of (B.5) and get the relation

$$\sum_{i,j\in\{0,1\}} \frac{\partial F}{\partial u_{n+i,m+j}} \dot{u}_{n+i,m+j} = 0, \tag{B.6}$$

which must be satisfied on all solutions of (B.1). Then we express in (B.6) the functions $u_{n+2,m+1}, u_{n+1,m+1}, u_{n-1,m+1}$ in terms of

$$u_{n+2,m}, u_{n+1,m}, u_{n,m}, u_{n-1,m}, u_{n,m+1}, \tag{B.7}$$

which can be considered in this problem as independent variables. Passing to the numerator of (B.6), we get a polynomial of the variables (B.7). By equating to zero the coefficients of this polynomial, we are led to a system of algebraic equations for the coefficients c_k of F.

Using computer algebra systems like Maple, Mathematica and Reduce, we can find all solutions of this system of algebraic equations and can choose among them the nondegenerate ones. Up to the transformation $u_{n,m} \to -u_{n,m}$, leaving (B.5) unchanged, we get two discrete equations of the form:

$$(u_{n+1,m+1} - 1)(u_{n,m+1} + 1) = (u_{n+1,m} + c)(u_{n,m} - c), \tag{B.8}$$

where $c = 1$ or $c = -1$, compatible with (B.5). Both of them are known integrable equations. The first one is integrable by the inverse scatterring method, while the second one is Darboux integrable, see details and references in [15].

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