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Integrable discrete autonomous quad-equations admitting, as generalized symmetries, known five-point differential-difference equations

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In this paper we construct the autonomous quad-equations which admit as symmetries the five-point differential-difference equations belonging to known lists found by Garifullin, Yamilov and Levi. The obtained equations are classified up to autonomous point transformations and some simple non-autonomous transformations. We discuss our results in the framework of the known literature. There are among them a few new examples of both sine-Gordon and Liouville type equations.

Keywords: Integrability; Generalized symmetries; Quad-equations.

2000 Mathematics Subject Classification: 37L20, 37K10, 39A14

1. Introduction

In this paper we consider in discrete *quad-equations*. Quad-equations are four-point relations of the form:

$$F(u_{n,m}, u_{n+1,m}, u_{n,m+1}, u_{n+1,m+1}) = 0, \quad (1.1)$$

for an unknown field $u_{n,m}$ depending on two discrete variables, i.e. $(n, m) \in \mathbb{Z}^2$. The function $F = F(x, y, z, w)$ is assumed to be an *irreducible multi-affine polynomial* [6]. We recall that a polynomial is said to be multi-affine if it is affine with respect to all its variable, and it is said to be irreducible if it only admits trivial factors, i.e. complex constants. Moreover, by omitting the subscript n, m we underline that it does not depend explicitly on the discrete variables, i.e. the equation (1.1) is assumed to be *autonomous*.

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It is known [20, 29, 31–33] that the *generalized symmetries* of quad-equations are given by differential-difference equations of the form:

$$\frac{du_{n,m}}{dt_1} = \varphi_{n,m} \left(u_{n+k_1,m}, \dots, u_{n-k'_1,m} \right), \tag{1.2a}$$

$$\frac{du_{n,m}}{dt_2} = \psi_{n,m} \left(u_{n,m+k_2}, \dots, u_{n,m-k'_2} \right), \tag{1.2b}$$

where $k_i, k'_i \in \mathbb{N}$. This means that one generalized symmetries depends only on shifts in the n direction, while the other depends only on shifts in the m direction. We note that usually differential-difference equations are also addressed by the amount of points of the lattice that they are involving. In this sense the equations in (1.2) are a $k_1 + k'_1 + 1$ -point differential-difference equation and a $k_2 + k'_2 + 1$ -point differential-difference equation respectively.

The vast majority of quad-equations known in literature admits as generalized symmetries three-point differential-difference equations in both directions [23, 29, 32, 36, 44, 45]. When the given quad-equation admits three-point generalized symmetries it can be interpreted as a Bäcklund transformation for these differential-difference equations [28, 29]. More recently, several examples with more complicated symmetry structure have been discovered [2, 10, 11, 14, 33].

In the aforementioned papers the problem which was addressed was to find the generalized symmetries of a given quad-equation. If the generalized symmetries are given by integrable differential-difference equations, then one can construct a whole hierarchy of generalized symmetries for the given quad-equation which is integrable according to the generalized symmetries method [46]. On the other hand, also the converse problem can be considered: fixed a differential-difference equation find the quad-equation admitting it as a generalized symmetry. This point of view was taken in [31], using one well-known five-point autonomous differential-difference equation. In [15] a similar analysis was carried out for a known class of integrable non-autonomous Volterra and Toda three-point differential-difference equations [30].

In this paper we are going to generalize the results of [15, 31]: we will start from a known integrable *autonomous five-point* differential-differential equation and construct the quad-equation admitting it as generalized symmetry in the n direction, i.e. of the form (1.2a). To this end we will use the classification of the five-point differential-difference equations given in [18, 19]. The equations belonging to the classification in [18, 19] have the following form:

$$\begin{aligned} \frac{du_k}{dt} = & A(u_{k+1}, u_k, u_{k-1})u_{k+2} + B(u_{k+1}, u_k, u_{k-1})u_{k-2} \\ & + C(u_{k+1}, u_k, u_{k-1}), \end{aligned} \tag{1.3}$$

i.e. are linear in $u_{k\pm 2}$. We recall that in papers [18, 19] integrability is defined as the existence of an infinite hierarchy of generalized symmetries. These equations are divided in two classes, which we now describe explicitly. The first class, Class I was presented in [18] and contains all the equations of the form (1.3) such that the condition:

$$A \neq \alpha(u_{k+1}, u_k) \alpha(u_k, u_{k-1}), \quad B \neq \beta(u_{k+1}, u_k) \beta(u_k, u_{k-1}), \tag{1.4}$$

holds true for any functions α, β . This Class contains seventeen equations. Equations belonging to this class are denoted by (E.x), where x is an Arabic number. The second class, Class II, was presented in [19] and contains all the equations of the form (1.3) such that the condition (1.4) does

not hold, i.e. there exist functions α, β such that

$$A = \alpha(u_{k+1}, u_k) \alpha(u_k, u_{k-1}), \quad B = \beta(u_{k+1}, u_k) \beta(u_k, u_{k-1}). \quad (1.5)$$

This class contains fourteen equations. Equations belonging to this class are denoted by (E.x'), where x is an Arabic number. For easier understanding of the results, we split the complete list into smaller Lists 1-6. In each List the equations are related to each other by autonomous non-invertible and non-point transformations or simple non-autonomous point transformations. Moreover, for sake of simplicity, since the equations are autonomous, in displaying the equations we will use the short-hand notation $u_i = u_{k+i}$.

List 1. Equations related to the double Volterra equation:

$$\frac{du_0}{dt} = u_0(u_2 - u_{-2}), \quad (E.1)$$

$$\frac{du_0}{dt} = u_0^2(u_2 - u_{-2}), \quad (E.2)$$

$$\frac{du_0}{dt} = (u_0^2 + u_0)(u_2 - u_{-2}), \quad (E.3)$$

$$\frac{du_0}{dt} = (u_2 + u_1)(u_0 + u_{-1}) - (u_1 + u_0)(u_{-1} + u_{-2}), \quad (E.4)$$

$$\begin{aligned} \frac{du_0}{dt} &= (u_2 - u_1 + a)(u_0 - u_{-1} + a) \\ &+ (u_1 - u_0 + a)(u_{-1} - u_{-2} + a) + b, \end{aligned} \quad (E.5)$$

$$\begin{aligned} \frac{du_0}{dt} &= u_2 u_1 u_0 (u_0 u_{-1} + 1) \\ &- (u_1 u_0 + 1) u_0 u_{-1} u_{-2} + u_0^2 (u_{-1} - u_1), \end{aligned} \quad (E.6)$$

$$\frac{du_0}{dt} = u_0 [u_1(u_2 - u_0) + u_{-1}(u_0 - u_{-2})], \quad (E.1')$$

$$\frac{du_0}{dt} = u_1 u_0^2 u_{-1} (u_2 - u_{-2}). \quad (E.2')$$

Transformations $\tilde{u}_k = u_{2k}$ or $\tilde{u}_k = u_{2k+1}$ turn equations (E.1)-(E.3) into the well-known Volterra equation and its modifications in their standard form. The other equations are related to the *double Volterra equation*(E.1) through some autonomous non-invertible non-point transformations. We note that equation (E.2') was presented in [4].

List 2. Linearizable equations:

$$\begin{aligned} \frac{du_0}{dt} &= (T - a) \left[\frac{(u_1 + au_0 + b)(u_{-1} + au_{-2} + b)}{u_0 + au_{-1} + b} + u_0 + au_{-1} + b \right] \\ &+ cu_0 + d, \end{aligned} \quad (E.7)$$

$$\frac{du_0}{dt} = \frac{u_2 u_0}{u_1} + u_1 - a^2 \left(u_{-1} + \frac{u_0 u_{-2}}{u_{-1}} \right) + cu_0. \quad (E.8)$$

In both equations $a \neq 0$, in (E.7) $(a + 1)d = bc$, and T is the translation operator $T f_k = f_{k+1}$.

Both equations of List 2 are related to the linear equation:

$$\frac{du_0}{dt} = u_2 - a^2 u_{-2} + \frac{c}{2} u_0 \tag{1.6}$$

through an autonomous non-invertible non-point transformations. We note that (E.7) is linked to (1.6) with a transformation which is implicit in both directions, see [18] for more details.

List 3. Equations related to a generalized symmetry of the Volterra equation:

$$\begin{aligned} \frac{du_0}{dt} &= u_0 [u_1(u_2 + u_1 + u_0) - u_{-1}(u_0 + u_{-1} + u_{-2})] \\ &\quad + cu_0(u_1 - u_{-1}), \end{aligned} \tag{E.3'}$$

$$\begin{aligned} \frac{du_0}{dt} &= (u_0^2 - a^2) [(u_1^2 - a^2)(u_2 + u_0) - (u_{-1}^2 - a^2)(u_0 + u_{-2})] \\ &\quad + c(u_0^2 - a^2)(u_1 - u_{-1}), \end{aligned} \tag{E.4'}$$

$$\frac{du_0}{dt} = (u_1 - u_0 + a)(u_0 - u_{-1} + a)(u_2 - u_{-2} + 4a + c) + b, \tag{E.5'}$$

$$\frac{du_0}{dt} = u_0 [u_1(u_2 - u_1 + u_0) - u_{-1}(u_0 - u_{-1} + u_{-2})], \tag{E.6'}$$

$$\frac{du_0}{dt} = (u_0^2 - a^2) [(u_1^2 - a^2)(u_2 - u_0) + (u_{-1}^2 - a^2)(u_0 - u_{-2})], \tag{E.7'}$$

$$\frac{du_0}{dt} = (u_1 + u_0)(u_0 + u_{-1})(u_2 - u_{-2}). \tag{E.8'}$$

These equations are related between themselves by some transformations, for more details see [19]. Moreover equations (E.3',E.4',E.5') are the generalized symmetries of some known three-point autonomous differential-difference equations [46].

List 4. Equations of the relativistic Toda type:

$$\frac{du_0}{dt} = (u_0 - 1) \left(\frac{u_2(u_1 - 1)u_0}{u_1} - \frac{u_0(u_{-1} - 1)u_{-2}}{u_{-1}} - u_1 + u_{-1} \right), \tag{E.9}$$

$$\begin{aligned} \frac{du_0}{dt} &= \frac{u_2 u_1^2 u_0^2 (u_0 u_{-1} + 1)}{u_1 u_0 + 1} - \frac{(u_1 u_0 + 1) u_0^2 u_{-1}^2 u_{-2}}{u_0 u_{-1} + 1} \\ &\quad - \frac{(u_1 - u_{-1})(2u_1 u_0 u_{-1} + u_1 + u_{-1}) u_0^3}{(u_1 u_0 + 1)(u_0 u_{-1} + 1)}, \end{aligned} \tag{E.10}$$

$$\frac{du_0}{dt} = (u_1 u_0 - 1)(u_0 u_{-1} - 1)(u_2 - u_{-2}). \tag{E.13'}$$

Equation (E.13') was known [12, 14] to be is a relativistic Toda type equation. Since in [18] it was shown that the equations of List 4 are related through autonomous non-invertible non-point transformations, it was suggested that (E.9) and (E.10) should be of the same type. Finally, we note that equation (E.9) appeared in [1] earlier than in [18].

List 5. Equations related to the Itoh-Narita-Bogoyavlensky (INB) equation:

$$\frac{du_0}{dt} = u_0(u_2 + u_1 - u_{-1} - u_{-2}), \tag{E.11}$$

$$\begin{aligned} \frac{du_0}{dt} &= (u_2 - u_1 + a)(u_0 - u_{-1} + a) \\ &\quad + (u_1 - u_0 + a)(u_{-1} - u_{-2} + a) \\ &\quad + (u_1 - u_0 + a)(u_0 - u_{-1} + a) + b, \end{aligned} \tag{E.12}$$

$$\frac{du_0}{dt} = (u_0^2 + au_0)(u_2u_1 - u_{-1}u_{-2}), \tag{E.13}$$

$$\frac{du_0}{dt} = (u_1 - u_0)(u_0 - u_{-1}) \left(\frac{u_2}{u_1} - \frac{u_{-2}}{u_{-1}} \right), \tag{E.14}$$

$$\frac{du_0}{dt} = u_0(u_2u_1 - u_{-1}u_{-2}), \tag{E.9'}$$

$$\frac{du_0}{dt} = (u_1 - u_0 + a)(u_0 - u_{-1} + a)(u_2 - u_1 + u_{-1} - u_{-2} + 2a) + b, \tag{E.10'}$$

$$\frac{du_0}{dt} = u_0(u_1u_0 - a)(u_0u_{-1} - a)(u_2u_1 - u_{-1}u_{-2}), \tag{E.11'}$$

$$\frac{du_0}{dt} = (u_1 + u_0)(u_0 + u_{-1})(u_2 + u_1 - u_{-1} - u_{-2}). \tag{E.12'}$$

Equation (E.11) is the well-known INB equation [8,27,34]. Equations (E.12) with $a = 0$ and (E.13) with $a = 0$ are simple modifications of the INB and were presented in [33] and [9], respectively. Equation (E.13) with $a = 1$ has been found in [40]. Up to an obvious linear transformation, it is equation (17.6.24) with $m = 2$ in [40]. Equation (E.9') is a well-known modification of INB equation (E.11), found by Bogoyalavlesky himself [8]. Finally, equation (E.11') with $a = 0$ was considered in [4]. All the equations in this list can be reduced to the INB equation using autonomous non-invertible non-point transformations. Moreover, equations (E.12),(E.14) and (E.9') are related through non-invertible transformations to the equation:

$$\frac{du_0}{dt} = (u_2 - u_0)(u_1 - u_{-1})(u_0 - u_{-2}). \tag{1.7}$$

We will also study below equation (1.7), see Remark 2.2 in Section 2.

List 6. Other equations:

$$\frac{du_0}{dt} = u_0^2(u_2u_1 - u_{-1}u_{-2}) - u_0(u_1 - u_{-1}), \tag{E.15}$$

$$\frac{du_0}{dt} = (u_0 + 1) \left[\frac{u_2u_0(u_1 + 1)^2}{u_1} - \frac{u_{-2}u_0(u_{-1} + 1)^2}{u_{-1}} + (1 + 2u_0)(u_1 - u_{-1}) \right], \tag{E.16}$$

$$\frac{du_0}{dt} = (u_0^2 + 1) \left(u_2\sqrt{u_1^2 + 1} - u_{-2}\sqrt{u_{-1}^2 + 1} \right), \tag{E.17}$$

$$\frac{du_0}{dt} = u_1u_0^3u_{-1}(u_2u_1 - u_{-1}u_{-2}) - u_0^2(u_1 - u_{-1}). \tag{E.14'}$$

Equation (E.15) has been found in [42] and it is called the discrete Sawada-Kotera equation [2, 42]. Equation (E.14') is a simple modification of the discrete Sawada-Kotera equation (E.15). Equation (E.16) has been found in [1] and is related to (E.15). On the other hand equation (E.17) has been found as a result of the classification in [18] and seems to be a new equation. It was shown in [16] that equation (E.17) is a discrete analogue of the Kaup-Kupershmidt equation. Then we will refer to equation (E.17) as the discrete

Kaup-Kupershmidt equation. No transformation into known equations of equation (E.17) is known.

As a result we will obtain several examples of autonomous nontrivial quad-equations. By nontrivial equations we mean non-degenerate, irreducible and nonlinear equations. Moreover we consider as trivial also the equations

$$u_{n+1,m+1}u_{n,m+1} = \mathcal{K}_1 u_{n+1,m}u_{n,m}, \tag{1.8a}$$

$$u_{n+1,m+1}u_{n,m} = \mathcal{K}_2 u_{n+1,m}u_{n,m+1}, \tag{1.8b}$$

which are equivalent to a linear one up to the transcendental transformation $u_{n,m} = \exp v_{n,m}$. The resulting equations belong to two different types of equations: they can be either *Liouville type* LT or *sine-Gordon type* sGT. Liouville type equations are quad-equation which are *Darboux integrable*, i.e. they possess *first integrals*, one containing only shifts in the first direction and the other containing only shifts in the second direction. This means that there exist two functions:

$$W_1 = W_{1,n,m}(u_{n+l_1,m}, u_{n+l_1+1,m}, \dots, u_{n+k_1,m}), \tag{1.9a}$$

$$W_2 = W_{2,n,m}(u_{n,m+l_2}, u_{n,m+l_2+1}, \dots, u_{n,m+k_2}), \tag{1.9b}$$

where $l_1 < k_1$ and $l_2 < k_2$ are integers, such that the relations

$$(T_n - \text{Id})W_2 = 0, \tag{1.10a}$$

$$(T_m - \text{Id})W_1 = 0 \tag{1.10b}$$

hold true identically on the solutions of (1.1). By T_n, T_m we denote the shift operators in the first and second directions, i.e. $T_n h_{n,m} = h_{n+1,m}$, $T_m h_{n,m} = h_{n,m+1}$, and by Id we denote the identity operator $\text{Id} h_{n,m} = h_{n,m}$. The number $k_i - l_i$, where $i = 1, 2$, is called the *order* of the first integral W_i . On the other hand a quad-equation is said to be of sine-Gordon type if it is not Darboux integrable and possess two generalized symmetries of the form (1.2). Such equations are usually integrable by the inverse scattering method.

We underline that, by construction, all the equations we are going to produce will possess a generalized symmetry in the n direction (1.2a) which is not sufficient for integrability. To prove that the equation, we are going to produce, are LT or sGT we will either identify them with known equations, or present their first integrals in the sense of (1.9) or a generalized symmetry in the m direction (1.2b).

The plan of the paper is then the following: In Section 2 we discuss the theoretical background which allows us to make the relevant computations. In Section 3 we enumerate all the quad-equations of the form (1.1) which corresponds to the equations of the Class I and II and describe their properties. Finally in Section 4 we give a summary of the work and some outlook for future works in the field.

2. The method

The most general autonomous multi-affine quad-equation has the following form:

$$\begin{aligned}
 F \equiv & u_{n+1,m+1}u_{n,m+1} (a_1u_{n,m}u_{n+1,m} + a_2u_{n,m} + a_3u_{n+1,m} + a_4) \\
 & + u_{n+1,m+1} (b_1u_{n,m}u_{n+1,m} + b_2u_{n,m} + b_3u_{n+1,m} + b_4) \\
 & + u_{n,m+1} (c_1u_{n,m}u_{n+1,m} + c_2u_{n,m} + c_3u_{n+1,m} + c_4) \\
 & + d_1u_{n,m}u_{n+1,m} + d_2u_{n,m} + d_3u_{n+1,m} + d_4 = 0.
 \end{aligned}
 \tag{2.1}$$

From (1.2a) and (1.3) we have that the most general symmetry in the n direction of the form (1.2a) belonging to Class I and II can be written as:

$$\begin{aligned}
 \dot{u}_{n,m} = & a(u_{n+1,m}, u_{n,m}, u_{n-1,m})u_{n+2,m} + b(u_{n+1,m}, u_{n,m}, u_{n-1,m})u_{n-2,m} \\
 & + c(u_{n+1,m}, u_{n,m}, u_{n-1,m}).
 \end{aligned}
 \tag{2.2}$$

We have then the following general result, analogous to Theorem 2 in [15]:

Theorem 2.1. *If the autonomous quad-equation (2.1) admits a generalized symmetry of the form (2.2) then it has the following form:*

$$\begin{aligned}
 \widehat{F} \equiv & \alpha u_{n+1,m+1}u_{n,m+1} + u_{n+1,m+1} (\beta_1u_{n,m} + \beta_2) + u_{n,m+1} (\gamma_1u_{n+1,m} + \gamma_2) \\
 & + \delta_1u_{n,m}u_{n+1,m} + \delta_2u_{n,m} + \delta_3u_{n+1,m} + \delta_4 = 0.
 \end{aligned}
 \tag{2.3}$$

Proof. By multi-linearity we can always solve (2.1) with respect to $u_{n+1,m+1}$ and write:

$$u_{n+1,m+1} = f(u_{n+1,m}, u_{n,m}, u_{n,m+1}).
 \tag{2.4}$$

From Theorem 2 in [31] we have that a quad-equation admits a generalized symmetry of the form (1.2a) with $k_1 = -k'_1 = 2$ if the following conditions are satisfied:

$$(T_n - T_n^{-1}) \log \frac{\partial f}{\partial u_{n+1,m}} = (1 - T_m) \log \frac{\partial \varphi}{\partial u_{n+2,m}},
 \tag{2.5a}$$

$$(T_n^{-2} - 1) \log \left(\frac{\partial f}{\partial u_{n,m}} / \frac{\partial f}{\partial u_{n,m+1}} \right) = (1 - T_m) \log \frac{\partial \varphi}{\partial u_{n-2,m}}.
 \tag{2.5b}$$

In (2.5) we suppressed the indices n, m in φ since we are dealing with autonomous differential-difference equations. Now from (2.2) we have:

$$\frac{\partial \varphi}{\partial u_{n+2,m}} = a(u_{n+1,m}, u_{n,m}, u_{n-1,m}), \quad \frac{\partial \varphi}{\partial u_{n-2,m}} = b(u_{n+1,m}, u_{n,m}, u_{n-1,m}).
 \tag{2.6}$$

From equation (2.6) we have differentiating (2.5a) w.r.t. $u_{n+2,m}$:

$$0 = \frac{\partial}{\partial u_{n+2,m}} T_n \log \frac{\partial f}{\partial u_{n+1,m}} = -2T_n \left(\frac{\partial^2 F / \partial u_{n+1,m+1} \partial u_{n+1,m}}{\partial F / \partial u_{n+1,m+1}} \right),
 \tag{2.7}$$

where we used the implicit function theorem and the fact that F is multi-affine (2.1). In the same way differentiating (2.5b) w.r.t $u_{n-2,m}$ we get:

$$0 = \frac{\partial}{\partial u_{n-2,m}} T_n^{-2} \log \left(\frac{\partial f}{\partial u_{n,m}} / \frac{\partial f}{\partial u_{n,m+1}} \right) = -2T_n^{-2} \left(\frac{\partial^2 F / \partial u_{n,m+1} \partial u_{n,m}}{\partial F / \partial u_{n,m+1}} \right).
 \tag{2.8}$$

Therefore we get the two conditions:

$$\frac{\partial^2 F}{\partial u_{n+1,m+1} \partial u_{n+1,m}} = \frac{\partial^2 F}{\partial u_{n,m+1} \partial u_{n,m}} = 0. \tag{2.9}$$

Working out explicitly the conditions in (2.9) and using (2.1) we obtain:

$$a_1 = a_2 = a_3 = b_1 = b_3 = c_1 = c_2 = 0. \tag{2.10}$$

Relabeling the parameters as

$$a_4 \rightarrow \alpha, \quad b_2 \rightarrow \beta_1, \quad b_4 \rightarrow \beta_2, \quad c_3 \rightarrow \gamma_1, \quad c_4 \rightarrow \gamma_2, \quad d_i \rightarrow \delta_i, \tag{2.11}$$

equation (2.3) follows. □

Remark 2.1. We note that the condition (2.9) is the same as formula (29) in [15] even though the class of considered differential-difference equation is different.

Remark 2.2. It can be proved in a similar way that Theorem 2.1 is valid for generalized symmetries of the form (1.2a) satisfying the conditions $k'_1 = k_1$ and

$$\frac{\partial^2 \varphi_{n,m}}{\partial u_{n+k_1}^2} = 0, \quad \frac{\partial^2 \varphi_{n,m}}{\partial u_{n-k_1}^2} = 0. \tag{2.12}$$

Theorem 2.1 tells us that the most general form of a quad-equation admitting a five-point generalized symmetry in the n direction of the form (2.2) is given by (2.3). At this point we can pick up any of the members of Class I and II and follow the scheme of [18, App. B], i.e. we fix a specific form of a , b and c in (2.2). We begin by imposing the exponential integrability conditions (2.5a) and (2.5b) and finally we impose the symmetry condition:

$$\sum_{i,j \in \{0,1\}} \varphi(u_{n+2+i,m+j}, \dots, u_{n-2+i,m+j}) \frac{\partial \widehat{F}}{\partial u_{n+i,m+j}} = 0 \tag{2.13}$$

which must be satisfied on all the solutions of (2.3). Then we express in (2.13) the functions $u_{n+2,m+1}$, $u_{n+1,m+1}$ and $u_{n-1,m+1}$ in terms of the independent variables

$$u_{n+2,m}, u_{n+1,m}, u_{n,m}, u_{n-1,m}, u_{n,m+1}. \tag{2.14}$$

Taking the numerator of (2.13) we obtain a polynomial in the independent variable (2.14). This polynomial must be identically zero, so we can equate to zero all its coefficients.

This lead us to a system of algebraic equation in the coefficients of the multi-affine function (2.3). Doing so we reduce the problem of finding a quad-equation admitting a given five-point symmetry of the form (2.2) to the problem of solving system of algebraic equations. Such system can be solved using a Computer Algebra System like Maple, Mathematica or Reduce. Amongst the possible solutions of the system we choose the non-degenerate ones. The non-degeneracy condition is the following one:

$$\frac{\partial F}{\partial u_{n+1,m+1}} \frac{\partial f}{\partial u_{n+1,m}} \frac{\partial f}{\partial u_{n,m+1}} \frac{\partial f}{\partial u_{n,m}} \neq 0. \tag{2.15}$$

Note that this non-degeneracy condition includes the requirement that a quad-equation must be given by an irreducible multi-affine polynomial, see the introduction.

Remark 2.3. Several equations in Class I and II, e.g. (E.5) or (E.5'), depend on some parameters. Depending on the value of the parameters there can be, in principle, different quad-equations admitting the given differential-difference equation as five-point generalized symmetries. When possible, in order to avoid ambiguities and simplify the problem, we use some simple autonomous transformations to fix the values of some parameters. The remaining free parameters are then treated as unknown coefficients in the system of algebraic equations. We will describe these subcases when needed in the next section.

The number of the resulting equations is then reduced using some point transformations. These point transformations are essentially of two different kind: non-autonomous transformations of the dependent variable $u_{n,m}$:

$$u_{n,m} \rightarrow (-1)^m u_{n,m}, \tag{T.1}$$

$$u_{n,m} \rightarrow \left(\frac{1 + i\sqrt{3}}{2} \right)^m u_{n,m}, \tag{T.2}$$

$$u_{n,m} \rightarrow i^m u_{n,m} \tag{T.3}$$

and mirror reflections of the lattice:

$$m \rightarrow 1 - m, \quad \text{i.e.} \quad u_{n,m} \longleftrightarrow u_{n,m+1}, \quad u_{n+1,m} \longleftrightarrow u_{n+1,m+1}, \tag{T.*1}$$

$$n \rightarrow 1 - n, \quad \text{i.e.} \quad u_{n,m} \longleftrightarrow u_{n+1,m}, \quad u_{n,m+1} \longleftrightarrow u_{n+1,m+1}. \tag{T.*2}$$

Now, in the next section we describe the results of this search.

3. Results

In this section we describe the results of the procedure outlined in Section 2. Specifically, as described in 2.3 we will explicit the particular cases in which the parametric equations can be divided.

3.1. List 1

Equation (E.1): To equation (E.1) correspond only degenerate quad-equations in the sense of condition (2.15).

Equation (E.2): To equation (E.2) correspond two nontrivial quad-equations:

$$u_{n+1,m+1}(au_{n,m+1} + u_{n,m}) - u_{n+1,m}(u_{n,m+1} + au_{n,m}) = 0, \quad a^2 = -1. \tag{3.1}$$

Up to transformation (T.1) we can reduce these two equations to only one, say the one with $a = -i$. Equation (3.1, $a = -i$) is a special case of equation (7) with $a_2 = -i$ of List 4 in [14]. This means that this is a LT equation, its first integrals being:

$$W_1 = u_{n+1,m}u_{n-1,m}, \quad W_2 = i^{n+1} \frac{u_{n,m+1} - u_{n,m}}{u_{n,m+1} + u_{n,m}}. \tag{3.2}$$

See [14] for more details.

Equation (E.3): To equation (E.3) correspond only degenerate quad-equations in the sense of condition (2.15).

Equation (E.4): To equation (E.4) correspond only degenerate quad-equations in the sense of condition (2.15) or the linear discrete wave equation:

$$u_{n+1,m+1} + u_{n,m+1} - u_{n+1,m} - u_{n,m} = 0. \quad (3.3)$$

Equation (E.5): Equation (E.5) depends on the parameters a and b . If $a \neq 0$ it is possible to scale it to one through a scaling transformation. This amounts to consider the cases $a = 1$ and $a = 0$.

To equation (E.5, $a = 1$) correspond only degenerate quad-equations in the sense of condition (2.15) or the linear discrete wave equation (3.3). The same holds true for equation (E.5, $a = 0$). So to all the instances of equation (E.5) correspond only trivial or linear equations.

Equation (E.6): To equation (E.6) correspond two quad-equation. One is the trivial exponential wave equation (1.8a), while the other one is the LT equation:

$$u_{n+1,m+1}u_{n,m+1} + u_{n+1,m}u_{n,m} + 1 = 0. \quad (3.4)$$

Equation (3.4) is equation (9) with $c_4 = 1$ of List 4 in [14]. Its first integrals are [14]:

$$W_1 = (-1)^m (2u_{n+1,m}u_{n,m} + 1), \quad W_2 = \left(\frac{u_{n,m+1}}{u_{n,m-1}} \right)^{(-1)^n}. \quad (3.5)$$

Equation (E.1'): To equation (E.1') correspond only degenerate quad-equations in the sense of condition (2.15) and the trivial linearizable equation (1.8a).

Equation (E.2'): Equation (E.2') is very rich, since it give raise to many different equations. We have two trivial linearizable equations of the form (1.8a), but also twelve nontrivial equations. We have four equations of the form:

$$(k_1u_{n,m} + k_3u_{n,m+1})u_{n+1,m} + (k_2u_{n,m} + k_4u_{n,m+1})u_{n+1,m+1} = 0, \quad (3.6)$$

with

$$k_1 = 1, \quad k_2 = -1, \quad k_3 = 1, \quad k_4 = 1, \quad (3.7a)$$

$$k_1 = 1, \quad k_2 = i, \quad k_3 = -i, \quad k_4 = -1, \quad (3.7b)$$

$$k_1 = -1, \quad k_2 = i, \quad k_3 = -i, \quad k_4 = 1, \quad (3.7c)$$

$$k_1 = -1, \quad k_2 = -1, \quad k_3 = 1, \quad k_4 = -1. \quad (3.7d)$$

Then four equations of the form:

$$u_{n,m}u_{n+1,m} + k_1u_{n,m+1}u_{n+1,m} + k_2u_{n,m+1}u_{n+1,m+1} = 0, \quad (3.8)$$

with

$$k_1 = -1 + i, \quad k_2 = -i, \quad (3.9a)$$

$$k_1 = 1 - i, \quad k_2 = -i, \quad (3.9b)$$

$$k_1 = -1 - i, \quad k_2 = i, \quad (3.9c)$$

$$k_1 = 1 + i, \quad k_2 = i. \quad (3.9d)$$

Finally we have four equations of the form:

$$u_{n,m}u_{n+1,m} + k_1u_{n,m}u_{n+1,m+1} + k_2u_{n,m+1}u_{n+1,m+1} = 0, \tag{3.10}$$

with:

$$k_1 = -1 + i, \quad k_2 = -i, \tag{3.11a}$$

$$k_1 = 1 - i, \quad k_2 = -i, \tag{3.11b}$$

$$k_1 = -1 - i, \quad k_2 = i, \tag{3.11c}$$

$$k_1 = 1 + i, \quad k_2 = i. \tag{3.11d}$$

Up to the transformations (T.1,T.3) and (T*.1) we can reduce equations (3.6,3.8,3.10) to only two equations, say (3.6) with the parameters (3.7d) and (3.8) with the parameters (3.9d):

$$(u_{n,m} - u_{n,m+1})u_{n+1,m} + (u_{n,m+1} + u_{n,m})u_{n+1,m+1} = 0, \tag{3.12a}$$

$$[u_{n,m} + (1 + i)u_{n,m+1}]u_{n+1,m} + iu_{n+1,m+1}u_{n,m+1} = 0. \tag{3.12b}$$

For a complete description of the transformations leading to (3.12) see Figure 1.

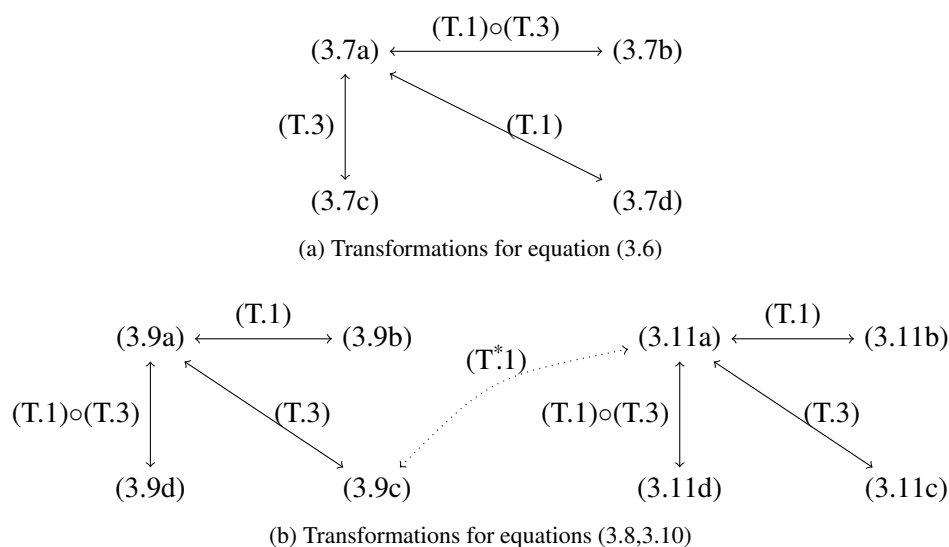


Figure 1: The relationship between the equations (3.6,3.8,3.10). By (T.1)◦(T.3) we mean the composition of the two transformations.

Equation (3.12a) is a LT equation. It can be identified with equation (7) with $a_2 = 1$ of List 4 of [14]. In particular its first integrals are given by:

$$W_1 = (-1)^m u_{n+1,m}u_{n-1,m}, \quad W_2 = i^n \frac{u_{n,m} + iu_{n,m+1}}{iu_{n,m} + u_{n,m+1}}. \tag{3.13}$$

Equation (3.12b) is a LT equation too. Its first integrals are given by:

$$W_1 = u_{n,m}u_{n+1,m}u_{n+2,m}u_{n+3,m}, \quad W_2 = (-i)^n \frac{u_{n,m} + iu_{n,m+1}}{u_{n,m} + u_{n,m+1}}. \tag{3.14}$$

Note that the W_2 first integral of (3.12b) is very similar to the W_2 first integral of (3.12a). Moreover, the W_1 first integral of (3.12b) is a generalization of the W_1 first integral of (3.12a) with more points. It is possible to show that equation (3.12b) do not belong to known families of Darboux integrable equations [5, 14, 24, 37, 38], because of the specific form of W_1 . Therefore we believe that equation (3.12b) is a new Darboux integrable equation.

3.2. List 2

Equation (E.7): Equation (E.7) depends on four parameters a, b, c and d linked among themselves by the condition $(a + 1)d = bc$. Therefore we have that using a linear transformation $u_{n,m} \rightarrow \alpha u_{n,m} + \beta$ we need to consider only three different cases: 1) $a \neq 0, a \neq -1, b = 0, d = 0$, 2) $a = -1, b = 1, c = 0, 3) a = -1, b = 0$ (recall that $a \neq 0$ in all cases).

Case $a \neq 0, a \neq -1, b = 0, d = 0$: For these values of the parameters we obtain either linear equations or exponential wave equations (1.8b) if also $a \neq 1$ and $c \neq 0$. For $a = 1$ and $c = 0$ we obtain the equation:

$$d_3^4 u_{n,m} u_{n+1,m} + d_3^3 d_4 (u_{n+1,m} + u_{n,m}) - c_3 d_3 d_4^2 (u_{n,m+1} + u_{n+1,m+1}) + d_3^2 d_4 [d_4 + c_3 (u_{n,m} u_{n+1,m+1} + u_{n,m+1} u_{n+1,m})] + c_3^2 d_4^2 u_{n,m+1} u_{n+1,m+1} = 0. \tag{3.15}$$

In equation (3.15) we have the condition $c_3 d_3 d_4 \neq 0$. Using a linear transformation of the form $u_{n,m} = A \hat{u}_{n,m} + B$ we can reduce equation (3.15) to two different cases.

Case 1: If $d_3^2 + c_3 d_4 = 0$ and if we chose $A = d_4/d_3 = -4B$, we have:

$$(u_{n+1,m+1} - u_{n+1,m})(u_{n,m} - u_{n,m+1}) + u_{n+1,m+1} + u_{n+1,m} + u_{n,m+1} + u_{n,m} = 0. \tag{3.16}$$

This equation is equation (1) of List 3 of [14], hence it is a LT equation with the following first integrals:

$$W_1 = \frac{2(u_{n+1,m} + u_{n,m}) + 1}{(u_{n+2,m} - u_{n,m})(u_{n+1,m} - u_{n-1,m})}, \tag{3.17a}$$

$$W_2 = (-1)^n \frac{u_{n,m-1} + u_{n,m+1} - 2(u_{n,m} - 1)}{u_{n,m+1} - u_{n,m-1}}. \tag{3.17b}$$

Case 2: If $d_3^2 + c_3 d_4 \neq 0$ we chose

$$A = 2 \frac{d_4}{d_3}, \quad B = \frac{d_3 d_4}{d_3^2 + c_3 d_4}, \tag{3.18}$$

and equation (3.15) reduces to:

$$\left(\frac{c_3 d_4}{d_3^2} u_{n+1,m+1} + u_{n+1,m} \right) \left(\frac{c_3 d_4}{d_3^2} u_{n,m+1} + u_{n,m} \right) + u_{n+1,m} + u_{n,m} + \frac{d_3^2}{d_3^2 + c_3 d_4} = 0. \tag{3.19}$$

This equation is equation (3) of List 3 of [14] with $a_3 = c_3 d_4/d_3^2$, hence it is a LT equation with the following first integrals:

$$W_1 = (-a_3)^{-m} \frac{(a_3 + 1)(u_{n+1,m} + u_{n,m}) + 1}{(u_{n+2,m} - u_{n,m})(u_{n+1,m} - u_{n-1,m})}, \tag{3.20a}$$

$$W_2 = \left[\frac{u_{n,m-1} + a_3 u_{n,m} + 1}{\sqrt{-a_3} (u_{n,m} + a_3 u_{n,m+1})} \right]^{(-1)^n}. \quad (3.20b)$$

Case $a = -1, b = 1, c = 0$: For this value of the parameters only degenerate or linear equations arise. In particular we have discrete wave-like equation.

Case $a = -1, b = 0$: For this value of the parameters we have several linear wave-like equations, but also one nontrivial equation:

$$\left(u_{n+1,m+1} + \frac{d_3}{c_3} \right) \left(u_{n,m} - \frac{b_4}{c_3} \right) - \left(u_{n,m+1} + \frac{d_3}{c_3} \right) \left(u_{n+1,m} - \frac{b_4}{c_3} \right) = 0, \quad (3.21)$$

where we have the following relationship between the parameters:

$$d = -c \frac{b_4}{c_3}, \quad c(b_4 + d_3) = 0. \quad (3.22)$$

The case $b_4 = -d_3$ (3.21) is equivalent to (1.8) up to a linear transformation. The case $b_4 \neq -d_3, c = d = 0$, and (3.21) is equivalent to (4) of List 4 of [14] up to a linear transformation and it is a LT equation.

Equation (E.8): To equation (E.8) correspond only degenerate quad-equations in the sense of condition (2.15) and the trivial linearizable equation (1.8b).

3.3. List 3

Equation (E.3'): To equation (E.3') correspond only degenerate quad-equations in the sense of condition (2.15) and the trivial linearizable equation (1.8a).

Equation (E.4'): Equation (E.4') depends on the parameters a and c . If $a \neq 0$ then it can always be rescaled to 1, hence we can consider two independent cases: $a = 1$ and $a = 0$.

However, to equation (E.4') with $a = 0$ correspond only degenerate quad-equations in the sense of condition (2.15) and the trivial linearizable equation (1.8a).

On the other hand, the case when $a = 1$ admits nontrivial solutions. Indeed, to equation (E.4') with $a = 1$ correspond eight nontrivial equations. Four of them are valid for every value of c :

$$(u_{n+1,m} + \sigma_1)(u_{n,m} - \sigma_1) = (u_{n+1,m+1} - \sigma_2)(u_{n,m+1} + \sigma_2), \quad \sigma_i = \pm 1. \quad (3.23)$$

Using the transformation $u_{n,m} \rightarrow -u_{n,m}$ we can reduce the (3.23, $-\sigma_1 = \sigma_2 = 1$) case to the (3.23, $\sigma_1 = -\sigma_2 = 1$) case and the (3.23, $\sigma_1 = \sigma_2 = -1$) case to the (3.23, $\sigma_1 = \sigma_2 = 1$) case respectively. This means that we have only the two following independent equations

$$(u_{n+1,m} + 1)(u_{n,m} - 1) = (u_{n+1,m+1} - 1)(u_{n,m+1} + 1), \quad (3.24a)$$

$$(u_{n+1,m} + 1)(u_{n,m} - 1) = (u_{n+1,m+1} + 1)(u_{n,m+1} - 1). \quad (3.24b)$$

Other four equations are found if $c = -4$:

$$(u_{n+1,m} + \sigma_1)(u_{n,m} - \sigma_1) = -(u_{n+1,m+1} - \sigma_2)(u_{n,m+1} + \sigma_2), \quad \sigma_i = \pm 1. \quad (3.25)$$

Using the transformation $u_{n,m} \rightarrow -u_{n,m}$ we can reduce the (3.25, $-\sigma_1 = \sigma_2 = 1$) case to the (3.25, $\sigma_1 = -\sigma_2 = 1$) case and the (3.25, $\sigma_1 = \sigma_2 = -1$) case to the (3.25, $\sigma_1 = \sigma_2 = 1$) case respectively. This means that we have only the two following independent equations

$$(u_{n+1,m} + 1)(u_{n,m} - 1) = -(u_{n+1,m+1} + 1)(u_{n,m+1} - 1), \quad (3.26a)$$

$$(u_{n+1,m} + 1)(u_{n,m} - 1) = -(u_{n+1,m+1} - 1)(u_{n,m+1} + 1). \quad (3.26b)$$

Equation (3.24a) is a well-known sGT equation. It corresponds to equation $T1^*$ in [32]. See references therein for a discussion of the appearance of this equation in literature. On the other hand equation (3.24b) is transformed into equation (3) of List 4 in [14] with the linear transformation $u_{n,m} = 2\hat{u}_{n,m} + 1$. This means that this equation is a known LT equation with the following first integrals:

$$W_1 = (u_{n+1,m} + 1)(u_{n,m} - 1), \quad W_2 = \frac{(u_{n,m+2} - u_{n,m})(u_{n,m+1} - u_{n,m-1})}{(u_{n,m+2} - u_{n,m+1})(u_{n,m} - u_{n,m-1})}. \quad (3.27)$$

For further details see [37].

Equation (3.26a) is a new example of LT equation and has first integrals:

$$W_1 = (-1)^m (u_{n+1,m} + 1)(u_{n,m} - 1), \quad (3.28a)$$

$$W_2 = \frac{(u_{n,m+3} - u_{n,m-1})(u_{n,m+1} - u_{n,m-3})}{(u_{n,m+3} - u_{n,m+1})(u_{n,m-1} - u_{n,m-3})}. \quad (3.28b)$$

The integral W_2 is a sixth order (seven-point) first integral. It can be proved that there is no first integral of lower order in this direction. Examples of equations with first integrals of such a high order, except for a special series of Darboux integrable equations presented in [13], are, at the best of our knowledge, new in the literature.

Now we are going to use a transformation theory presented in [37, 47]. We can rewrite (3.26a) in a special form and introduce a new function $v_{n,m}$ as follows:

$$\frac{u_{n+1,m} + 1}{u_{n+1,m+1} + 1} = -\frac{u_{n,m+1} - 1}{u_{n,m} - 1} = v_{n+1,m}. \quad (3.29)$$

So we have

$$v_{n,m} = \frac{u_{n,m} + 1}{u_{n,m+1} + 1}, \quad v_{n+1,m} = -\frac{u_{n,m+1} - 1}{u_{n,m} - 1}, \quad (3.30a)$$

$$u_{n,m} = 1 + 2\frac{v_{n,m} - 1}{v_{n+1,m}v_{n,m} + 1}, \quad u_{n,m+1} = -1 + 2\frac{v_{n+1,m} + 1}{v_{n+1,m}v_{n,m} + 1}. \quad (3.30b)$$

This is a non-point transformation which is invertible on the solutions of (3.26a). Formula (3.30) allows to rewrite (3.26a) as

$$(v_{n+1,m+1} + 1)(v_{n,m} - 1) = (v_{n+1,m}^{-1} + 1)(v_{n,m+1}^{-1} - 1), \tag{3.31}$$

and its first integrals as:

$$W_1 = (-1)^m \frac{(v_{n+1,m} + 1)v_{n,m}(v_{n-1,m} - 1)}{(v_{n+1,m}v_{n,m} + 1)(v_{n,m}v_{n-1,m} + 1)}, \tag{3.32a}$$

$$W_2 = \frac{(v_{n,m+3}v_{n,m+2}v_{n,m+1}v_{n,m} - 1)(v_{n,m+1}v_{n,m}v_{n,m-1}v_{n,m-2} - 1)}{(v_{n,m+3}v_{n,m+2} - 1)v_{n,m+1}v_{n,m}(v_{n,m-1}v_{n,m-2} - 1)}. \tag{3.32b}$$

Both first integrals (3.32) are of lowest possible order in their directions. Example (3.31) is still exceptional, even though the order of W_2 has been lowered to five.

Equation (3.26b) is a sGT equation, as we can prove that it possesses the following second order autonomous generalized symmetries in both directions:

$$\frac{du_{n,m}}{dt_1} = (u_{n,m}^2 - 1) [(u_{n+1,m}^2 - 1)(u_{n+2,m} + u_{n,m}) - (u_{n-1,m}^2 - 1)(u_{n,m} + u_{n-2,m}) - 4(u_{n+1,m} - u_{n-1,m})], \tag{3.33a}$$

$$\frac{du_{n,m}}{dt_2} = (u_{n,m}^2 - 1)(T_m - 1) \left(\frac{u_{n,m+1} + u_{n,m}}{U_{n,m}} + \frac{u_{n,m-1} + u_{n,m-2}}{U_{n,m-1}} \right), \tag{3.33b}$$

where

$$U_{n,m} = (u_{n,m+1} + u_{n,m})(u_{n,m} + u_{n,m-1}) - 2(u_{n,m}^2 - 1). \tag{3.34}$$

It can be proved by direct computation using the method of [20, 31, 32] that equation (3.26b) does not admit any autonomous generalized symmetries of lower order. Equation (3.26b) seems to be a new interesting example and deserves a separate study, see [17].

Here we limit ourselves to observe that equation (3.26b) is an integrable equation according to the algebraic entropy test [7, 41, 43]. Indeed, computing the degree of the iterates of equation (3.26b) by using the library `ae2d.py` [21, 22], we obtain the following sequence:

$$1, 2, 4, 7, 11, 16, 22, 29, 37, 46, 56, 67, 79, \dots \tag{3.35}$$

The sequence (3.35) is fitted by the following generating function:

$$g(z) = \frac{z^2 - z + 1}{(1 - z)^3}. \tag{3.36}$$

Since the generating function (3.36) has only one pole in $z_0 = 1$ and it lies on the unit circle, equation (3.26b) is integrable according to the algebraic entropy criterion. Moreover, due to the presence of $(z - 1)^3$ in the denominator of the generating function (3.36) we have that growth of equation (3.26b) is *quadratic*. According to the classification of discrete equations using algebraic entropy [26] we have that equation (3.26b) is supposed to be genuinely integrable and not linearizable.

Equation (E.5'): To equation (E.5') with $a \neq 0$ corresponds one quad-equation:

$$\begin{aligned} & \left(u_{n,m} - u_{n,m+1} + \frac{b_4}{c_3}\right) \left(u_{n+1,m} - u_{n+1,m+1} + \frac{b_4}{c_3}\right) \\ & + \frac{b_4 + d_3}{c_3} \left(u_{n,m+1} - u_{n+1,m} - \frac{b_4}{c_3} - a\right) = 0, \quad b_4 + d_3 \neq 0. \end{aligned} \tag{3.37}$$

When $a = 0$, to equation (E.5') corresponds two quad-equations. One of them is just (3.37) with $a = 0$, and the second one is:

$$\begin{aligned} & \left(u_{n,m} + u_{n,m+1} - \frac{b_4}{c_3}\right) \left(u_{n+1,m} + u_{n+1,m+1} - \frac{b_4}{c_3}\right) \\ & + \frac{b_4 - d_3}{c_3} \left(u_{n,m+1} + u_{n+1,m} - \frac{b_4}{c_3}\right) = 0, \quad b_4 - d_3 \neq 0, \end{aligned} \tag{3.38}$$

but in this case we have $b = c = 0$.

For any a equation (3.37) is transformed into (T4) of [32] by the following non-autonomous linear point transformation:

$$u_{n,m} = \widehat{u}_{n,m} \frac{b_4 + d_3}{c_3} - n \left(\frac{b_4 + d_3}{2c_3} + a\right) + m \frac{b_4 - d_3}{2c_3}, \tag{3.39}$$

where $\widehat{u}_{n,m}$ satisfies (T4):

$$\left(\widehat{u}_{n,m} - \widehat{u}_{n,m+1} + \frac{1}{2}\right) \left(\widehat{u}_{n+1,m} - \widehat{u}_{n+1,m+1} + \frac{1}{2}\right) + \widehat{u}_{n,m+1} - \widehat{u}_{n+1,m} = 0. \tag{T4}$$

We recall that equation (T4) is a sGT equation, with two nontrivial three-point generalized symmetries [32].

In the same way equation (3.38) is another sGT equation which is transformed by linear transformation of $u_{n,m}$ into the special form:

$$(u_{n,m} + u_{n,m+1} + 1)(u_{n+1,m} + u_{n+1,m+1} + 1) = 2(u_{n,m+1} + u_{n+1,m} + 1). \tag{3.40}$$

This equation is a new sGT quad-equation which possesses five-point autonomous generalized symmetries:

$$\frac{du_{n,m}}{d\theta_1} = (u_{n+2,m} - u_{n-2,m})(u_{n+1,m} - u_{n,m})(u_{n,m} - u_{n-1,m}), \tag{3.41a}$$

$$\frac{du_{n,m}}{d\theta_2} = \frac{[(u_{n,m+1} + u_{n,m})^2 - 1][(u_{n,m} + u_{n,m-1})^2 - 1]}{u_{n,m+1} + 2u_{n,m} + u_{n,m-1}} (T_m - \text{Id}) \left(\frac{1}{U_{n,m}}\right), \tag{3.41b}$$

where

$$U_{n,m} = (u_{n,m+1} + u_{n,m-1})(u_{n,m} + u_{n,m-2}) + 2(u_{n,m+1}u_{n,m-1} + u_{n,m}u_{n,m-2} + 1). \tag{3.42}$$

We note that equation (3.38) possesses also the following non-autonomous three-point generalized symmetry in the n -direction:

$$\frac{du_{n,m}}{d\theta_3} = (-1)^m (u_{n+1,m} - u_{n,m})(u_{n,m} - u_{n-1,m}), \tag{3.43}$$

and the non-autonomous point symmetry

$$\frac{du_{n,m}}{d\theta_4} = (-1)^m. \quad (3.44)$$

On the other side it is possible to prove by direct computation using the method of [20, 31, 32] that no three-point generalized symmetry exists in the m -direction. Hence the generalized symmetry (3.41b) is the lowest order generalized symmetry of equation (3.38) in the m -direction.

We remark that equation (3.38) through the following nontrivial non-invertible transformation:

$$\widehat{u}_{n,m} = u_{n,m+1} + u_{n,m}, \quad (3.45)$$

is transformed into (3.26b), i.e. we have that $\widehat{u}_{n,m}$ satisfies equation (3.26b). The symmetries (3.41a,3.41b) are transformed into symmetries (3.33a,3.33b) too, but here $\theta_1 = 4t_1$, $\theta_2 = 2t_2$. However, in case of (3.41a), we can check this fact only by using discrete equation (3.40). This means that only common solutions of (3.41a) and (3.40) are transformed into solutions of (3.33a).

Finally, we underline that equation (3.41b) is a particular case of known example of integrable five-point differential-difference equation presented in [3]. According to this remark we have that equation (3.33b) is a nontrivial modification of a known example.

Equation (E.6'): To equation (E.5) correspond only degenerate quad-equations in the sense of condition (2.15) and the trivial linearizable equation (1.8a).

Equation (E.7'): Equation (E.7') depends on the parameter a . If $a \neq 0$ then it can always be rescaled to 1, hence we can consider two independent cases: $a = 1$ and $a = 0$. However, in both cases to equation (E.7') corresponds only degenerate quad-equations in the sense of condition (2.15).

Equation (E.8'): To equation (E.5) correspond only degenerate quad-equations in the sense of condition (2.15) and the trivial linearizable equation (1.8a).

3.4. List 4

Equation (E.9) To equation (E.9) correspond only degenerate quad-equations in the sense of condition (2.15).

Equation (E.10) To equation (E.10) correspond only degenerate quad-equations in the sense of condition (2.15) and equations of the form of the linear discrete wave equations (3.3).

Equation (E.13'): To equation (E.13') correspond two nontrivial equations:

$$u_{n+1,m+1}(u_{n,m} - u_{n,m+1}) - u_{n+1,m}(u_{n,m} + u_{n,m+1}) + 2 = 0, \quad (3.46a)$$

$$u_{n+1,m}(u_{n,m+1} - u_{n,m}) - u_{n+1,m+1}(u_{n,m+1} + u_{n,m}) + 2 = 0. \quad (3.46b)$$

Equation (3.46b) can be reduced to equation (3.46a) using the transformation (T*.1). Using the scaling $u_{n,m} = \sqrt{2}\widehat{u}_{n,m}$ equation (3.46a) is reduced to:

$$\widehat{u}_{n+1,m+1}(\widehat{u}_{n,m} - \widehat{u}_{n,m+1}) - \widehat{u}_{n+1,m}(\widehat{u}_{n,m} + \widehat{u}_{n,m+1}) + 1 = 0, \quad (3.47)$$

which is a known sGT equation. Namely equation (3.47) is equation (39) of [14], see [12] for more details. Indeed, from [14] we know that actually equation (3.46a) has a more complicated symmetry

structure in the n direction and its general five-point symmetry is the following *non-autonomous* one:

$$\frac{du_{n,m}}{d\mathcal{E}_1} = (u_{n+1,m}u_{n,m} - 1)(u_{n,m}u_{n-1,m} - 1)(a_n u_{n+2,m} - a_{n-1} u_{n-2,m}), \quad (3.48)$$

where a_n is a two-periodic function, i.e. a solution of the difference equation $a_{n+2} = a_n$. We note that $a_n = 1$ is a solution which corresponds to equation (E.13'). Moreover equation (3.46a) has a non-autonomous three-point symmetry in the m direction:

$$\frac{du_{n,m}}{d\mathcal{E}_2} = (-1)^n \frac{u_{n,m+1}u_{n,m-1} + u_{n,m}^2}{u_{n,m+1} + u_{n,m-1}}. \quad (3.49)$$

No autonomous three-point symmetry exists for equation (3.46a).

3.5. List 5

Equation (E.11) To equation (E.11) correspond only degenerate quad-equations in the sense of condition (2.15).

Equation (E.12) Equation (E.12) depends on two parameters a and b . If $a \neq 0$ it can always be scaled to $a = 1$, so for this equations we get two different cases: when $a = 1$ and when $a = 0$.

Case $a = 1$: For this value of the parameter only degenerate or linear equations arise. In particular we have a discrete wave equation (3.3).

Case $a = 0$: For this value of the parameter only degenerate or linear equations arise. In particular we have a discrete wave equation (3.3).

Equation (E.13) Equation (E.13) depends only on the parameter a . If $a \neq 0$ it can be scaled to $a = 1$ always. Therefore we have to consider the cases $a = 1$ and $a = 0$.

Case $a = 1$: For this value of the parameter there are two nontrivial equations:

$$u_{n+1,m+1}(u_{n,m+1} + u_{n,m} + 1) + u_{n,m}(u_{n+1,m} + 1) = 0, \quad (3.50a)$$

$$u_{n,m+1}(u_{n+1,m+1} + u_{n+1,m} + 1) + u_{n+1,m}(u_{n,m} + 1) = 0. \quad (3.50b)$$

Up to transformation (T*.1) we have that equation (3.50b) reduces to equation (3.50a). Therefore we have only one independent equation. Equation (3.50a) is a sGT equation and appeared as formula (58) in [33].

Case $a = 0$: For this value of the parameter there are six nontrivial equations:

$$(u_{n,m} + u_{n,m+1})u_{n+1,m+1} + u_{n,m}u_{n+1,m} = 0, \quad (3.51a)$$

$$(u_{n,m} + u_{n,m+1})u_{n+1,m} + u_{n,m+1}u_{n+1,m+1} = 0, \quad (3.51b)$$

$$\left(\frac{1+i\sqrt{3}}{2}u_{n,m} + \frac{1-i\sqrt{3}}{2}u_{n,m+1} \right) u_{n+1,m+1} - u_{n,m}u_{n+1,m} = 0, \quad (3.51c)$$

$$\left(\frac{1-i\sqrt{3}}{2}u_{n+1,m} + \frac{1+i\sqrt{3}}{2}u_{n+1,m+1} \right) u_{n,m+1} - u_{n,m}u_{n+1,m} = 0, \quad (3.51d)$$

$$\left(\frac{1+i\sqrt{3}}{2}u_{n+1,m} + \frac{1-i\sqrt{3}}{2}u_{n+1,m+1} \right) u_{n,m+1} - u_{n,m}u_{n+1,m} = 0, \quad (3.51e)$$

$$\left(\frac{1-i\sqrt{3}}{2}u_{n,m} + \frac{1+i\sqrt{3}}{2}u_{n,m+1} \right) u_{n+1,m+1} - u_{n,m}u_{n+1,m} = 0. \quad (3.51f)$$

Up to transformations (T.1), (T.2) and (T*1) we have that all the equations in (3.51) can be reduced to (3.51a). The precise relationship between these equations is illustrated in Figure 2. Equation (3.51a) is a known LT equation and it can be identified with (6) with $a_2 = 1$ of List 4 of [14]. Its first integrals are:

$$W_1 = u_{n+1,m}u_{n,m}u_{n-1,m}, \quad W_2 = d^{-n} \frac{u_{n,m} - du_{n,m+1}}{du_{n,m} - u_{n,m+1}}, \quad d \equiv -\frac{1+i\sqrt{3}}{2}. \quad (3.52)$$

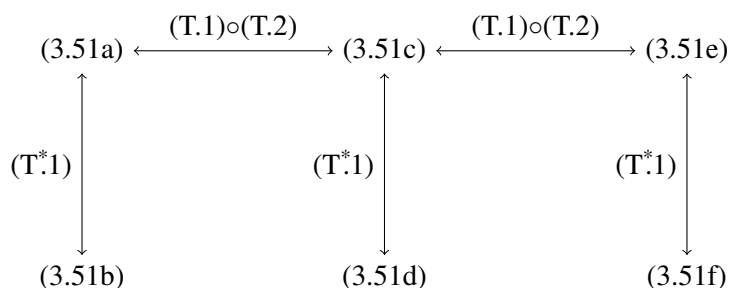


Figure 2: The relationship between the equations (3.51). By (T.1)∘(T.2) we mean the composition of the two transformations.

Equation (E.14): To equation (E.14) correspond only degenerate quad-equations in the sense of condition (2.15).

Equation (1.7): To equation (1.7) correspond three nontrivial quad-equations^a:

$$\left(u_{n+1,m+1} + \frac{c_4}{a_4} \right) \left(u_{n,m+1} + \frac{c_4}{a_4} \right) - \left(u_{n+1,m} - \frac{d_3}{a_4} \right) \left(u_{n,m} - \frac{d_3}{a_4} \right) = 0, \quad a_4 \neq 0, \quad (3.53a)$$

$$\left(u_{n+1,m+1} + u_{n+1,m} + \frac{c_4}{c_3} \right) \left(u_{n,m+1} + u_{n,m} + \frac{c_4}{c_3} \right) = \frac{c_4^2 - c_3 d_4}{c_3^2}, \quad c_3 \neq 0, c_4^2 - c_3 d_4 \neq 0, \quad (3.53b)$$

$$\left(u_{n+1,m+1} - u_{n+1,m} - \frac{c_4}{c_3} \right) \left(u_{n,m+1} - u_{n,m} - \frac{c_4}{c_3} \right) = \frac{c_4^2 + c_3 d_4}{c_3^2}, \quad c_3 \neq 0, c_4^2 + c_3 d_4 \neq 0. \quad (3.53c)$$

^aBy direct computations several sub-cases of equations (3.53) are found. We omit them since they are not independent cases.

Equation (3.53a) is a LT equation. Using the transformation

$$\widehat{u}_{n,m} = u_{n,m} + \frac{c_4}{a_4} \tag{3.54}$$

we can bring it into equation (1) of List 4 of [14]. Its first integrals are given by:

$$W_1 = (u_{n+2,m} - u_{n,m})(u_{n+1,m} - u_{n-1,m}), \quad W_2 = \left(\frac{a_4 u_{n,m+1} + c_4}{a_4 u_{n,m} - d_3} \right)^{(-1)^n}. \tag{3.55}$$

Equation (3.53b) under the non-autonomous transformation

$$\widehat{u}_{n,m} = (-1)^m \left(u_{n,m} + \frac{c_4}{2c_3} \right), \tag{3.56}$$

is mapped into a particular case of equation (3.53c). Equation (3.53c) itself is of LT. It can be identified with equation (5) of List 4 of [14]. This implies that its first integrals are given by:

$$W_1 = u_{n+1,m} - u_{n-1,m}, \quad W_2 = (-1)^n \frac{u_{n,m+1} - u_{n,m} + \kappa - \delta}{u_{n,m+1} - u_{n,m} + \kappa + \delta}, \quad \kappa = \frac{c_4}{c_3}, \quad \delta = \frac{c_4^2 + c_3 d_4}{c_3^2}. \tag{3.57}$$

Equation (E.9’): To equation (E.9’) correspond only degenerate quad-equations in the sense of condition (2.15) and the trivial linearizable equation (1.8a).

Equation (E.10’): Equation (E.10’) depends on the parameter a . If $a \neq 0$ then it can always be rescaled to 1, hence we can consider two independent cases: $a = 1$ and $a = 0$. However in both cases to equation (E.9’) correspond only degenerate quad-equations in the sense of condition (2.15) or linear equations of the form of the discrete wave equation (3.3).

Equation (E.11’): Equation (E.11’) depends on the parameter a . If $a \neq 0$ then it can always be rescaled to 1, hence we can consider two independent cases: $a = 1$ and $a = 0$.

To equation (E.11’) when $a = 1$ correspond four nontrivial equations:

$$u_{n,m}u_{n+1,m} + u_{n,m+1}u_{n+1,m} + u_{n,m+1}u_{n+1,m+1} = 1, \tag{3.58a}$$

$$u_{n,m}u_{n+1,m} - u_{n,m+1}u_{n+1,m} + u_{n,m+1}u_{n+1,m+1} = 1, \tag{3.58b}$$

$$u_{n,m}u_{n+1,m} + u_{n,m}u_{n+1,m+1} + u_{n,m+1}u_{n+1,m+1} = 1, \tag{3.58c}$$

$$u_{n,m}u_{n+1,m} - u_{n,m}u_{n+1,m+1} + u_{n,m+1}u_{n+1,m+1} = 1. \tag{3.58d}$$

Up to transformations (T*.1,T.1) we have that all the equations in (3.58) can be reduced to (3.58a), see Figure 3. Equation (3.58a) is a known equation. It was introduced in [25], while in [10] it was proved that it is a sGT equation^b.

To equation (E.11’) when $a = 0$ correspond twelve nontrivial equations. Six of them are those of presented in formula (3.51), but we also have the following six other equations:

$$u_{n,m+1}u_{n+1,m+1} + u_{n+1,m}(u_{n,m} - u_{n,m+1}) = 0, \tag{3.59a}$$

$$(u_{n,m} - u_{n,m+1})u_{n+1,m+1} - u_{n,m}u_{n+1,m} = 0, \tag{3.59b}$$

^bTherein it is given by equation (1.9).

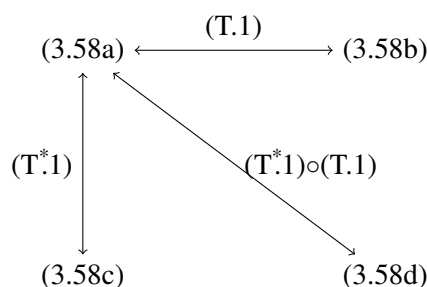


Figure 3: The relationship between the equations (3.58). By $(T^*.1) \circ (T.1)$ we mean the composition of the two transformations.

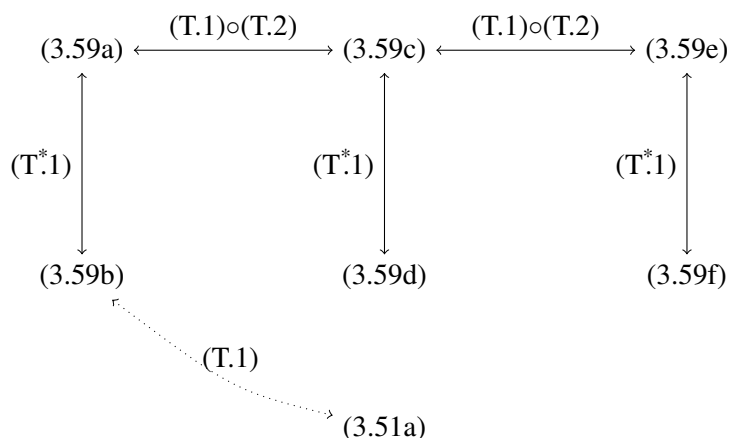


Figure 4: The relationship between the equations (3.59) and (3.51a). By $(T.1) \circ (T.2)$ we mean the composition of the two transformations.

$$\left(\frac{1-i\sqrt{3}}{2} u_{n,m+1} + u_{n,m} \right) u_{n+1,m} - \frac{1+i\sqrt{3}}{2} u_{n,m+1} u_{n+1,m+1} = 0, \quad (3.59c)$$

$$\left(u_{n,m+1} + \frac{1-i\sqrt{3}}{2} u_{n,m} \right) u_{n+1,m+1} - \frac{1+i\sqrt{3}}{2} u_{n+1,m} u_{n,m} = 0, \quad (3.59d)$$

$$\left(\frac{1+i\sqrt{3}}{2} u_{n,m+1} + u_{n,m} \right) u_{n+1,m} - \frac{1-i\sqrt{3}}{2} u_{n,m+1} u_{n+1,m+1} = 0, \quad (3.59e)$$

$$\left(\frac{1-i\sqrt{3}}{2} u_{n,m+1} + u_{n,m} \right) u_{n+1,m+1} + \frac{1+i\sqrt{3}}{2} u_{n+1,m} u_{n,m} = 0. \quad (3.59f)$$

Using the transformations $(T^*.1, T.1, T.2)$ we have that the only independent equation is (3.51a). See Figure 4 for the details about the needed transformations. As discussed above equation (3.51a) is a LT equation with known first integrals (3.52).

Equation (E.12'): To equation (E.12') correspond only degenerate quad-equations in the sense of condition (2.15) or the trivial linearizable equation (1.8a).

3.6. List 6

Equation (E.15): To equation (E.15) correspond only degenerate quad-equations in the sense of condition (2.15).

Equation (E.16): To equation (E.16) correspond only degenerate quad-equations in the sense of condition (2.15).

Equation (E.17): Equation (E.17) is not rational, so we are not going to consider it as we are dealing with multi-affine quad-equations (2.3). It can be remarked that equation (E.17) has as a rational form, but it is quadratic in $u_{k\pm 2}$, so it is outside the main class of differential-difference equations.

Equation (E.14'): To equation (E.14') correspond only degenerate quad-equations in the sense of condition (2.15) or equations of the form of the linear discrete wave equation (3.3).

4. Summary and outlook

In this paper we have constructed all the possible autonomous quad-equations (1.1) admitting, as generalized symmetries, some known five-point differential-difference equations belonging to a class recently classified in [18, 19].

In section 2 we gave a general result on all the quad-equations admitting a generalized symmetry of the form (2.2), resulting in the simplified form (2.3). Then in section 3 we considered all the explicit examples of five-point differential-difference equations and found all the corresponding quad-equations. We discussed the properties of these equations, highlighting which are LT and which are sGT equations. We present a summary of these properties in Table 1. Referring to the table, we have that the majority of the differential-difference equations of the form (1.3) as classified in [18, 19] give rise to degenerate or trivial quad-equations. However eight of them give also rise to LT equations. Most of these LT equations were known, but we also obtained two new LT equations, namely equations (3.12b) and (3.26a). Moreover, as expected, sGT equations are even rarer, appearing only in in four examples. Among these sGT equations there are two new examples, namely equations (3.26b) and (3.40). In this paper we limited ourselves to present the result of the algebraic entropy test for equation (3.26b), which suggests integrability. More details and generalizations, including $L - A$ pairs, on equation (3.26b) can be found in [17].

It should be remarked that in [35, 39, 40] some more general completely discrete equations consistent with the INB-like five-point differential-differential equations were studied. In the mentioned papers a natural discretization based on the Bäcklund transformation was presented and it had the form:

$$Q(u_{n,m+1}, u_{n+1,m+1}, u_{n+2,m+1}, u_{n,m}, u_{n+1,m}, u_{n+2,m}) = 0. \quad (4.1)$$

This is an equation on two adjacent quads. Quad equations (1.1) like we consider in this paper are also determined by Bäcklund transformations, but of a quite different form. Quad equations (1.1) might be special reductions of equations of the form (4.1) as well of the corresponding Bäcklund transformations.

Class I		Class 2	
Equations	Properties	Equations	Properties
(E.1)	Trivial	(E.1')	Trivial
(E.2)	LT	(E.2')	LT*
(E.3)	Trivial	(E.3')	Trivial
(E.4)	Trivial	(E.4', $a = 1$)	sGT*, LT*
(E.5, $a = 1$)	Trivial	(E.4', $a = 0$)	Trivial
(E.5, $a = 0$)	Trivial	(E.5')	sGT*
(E.6)	LT	(E.6')	Trivial
(E.7, $a \neq 0, \pm 1, b = d = 0, c \neq 0$)	Trivial	(E.7', $a = 1$)	Trivial
(E.7, $a = 1, b = c = d = 0$)	LT	(E.7', $a = 0$)	Trivial
(E.7, $a = -1, b = 1, c = 0$)	Trivial	(E.8')	Trivial
(E.7, $a = -1, b = 0$)	LT	(E.9')	Trivial
(E.8)	Trivial	(E.10', $a = 1$)	Trivial
(E.9)	Trivial	(E.10', $a = 0$)	Trivial
(E.10)	Trivial	(E.11', $a = 1$)	sGT
(E.11)	Trivial	(E.11', $a = 0$)	LT
(E.12, $a = 1$)	Trivial	(E.12')	Trivial
(E.12, $a = 0$)	Trivial	(E.13')	sGT
(E.13, $a = 1$)	sGT	(E.14')	Trivial
(E.13, $a = 0$)	LT		
(E.14)	Trivial		
(E.15)	Trivial		
(E.16)	Trivial		
(E.17)	Not rational		
(1.7)	LT		

Table 1: Summary of the properties of the corresponding discrete quad-equations. With * we underline the presence of new quad-equations.

As we remarked at the beginning of this section, in this paper we only dealt with autonomous differential-difference equations and autonomous quad-equations. We are now working on weakening this condition for the fully discrete equations to present a classification of non-autonomous quad-equations admitting the differential-difference equations found in [18, 19] as five-point generalized symmetries. This classification can be performed with the method presented in [15] and applied to some known autonomous and non-autonomous three-point differential-difference equations of Volterra and Toda type.

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