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SYMMETRY APPROACH TO THE INTEGRABILITY PROBLEM[1]

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We review the results of the twenty-year development of the symmetry approach to classifying integrable models in mathematical physics. The generalized Toda chains and the related equations of the nonlinear Schrödinger type, discrete transformations, and hyperbolic systems are central in this approach. Moreover, we consider equations of the Painlevé type, master symmetries, and the problem of integrability criteria for (2+1)-dimensional models. We present the list of canonical forms for (1+1)-dimensional integrable systems. We elaborate the effective tests for integrability and the algorithms for reduction to the canonical form.

Contents

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1. Introduction

Classifying integrable equations, we obtain key equations of a given type, whose list is itself interesting. When considering a list, one often wants to change the integrability definition to a "more reliable" or "more elementary" one, but this results in an additional verification, not a change, of the list. We discuss this using the example of the key solitonic equations.

We begin with the known list of six second-order ordinary differential equations (ODEs) that satisfy the Painlevé test,

$$y'' = a(x,y){y'}^2 + b(x,y)y' + c(x,y).$$

First, we refer to this list in connection with the discrete symmetry theory developed in [1–3] for the spectral problem

$$\Psi_{xx} = U(x,\lambda)\Psi \tag{1.1}$$

with the potential U of the second order in λ. This theory, whose simplest variant results in the Toda chain, allows reformulating Eqs. (P_3)–(P_6) (see below) as the conditions for the invariance of the spectral problem w.r.t. the chain of Darboux transformations (Sec. 3). Second, we demonstrate that Eqs. (P_1)–(P_5) can be considered as the stationary equations for the master symmetries of the Korteweg–de Vries (KdV) and nonlinear Schrödinger (NS) equations (Sec. 2). We recall that the equations determining the symmetries and master symmetries of the evolution equation $u_t = G$ can be formally written in the form

$$[D_{t'}, D_t] = 0, \qquad \big[[D_\tau, D_t], D_t\big] = 0, \tag{1.2}$$

where the evolutionary differentiation D_t pertains to the equation itself and $D_{t'}$ and D_τ to its symmetry and master symmetry. The corresponding infinitesimal symmetries of spectral problem (1.1) are determined by the common formula

$$D_\tau(\Psi) = A(x,\lambda)\Psi_x + B(x,\lambda)\Psi,$$

where $D_\tau = \partial_\tau + \omega(\lambda)\partial_\lambda$ and $A(x,\lambda) = a_1(x)\lambda + a_0(x)$. In other words, we obtain the KdV or NS equations for $\omega = 0$ (the isospectral case) and their master symmetries for $\omega \neq 0$ (cf. [4–7]).

Master symmetries for many equations (e.g., for the KdV, NS, and Toda chain equations) are nonlocal. However, the Landau–Lifshitz model, which is a universal equation in the NS class, has a local master symmetry [8]. In Sec. 6, we present the general locality criterion and find local master symmetries for the universal equations of other classes.

In this paper, we introduce the new notion of B-integrable equations for which there exists a change of variables relating the equation to its master symmetry. In the (1+1)-dimensional case, this is not very useful, because it leads to Burgers-type equations. In contrast, master symmetries of (1+2)-dimensional equations are often just trivial deformations. We observe this in Sec. 7 in the examples of Kadomtsev–Petviashvili and Davey–Stewartson equations. This property may be useful for classifying two-dimensional integrable equations. We note that definition (1.2) of symmetries and master symmetries does not assume the existence of the L–A pair.

Except for infinitesimal symmetries, a useful classification criterion is discrete symmetries (or the Bäcklund–Darboux transformations). In Secs. 4 and 5, we use this theory to classify integrable generalizations of the Toda chain. Their connection with the equations of the NS type, with hyperbolic systems of the Pohlmeyer–Lund–Regge type, and with analogues of the Ablowitz–Ladik chain then becomes obvious. The Sklyanin chain also appears naturally within this approach.

Twenty years of classifying integrable evolution equations with one spatial variable has resulted in the list of integrable equations, in the formulation of effective integrability tests, and in the development of algorithms for reducing equations to the canonical form (see [9–14]). In Secs. 4 and 5 and in the appendices, we present the lists of integrable equations from the most interesting classes (the classes of KdV, NS, Boussinesq, Toda, and Volterra equations) and the corresponding integrability conditions (necessary conditions for the existence of higher symmetries and conservation laws). We also briefly discuss the general problem of classifying scalar evolution equations of an arbitrary order in Appendix 2.

Notation. Because equations on a lattice are important for us, we use a special notation for them, writing u, u_1, and u_{-1} instead of u_n, u_{n+1}, and u_{n-1}; for double discrete equations, we write u, $u_{1,1}$, and $u_{0,-1}$ instead of $u_{m,n}$, $u_{m+1,n+1}$, and $u_{m,n-1}$, etc., unless it can lead to a discrepancy as it does for formulas with an explicit dependence on a discrete variable and for summation formulas. An analogue of the total derivative operator D_x is the shift operator $T_n\colon u_n \mapsto u_{n+1}$.

We label equations by Latin letters with indices, e.g., we use (P_i), $i = 1, 2, \ldots$, for the Painlevé equations and refer to the complete list as (P).

2. Stationary solutions of deformations of integrable equations

In this section, we discuss the relation between deformations[4] of integrable equations and the Painlevé equations

$$y'' = 6y^2 + x, \tag{P$_1$}$$

$$y'' = 2y^3 + xy + \alpha, \tag{P$_2$}$$

$$y'' = \frac{(y')^2}{y} - \frac{y'}{x} + \frac{1}{x}(\alpha y^2 + \beta) + \gamma y^3 + \frac{\delta}{y}, \tag{P$_3$}$$

$$y'' = \frac{(y')^2}{2y} + \frac{3}{2}y^3 + 4xy^2 + 2(x^2 - \alpha)y + \frac{\beta}{y}, \tag{P$_4$}$$

$$y'' = \left(\frac{1}{2y} + \frac{1}{y-1}\right)(y')^2 - \frac{y'}{x} + \frac{(y-1)^2}{x^2}\left(\alpha y + \frac{\beta}{y}\right) + \gamma\frac{y}{x} + \delta\frac{y(y+1)}{y-1}, \tag{P$_5$}$$

$$\begin{aligned} y'' = {} & \frac{1}{2}\left(\frac{1}{y} + \frac{1}{y-1} + \frac{1}{y-x}\right)(y')^2 - \left(\frac{1}{x} + \frac{1}{x-1} + \frac{1}{y-x}\right)y' + {} \\ & + \frac{y(y-1)(y-x)}{x^2(x-1)^2}\left(\alpha + \beta\frac{x}{y^2} + \gamma\frac{x-1}{(y-1)^2} + \delta\frac{x(x-1)}{(y-x)^2}\right). \end{aligned} \tag{P$_6$}$$

We now demonstrate that stationary solutions of the deformations of the NS equation,

$$\begin{aligned} u_t &= c_1\bigl(x(u_{xx} + 2u^2v) + 2u_x + 2uD_x^{-1}(uv)\bigr) + c_2(u_{xx} + 2u^2v) + c_3(xu_x + u) + c_4xu + c_5u_x, \\ v_t &= -c_1\bigl(x(v_{xx} + 2uv^2) + 2v_x + 2vD_x^{-1}(uv)\bigr) - c_2(v_{xx} + 2uv^2) + c_3(xv_x + v) - c_4xv + c_5v_x, \end{aligned} \tag{2.1}$$

and of the KdV equation,

$$q_t = c_1\bigl(x(q_{xxx} - 6qq_x) + 4q_{xx} - 8q^2 - 2q_xD_x^{-1}(q)\bigr) + c_2(q_{xxx} - 6qq_x) + c_3(xq_x + 2q) + c_4, \tag{2.2}$$

result in Eqs. (P_1)–(P_5) depending on the choice of the coefficients c_i. For $c_1 = 0$ and $c_2 \neq 0$, these deformations are trivial because they can be reduced by pointwise transformations (scaling, shift, and Galileo transformations) to the initial NS or KdV equation. Their stationary solutions then become automodel solutions either of the NS equation (described by Eqs. (P_2) and (P_4) [15]) or of the KdV equation ((P_1) and (P_2) [6, 16]). In the general case where $c_1 \neq 0$, the problem can be therefore considered the generalization of the

[4] We use the notation in [5, 7].

well-known problem of enumerating automodel reductions. We then obtain additional equations (P_3) and (P_5) with arbitrary parameters for the NS equation and the same equations with degenerate parameters for the KdV equation.

These results can be uniformly formulated as follows. The compatibility condition for the linear problems

$$\Psi_{xx} = U(x,\lambda)\Psi, \qquad D_t(\Psi) = A(x,\lambda)\Psi_x + B(x,\lambda)\Psi \tag{2.3}$$

results in the relation $2B_x + A_{xx} = 0$ and the equation

$$2D_t(U) = 4UA_x + 2U_xA - A_{xxx}. \tag{2.4}$$

We consider a potential U of at most the second order in λ; the case $U = q - \lambda$ corresponds to the Schrödinger spectral problem,

$$\psi_{xx} = (q - \lambda)\psi, \tag{2.5}$$

and the KdV hierarchy, while the case $U = q - 2\lambda z + \lambda^2$ corresponds to the spectral problem

$$\psi_{xx} + (z - \lambda)\psi_x + p\psi = 0, \tag{2.6}$$

which is gauge equivalent to the spectral Zakharov–Shabat problem, and the hierarchy related to the NS hierarchy by a differential substitution. If the spectral parameter depends on t, i.e.,

$$D_t = \partial_t + \omega(\lambda)\partial_\lambda,$$

then Eq. (2.4) determines the deformations of these hierarchies. These deformations are classical symmetries of the Galileo transformation and dilation and master symmetries. Deformed equations are nonlocal in general, i.e., they can contain the integration over x.

Stationary solutions of deformations (2.4) are ODEs that admit a representation of the form

$$2\omega U_\lambda = 4UA_x + 2U_xA - A_{xxx}. \tag{2.7}$$

It is well known that stationary equations for the KdV and NS hierarchies are Liouville integrable and determine finite-gap and solitonic solutions [17–20] because a set of first integrals

$$4UA^2 + A_x^2 - 2AA_{xx} = \Lambda(\lambda) \tag{2.8}$$

appears at $\omega = 0$, which allows reducing the order of the equation and eventually expressing the solutions in theta functions. Some of these integrals are also preserved in the case $\omega \neq 0$ because the relation

$$4\omega U_\lambda A = D_x(4UA^2 + A_x^2 - 2AA_{xx})$$

implies that if λ_0 is a zero of order r of the function $\omega(\lambda)$, then the quantities

$$\frac{d^k}{d\lambda^k}(4UA^2 + A_x^2 - 2AA_{xx})|_{\lambda=\lambda_0}, \quad k = 0, \dots, r-1, \tag{2.9}$$

are constants. However, the number of these first integrals is insufficient to ensure Liouville integrability.

In the simplest case $A = a_1\lambda + a$, we reproduce the Garnier result, presented in Table 1. There and in what follows, the star means that a parameter can be arbitrary, and the unity means that a parameter can be arbitrary but must be nonzero (we recall that the parameter values in Eqs. (P_3) and (P_5) can be changed by scaling). Comparing the first and second rows in Table 1, we note that a potential of the second order in λ results in the Painlevé equations with the arbitrary parameters and a linear potential results in degenerate cases.

Table 1

U \ $\omega(\lambda)$	1	λ	λ^2	$\lambda(\lambda-\lambda_0)$
$q-2\lambda z+\lambda^2$	(P_2)	(P_4)	$(P_3(*,\,*,1,*))$	$(P_3(*,\,*,*,1))$
$q-\lambda$	(P_1)	(P_{34})	$(P_3(1,\,*,0,*))$	$(P_5(*,\,*,1,0))$

Auxiliary linear problems for the Painlevé equations

Spectral problem (2.3) with a potential of the second order in λ therefore suffices for constructing almost the entire list of Painlevé equations. In Fuchs representation (2.7) for Eq. (P_6), the potential U and the function A are rational in λ.

Theorem 1. *If $A = a_1\lambda + a$ and $\omega = w_2\lambda^2 + w_1\lambda + w_0$, then Eq.* (2.7) *with a potential U of the second order in λ results in Painlevé equations* (P_1)–(P_5) *according to Table* 1.

Proof. The proof is a direct consideration of eight possible cases. We first consider a potential of the type $U = q - 2\lambda z + \lambda^2$; Eq. (2.7) is then equivalent to the system

$$\begin{aligned}
&a_1' = w_2, \qquad a' = (za_1)' + w_1,\\
&4za' + 2z'a - 2qa_1' - q'a_1 - 2w_1 z + 2w_0 = 0,\\
&a''' = 4qa' + 2q'a + 4w_0 z
\end{aligned} \tag{2.10}$$

(here and hereafter, the prime denotes the derivative w.r.t. x). Analyzing its solutions, we obtain the following results. The choice

$$\omega = 1, \qquad U = \frac{3}{4}y^2 + \frac{x}{4} - \lambda y + \lambda^2, \qquad A = 8\lambda + 4y$$

results in Eq. (P_2). In other cases, we set $w_0 = 0$ without lack of generality; the potential U is expressed through A by the formulas

$$q = \frac{2aa'' - (a')^2 - c_1}{4a^2}, \qquad z = \frac{1}{a_1}(a - w_1 x + c_2).$$

To decrease the order of the equation w.r.t. the function a, we use the additional first integral of form (2.9). Choosing

$$\omega = -\lambda, \qquad a_1 = 1, \qquad a = \frac{y}{2}, \qquad c_2 = 0,$$

we obtain Eq. (P_4) with the arbitrary parameters α and $\beta = 2c_1$. Choosing

$$\omega = w_2\lambda^2, \qquad a_1 = w_2 x$$

and substituting $a(x) = \tilde{x}y(\tilde{x})$ and $x = \tilde{x}^2$, we obtain Eq. (P_3) with the arbitrary parameters $\alpha = 16c_2/w_2^2$, β, and $\delta = 4c_1$ and the nonzero parameter $\gamma = 16/w_2^2$. Eventually, choosing

$$\omega = \lambda(\lambda - \lambda_0), \qquad a_1 = x, \qquad a = -\frac{\lambda_0 xy}{y-1},$$

we obtain Eq. (P_5) with the nonzero parameter $\delta = -2\lambda_0^2$ and the arbitrary parameters α, $\beta = -c_1/\delta$, and $\gamma = 4\lambda_0 c_2$.

We thus obtain the first row in Table 1. Further, in the case of the Schrödinger operator, we have $U = q - \lambda$, and Eq. (2.7) becomes equivalent to the system

$$2a_1' = w_2, \qquad 2a' = 2qa_1' + q'a_1 + w_1, \qquad a''' = 4qa' + 2q'a + 2w_0. \tag{2.11}$$

Equation (P_1) is obtained for $w_2 = w_1 = 0$,

$$\omega = 1, \qquad U = 2y - \lambda, \qquad A = 2(\lambda + y).$$

In the other cases, taking the shift of λ into account, we can set $w_0 = 0$, which allows integrating the last equation of system (2.11) once more and reducing it to the form

$$4a' = 2w_2q + (w_2x + c)q' + 2w_1, \qquad 2aa'' - (a')^2 = 4qa^2 + c_1. \tag{2.12}$$

We can integrate the first equation for $w_2 = 0$ to obtain $4a = cq + 2w_1x + c_2$; making the linear transformations

$$\omega = \lambda, \qquad U = u - \frac{x}{2} - \lambda, \qquad A = 2\lambda + u,$$

we obtain

$$u'' = \frac{(u')^2 - k^2}{2u} + 2u^2 - xu. \tag{P_{34}}$$

If $w_2 \neq 0$, then we can set $w_2 = 1$ without lack of generality. To express q through a, we use the second equation in system (2.12), which gives

$$U = \frac{2aa'' - (a')^2 - c_1}{4a^2} - \lambda, \qquad A = \frac{x\lambda}{2} + a.$$

We can decrease the order of the equation w.r.t. the function a by using the additional first integral of form (2.9). For the root $\omega = \lambda^2$ of higher order, this function satisfies the degenerate equation (P_3(1,∗,0,∗)) with the parameters $\alpha = 4$ and $\gamma = 0$ and arbitrary β and $\delta = c_1$.

To reduce the equation to the canonical form in the most complex case $\omega = \lambda(\lambda - 1)$, we substitute

$$2a(x) = x\big(y(x^2) - 1\big)^{-1}.$$

The function $y(x)$ thus determined satisfies the degenerate equation (P_5(∗,∗,1,0)) with the parameters $\alpha = -c_1/2$, $\gamma = -1/2$, and $\delta = 0$.

We recall that Eq. (P_{34}) is related to (P_2) through the differential substitutions

$$\begin{aligned} y &= \frac{u' \pm k}{2u}, &\qquad \alpha &= \mp k - \frac{1}{2}, \\ u &= y^2 \pm y' + \frac{x}{2}, &\qquad k &= \alpha \pm \frac{1}{2}, \end{aligned}$$

and, analogously, Eq. (P_5(∗,∗,1,0)) is related to Eq. (P_3(∗,∗,1,1)) by a differential substitution [21].

To relate the obtained results to the NS and KdV equations, we consider nonstationary equation (2.4). In a simpler case $U = q - \lambda$, it is equivalent to the system

$$\begin{aligned} &2a_{1,x} = w_2, \qquad 2a_x = 2qa_{1,x} + q_xa_1 + w_1, \\ &2q_t = -a_{xxx} + 4qa_x + 2q_xa + 2w_0. \end{aligned}$$

Using the first two equations to express a and a_1 through q, we obtain Eq. (2.2) up to a redefinition of the coefficients.

In the case $U = q - 2\lambda z + \lambda^2$, Eq. (2.4) is equivalent to the system

$$\begin{aligned}
&a_{1,x} = w_2, \qquad a_x = (za_1)_x + w_1,\\
&2z_t = 4za_x + 2z_x a - 2qa_{1,x} - q_x a_1 - 2w_1 z + 2w_0,\\
&2q_t = -a_{xxx} + 4qa_x + 2q_x a + 4w_0 z,
\end{aligned}$$

where the first two equations can be integrated to obtain $a_1 = w_2 x + 2k_2$ and $a = za_1 + w_1 x + k_1 - w_2/2$ and the last two equations become an evolution system. This system becomes a deformation of NS equation (2.1) upon the substitution

$$z = -\frac{v_x}{2v}, \qquad q = -uv - \frac{1}{2}\left(\frac{v_{xx}}{v} - \frac{3v_x^2}{2v^2}\right),$$

which relates the equation $\Psi_{xx} = U\Psi$ to the Zakharov–Shabat spectral problem

$$\begin{aligned}
&\begin{pmatrix}\psi_1\\ \psi_2\end{pmatrix}_x = \begin{pmatrix}-\lambda & -v\\ u & \lambda\end{pmatrix}\begin{pmatrix}\psi_1\\ \psi_2\end{pmatrix},\\
&\psi_1 = v^{1/2}\Psi, \qquad \psi_2 = -v^{-1/2}\big(\Psi_x + (\lambda - z)\Psi\big).
\end{aligned}$$

3. Periodic closings of integrable chains

3.1. The general scheme. The Darboux transformation of spectral problem (2.3) is

$$\overline{\Psi} = A\Psi_x + B\Psi, \tag{3.1}$$

where A and B are polynomials in λ. We now show that the condition for invariance w.r.t. the combination of the Darboux transformation and the shift of the spectral parameter

$$\overline{U}(\lambda) = U(\lambda + \varepsilon) \tag{3.2}$$

again results in the Painlevé equations and their higher analogues. To make the presentation less cumbersome, we only formally derive the Painlevé equations using the chain technique, which determines the representation of a general Darboux transformation as a product of elementary transformations (with linear A and B), in Secs. 3.2 and 3.3. In Sec. 3.4, we use several examples to investigate the spectral properties of the obtained potentials.

The compatibility condition for (3.1) and (2.3) results in the equations

$$(\overline{U} - U)A = 2B_x + A_{xx}, \qquad AB_x - A_x B + UA^2 - B^2 = \mu(\lambda), \tag{3.3}$$

where $\mu(\lambda)$ is the integration constant. In the case $\overline{U} = U$, we again obtain Eq. (2.8), i.e., the class of solutions invariant w.r.t. the Darboux transformation coincides with the class of stationary solutions [1, 20]. Condition (3.2) determines the difference analogue of stationary solutions of deformations (2.7). Depending on the power of the polynomials A and B, we obtain several known exactly solvable quantum mechanical models and potentials expressed through the Painlevé transcendents, which can be therefore treated as the generalizations of these models.

Example 1: Harmonic oscillator. In the case

$$U = q - \lambda, \qquad A = a_1\lambda + a, \qquad B = b_1\lambda + b, \qquad \mu = m_3\lambda^3 + m_2\lambda^2 + m_1\lambda + m_0,$$

Eqs. (3.3) and (3.2) result in the system

$$\begin{aligned}
&a_1^2 = -m_3, \qquad 2b_1' = -\varepsilon a_1, \qquad a'' + 2b' = -\varepsilon a,\\
&a_1 b_1' = b_1^2 + 2a_1 a - a_1^2 q + m_2,\\
&a_1 b' + a b_1' - a' b_1 = 2b_1 b + a^2 - 2a_1 a q + m_1,\\
&ab' - a'b = b^2 - a^2 q + m_0.
\end{aligned} \tag{3.4}$$

For $a_1 = m_3 = 0$ and $a \neq 0$, we can easily solve this system and obtain (up to linear changes of x and λ)

$$\mu = -\lambda^2 + \lambda + m_0, \qquad \varepsilon = -2, \qquad A = -x, \qquad B = \lambda - \frac{x^2}{2}, \qquad q = \frac{x^2}{4} - \frac{1}{2} + \frac{m_0}{x^2}.$$

Example 2: The fourth Painlevé equation. In the case $a_1 \neq 0$, we set

$$\varepsilon = -2, \qquad \mu = -\lambda(\lambda - \mu_1)(\lambda - \mu_2), \qquad a_1 = 1, \qquad b_1 = x,$$

without lack of generality. System (3.4) is then reduced to Eq. (P_4) in the variable $y = b/a - x$ with the parameters $\alpha = (\mu_1 + \mu_2)/2 - 1$ and $\beta = -(\mu_1 - \mu_2)^2/2$. We then obtain

$$q = (y + x)^2 - y' - 1, \qquad 2a = q + 1 - x^2 - \mu_1 - \mu_2.$$

Example 3: The Morse potential. Given a potential $U = q - 2\lambda z + \lambda^2$ of the second order in λ, in the simplest case

$$A = a, \qquad B = b_1 \lambda + b, \qquad \mu = m_2 \lambda^2 + m_1 \lambda + m_0,$$

we obtain the system

$$\begin{aligned}
&a^2 - b_1^2 = m_2, \qquad b_1' = \varepsilon a, \qquad a'' + 2b' = -2\varepsilon a z + \varepsilon^2 a,\\
&a b_1' - a' b_1 = 2a^2 z + 2bb_1 + m_1, \qquad ab' - a'b = b^2 - a^2 q + m_0.
\end{aligned} \tag{3.5}$$

Hence,

$$q = \frac{b^2}{a^2} - \left(\frac{b}{a}\right)' + \frac{m_0}{a^2}, \qquad z = -\frac{b'}{\varepsilon a},$$

where we have

$$a' = \sigma \varepsilon a, \qquad b_1 = \sigma a, \qquad b = ka - \frac{\sigma m_1}{4a}, \qquad \sigma = \pm 1$$

in the case $m_2 = 0$ and

$$a' = \varepsilon b_1, \qquad b_1' = \varepsilon a, \qquad b_1^2 = a^2 - 1, \qquad b = ka + \frac{1}{2}(m_1 - \varepsilon) b_1$$

in the case $m_2 = 1$.

3.2. The Schrödinger operator. In our scheme, it is convenient to represent the general Darboux transformation as a product of elementary transformations, which results in representing the Painlevé equations as periodic closings of integrable chains. We can show that an arbitrary Darboux transformation for the potential $U = q - \lambda$ can be factored into elementary transformations determined by formulas (3.1) and (3.3) with $A = 1$ and $B = f(x)$, which gives

$$\bar{q} = q + 2f', \qquad f' - f^2 + q = \mu. \tag{3.6}$$

Applying this transformation to the potential $q_1 = q$ N times, we obtain the potential

$$q_{N+1} = q_1 + 2(f_1 + \cdots + f_N)',$$

where $f_1' - f_1^2 + q_1 = \mu_1$ and the functions f_n are related by the chain of equations

$$f_{n+1}' + f_n' = f_{n+1}^2 - f_n^2 + \nu_n, \tag{3.7}$$

where $\nu_n = \mu_{n+1} - \mu_n$. The closing condition $q_{N+1} = q_1 + \varepsilon$ is equivalent to the periodicity condition

$$f_{N+1} = f_1, \qquad \nu_{N+1} = \nu_1, \qquad \nu_1 + \cdots + \nu_N = \varepsilon, \tag{3.8}$$

which transforms this chain into a finite-dimensional dynamic system.

For $N = 2$ and $N = 3$, we obtain the respective potentials in Examples 1 and 2 (Sec. 3.1). It is instructive to repeat the calculations for $N = 3$ because we can then consider system (3.7), (3.8) as a convenient form of Eq. (P_4).

To reduce the system to the normal form, it is convenient to pass to the variables $g_n = f_{n+1} + f_n$. We then obviously have

$$g_1' = g_1(g_2 - g_3) + \nu_1, \qquad g_2' = g_2(g_3 - g_1) + \nu_2, \qquad g_3' = g_3(g_1 - g_2) + \nu_3, \tag{3.9}$$

where we can set $\varepsilon = -2$ and $g_1 + g_2 + g_3 = -2x$ without lack of generality. Excluding g_2 and g_3, we find that the function $y = g_1$ satisfies Eq. (P_4), where $2\alpha = \nu_2 - \nu_3$ and $2\beta = -\nu_1^2$.

Deriving the fifth Painlevé equation, which corresponds to the case $N = 4$, is more involved. Properties of system (3.7), (3.8) strongly depend on the evenness of N. For even N, this system cannot be resolved w.r.t. derivatives, and its solutions may even contain a functional arbitrariness, as in the example of the reduction $f_j = -f_{N+1-j}$, which results in a nonclosed chain of length $N/2$. However, this is due to occasional coincidence of the parameters ν_j; in the general case, taking the constraint

$$2(f_1^2 - f_2^2 + \cdots - f_N^2) = \nu_1 - \nu_2 + \cdots - \nu_N = K \tag{3.10}$$

into account allows obtaining a closed dynamic system (in particular, the system is always well defined for $\varepsilon \neq 0$). It is convenient to introduce the auxiliary variable $p = f_N - f_1$ in what follows. All the variables f_j are obviously linearly expressed through g_j and p; in particular, for $N = 4$, we obtain the equations

$$g_1' = g_1 p + \nu_1, \qquad g_2' = -g_2(g_2 - g_1 + p) + \nu_2$$

in g_1 and g_2; constraint (3.10) (with the relation $\sum g = -\varepsilon x$ taken into account) becomes

$$2\varepsilon x p + (2g_1 + \varepsilon x)(4g_2 + \varepsilon x) = 2K.$$

Excluding p and g_2 from these three equations, we obtain a second-order equation in the variable g_1. The substitution

$$2g_1(x) = \frac{\varepsilon x}{y(x^2) - 1}$$

reduces it to Eq. (P_5), whose parameters are expressed through the parameters ν_j,

$$\alpha = \frac{\nu_1^2}{2\varepsilon^2}, \qquad \beta = -\frac{\nu_3^2}{2\varepsilon^2}, \qquad \gamma = \frac{\nu_4 - \nu_2}{4}, \qquad \delta = -\frac{\varepsilon^2}{32},$$

and are subject to the single restriction $\delta \neq 0$.

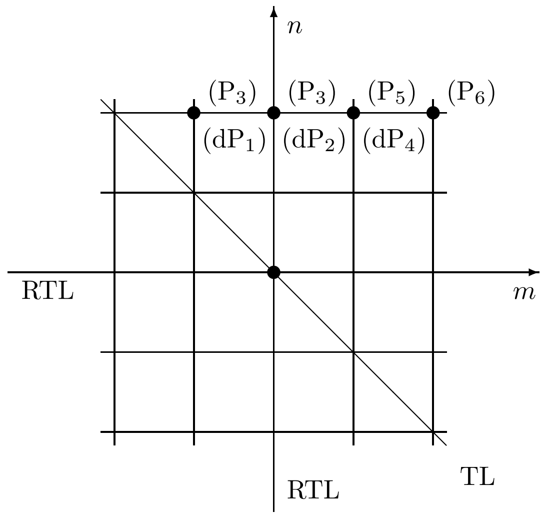


Fig. 1. Painlevé equations and periodic closings of Eqs. (3.14) and (4.11).

3.3. The operator with a second-order potential. In contrast to the Schrödinger operator, the Dirac operator admits a pair of essentially different Darboux transformations whose iterations generate the two-dimensional lattice in Fig. 1. We enumerate its sites such that the shifts along the axes m and n are on an equal footing and are equivalent to the shift in the relativistic Toda lattice (RTL), while the shift in the direction $m+n=0$ is described by the standard Toda lattice (TL). A two-dimensional lattice admits much more variants of closing than the dressing chain does. We show that the periodic closing when one of the sites lying in a side of the 4×4 square in Fig. 1 is identified with the central site results in Eqs. (P$_3$) (with both degenerate and arbitrary parameters), (P$_5$), and (P$_6$).

Lattice variables are related not only through differential but also purely algebraic equations, which can be interpreted as a nonlinear superposition principle for two Darboux transformations. The condition of periodic closing transforms these equations into difference analogues of the Painlevé equations, which are labeled as (dP) in Fig. 1. This scheme was partially realized in [22, 23].

For convenience, we consider the spectral problem

$$\psi'' + (z-\lambda)\psi' + p\psi = 0$$

or

$$\Psi' = U\Psi, \qquad \Psi = \begin{pmatrix} \psi \\ \psi' \end{pmatrix}, \qquad U = \begin{pmatrix} 0 & 1 \\ -p & \lambda - z \end{pmatrix}$$

in the matrix form. An arbitrary Darboux transformation can be factored into the product of elementary transformations

$$\Psi_{-1,1} = W\Psi, \qquad \Psi_{1,0} = M\Psi,$$

where the matrices W and M are

$$W = \begin{pmatrix} \lambda - z & -1 \\ p_{-1,1} & 0 \end{pmatrix}, \qquad M = \begin{pmatrix} 1 & \dfrac{1}{f} \\ -g & \dfrac{\lambda-\mu-g}{f} \end{pmatrix}.$$

We label the transformed wave functions with indices; this is convenient for further constructing a two-dimensional lattice in which all variables are enumerated by the pair of integers m, n. We usually use the concise notation for shifts: $u = u_{m,n}$, $u_{i,j} = u_{m+i,n+j}$.

The compatibility conditions

$$W' = U_{-1,1}W - WU, \qquad M' = U_{1,0}M - MU$$

are equivalent to the relations

$$z' = p - p_{-1,1}, \qquad p'_{-1,1} = p_{-1,1}(z - z_{-1,1}), \tag{3.11}$$

$$f' = f^2 + (\mu - z)f + p, \qquad g = \frac{p}{f}, \qquad z_{1,0} = f + g + \mu, \qquad p_{1,0} = g' + p. \tag{3.12}$$

The first Darboux transformation is therefore explicitly given; its iterations generate the Toda chain

$$q'' = e^{q_{1,-1}-q} - e^{q-q_{-1,1}}, \qquad z = q', \qquad p = e^{q_{1,-1}-q}. \tag{3.13}$$

The second transformation is reduced to solving the Riccati equation for the function $f = -(\psi'/\psi)\big|_{\lambda=\mu}$. Iterating this equation with the variables z and p excluded from formulas (3.12), we obtain the chain

$$f' = f(f + g - f_{-1,0} - g_{-1,0} + \nu), \qquad g' = f_{1,0}g_{1,0} - fg, \tag{3.14}$$

where $\nu = \mu - \mu_{-1,0}$, which is one representation of the relativistic Toda chain.

The commutativity of the Darboux transformations,

$$W_{1,0}M = M_{-1,1}W,$$

results in the constraints

$$fg = f_{0,-1}g_{-1,0}, \qquad f_{1,0} + g_{0,1} + \nu_1 = f + g, \qquad \mu_{-1,1} = \mu, \tag{3.15}$$

which we must take into account when rewriting chain (3.14) either in the symmetric form,

$$f' = fg - f_{0,1}g_{0,1}, \qquad g' = g(f_{0,-1} + g_{0,-1} - f - g - \nu), \tag{3.16}$$

or in the form of the doubled Volterra chain,

$$f' = f(g - g_{-1,1}), \qquad g' = g(f_{1,-1} - f). \tag{3.17}$$

Chain (3.16) is generated by the Darboux transformation

$$\Psi_{0,1} = N\Psi, \qquad N = W_{1,0}M = \begin{pmatrix} \lambda - \mu - f & -1 \\ fg_{-1,1} & g_{-1,1} \end{pmatrix},$$

whereas the equation $N' = U_{0,1}N - NU$ is equivalent to the relations

$$\begin{aligned} &f' = f^2 + (\mu - z)f + p, && g_{-1,1} = z - f - \mu, \\ &p_{0,1} = fg_{-1,1}, && z_{0,1} = f + g_{-1,1} + \mu - g'_{-1,1}. \end{aligned}$$

The parameter set μ is actually one-dimensional, and we can therefore label them with a single index, $\mu_{i,j} = \mu_{i+j}$ and $\nu_{i,j} = \nu_{i+j}$.

The periodic closings $f_{m-i,n+i} = f$ and $g_{m-i,n+i} = g$ in Volterra chain (3.17) as a rule result in solutions expressed through theta functions. However, we can organize a quasi–classical closure directly in

Toda chain (3.13) as soon as we set $q_{m+i,n-i} = q + 2\varepsilon x$, then obtaining the Painlevé-type equations. In the simplest nontrivial case $i = 2$, we obtain the equation

$$u'' = 2e^{2\varepsilon x - u} - 2e^{u}$$

in the function $u = q_{1,-1} - q$, which can be reduced by the substitution $e^{u}(x) = e^{\varepsilon x} y(e^{\varepsilon x})$ to the degenerate equation (P_3) with the parameter values

$$\alpha = -2\varepsilon^{-2}, \qquad \beta = 2\varepsilon^{-2}, \qquad \gamma = \delta = 0.$$

We now consider the periodic closings

$$f_{m+i,n+2} = f_{m,n}, \qquad g_{m+i,n+2} = g_{m,n}, \qquad \nu_{m+i+2} = \nu_m, \quad i = -1, 0, 1, 2 \tag{3.18}$$

(obviously, we can restrict the consideration to the upper side of the square in Fig. 1 because the axes m and n are symmetric). These four cases were previously considered in different notations: $i = -1, 0$ in [22], $i = 0, 1$ in [23], and $i = 0, 2$ in [2]; however, only a discrete variable dynamics was considered in [23], while a continuous variable dynamics was considered in [2, 22].

First-order closings. It is instructive to consider the simplest closings pertaining to the 2×2 square:

$$f_{m+i,n+1} = f_{m,n}, \qquad g_{m+i,n+1} = g_{m,n}, \qquad \nu_{m+i+1} = \nu_m, \quad i = 0, 1.$$

For $i = 0$, Eqs. (3.16) imply $f' = 0$ and $g' = -\nu g$, and we obtain $g_{1,0} = g$ and $f_{1,0} = f - \nu$ from (3.15).

Analogously, using Eqs. (3.14) and (3.16) for $i = 1$, we have

$$f - g_{0,1} = c = \text{const}, \qquad \frac{f}{g_{0,1}} = E, \qquad E' = \varepsilon E, \qquad \varepsilon = \nu + \nu_1,$$

while Eq. (3.15) implies the equalities $T_n(E) = E$ and $T_n(c) = c + \nu$, whence $c_{2k} = c_0 + k\varepsilon$ and $c_{2k+1} = c_0 + k\varepsilon + \nu$.

As is shown below, in all cases (3.18), the order of the ODE systems can be decreased to the second order. As above, this can be done using a pair of first integrals, one of which does not depend on x but depends linearly on the discrete variable while the second depends exponentially on x and is invariant w.r.t. shifts.

Equation ($\mathrm{P}_3(*,1,1,0)$). Using Eqs. (3.14), (3.16), and (3.17) in the simplest case $i = -1$, we obtain the system

$$\begin{aligned} f' &= fg - f_{0,1}g_{0,1}, & g' &= g(f_{0,1} - f), \\ f'_{0,1} &= f_{0,1}(g_{0,1} - g), & g'_{0,1} &= g_{0,1}(f + g - f_{0,1} - g_{0,1} - \nu), \end{aligned}$$

with the two first integrals

$$f + f_{0,1} + g = c = \text{const}, \qquad g g_{0,1} f_{0,1} = E^2, \qquad 2E' = -\nu E.$$

Using these integrals, we can reduce the system to the equivalent system

$$f' = fg - \frac{E^2}{g}, \qquad g' = g(c - g - 2f).$$

Excluding f and substituting $g(x) = Ey(E)$, we obtain Eq. (P_3) with the parameters

$$\alpha = -4c\nu^{-2}, \qquad \beta = 8\nu^{-2}, \qquad \gamma = 4\nu^{-2}, \qquad \delta = 0.$$

We now consider the evolution in the discrete time n. It is given by the system

$$f_{0,2}g_{0,2} = f_{0,1}g, \qquad f_{0,2} + g_{0,1} + \nu = f + g,$$

while the quantities c and E determine the first integrals of this system as well:

$$T_n(c) = c - \nu, \qquad T_n(E^2) = E^2.$$

Using these first integrals, we can exclude the variables $f_{0,2}$ and f from the second equation in the system and then set $c_n = c_0 - \nu n$ and substitute $g = E/u$, which results in the difference equation

$$u_{n+1} + u_{n-1} = \frac{c_0 - \nu n}{u_n} - \frac{x}{u_n^2}, \tag{3.19}$$

known as the discrete equation (P_1) [22, 24].

Equations (P_3(∗,∗,1,1)) *and* (P_5(∗,∗,1,0)). The closing $f = f_{0,2}$ and $g = g_{0,2}$ results in the system

$$\begin{aligned} f' &= fg - f_{0,1}g_{0,1}, & g' &= g(f_{0,1} + g_{0,1} - f - g - \nu), \\ f'_{0,1} &= f_{0,1}g_{0,1} - fg, & g'_{0,1} &= g_{0,1}(f + g - f_{0,1} - g_{0,1} - \nu_1), \end{aligned}$$

with the first integrals

$$f + f_{0,1} = c, \qquad gg_{0,1} = E^2, \qquad 2E' = -\varepsilon E, \qquad \varepsilon = \nu + \nu_1.$$

Using these integrals to excluding the variables $f_{0,1}$ and $g_{0,1}$, we obtain the system

$$f' = fg + \frac{E^2(f - c)}{g}, \qquad g' = E^2 - (2f + \nu - c)g - g^2, \tag{3.20}$$

which can be reduced to Eq. (P_3) by the substitution $g(x) = Ey(E)$, while

$$\alpha = 4(\nu - c)\varepsilon^{-2}, \qquad \beta = 4(c - \nu_1)\varepsilon^{-2}, \qquad \gamma = 4\varepsilon^{-2}, \qquad \delta = -4\varepsilon^{-2}.$$

Another substitution, $f(x)g(x) = E^2/\big(1 - y(E^2)\big)$, reduces system (3.20) to the degenerate equation (P_5) with the parameters

$$\alpha = \frac{c^2}{2\varepsilon^2}, \qquad \beta = -\frac{\nu_1^2}{2\varepsilon^2}, \qquad \gamma = -\frac{2}{\varepsilon^2}, \qquad \delta = 0.$$

Because Eqs. (P_3(∗,∗,1,1)) and (P_5(∗,∗,1,0)) are related to the same system, we can obtain the transformation that connects these equations [21].

The evolution in the discrete time m is determined by the system

$$\begin{aligned} f_{1,0}g_{1,0} &= f_{1,1}g, & f_{1,0} + g_{0,1} + \nu_1 &= f + g, \\ f_{1,1}g_{1,1} &= f_{1,0}g_{0,1}, & f_{1,1} + g + \nu &= f_{0,1} + g_{0,1}, \end{aligned}$$

by virtue of which,

$$T_m(c) = c - \varepsilon, \qquad T_m(E^2) = E^2.$$

Using the first equation of the system and the relation $f_{1,0} + f_{1,1} = c - \varepsilon$ to exclude the variables f and substituting as in the continuous case, we obtain the discrete equation (P_2) [22, 24] in the form

$$\frac{c_{m+1}y_{m+1}}{y_{m+1} + y_m} + \frac{c_m y_m}{y_m + y_{m-1}} = x\left(\frac{1}{y_m} - y_m\right) + c_m - \nu, \qquad c_m = c_0 - \varepsilon m. \tag{3.21}$$

Equation (P_5(∗,∗,∗,1)). The closing $f = f_{1,2}$ and $g = g_{1,2}$ results in the equations

$$f' = fg - f_{0,1}g_{0,1}, \qquad g' = g(f_{1,1} + g_{1,1} - f - g - \nu),$$
$$f'_{1,1} = f_{1,1}g_{1,1} - fg, \qquad g'_{0,1} = g_{0,1}(f + g - f_{0,1} - g_{0,1} - \nu_1),$$

where the "unnecessary" variables $f_{0,1}$, $g_{1,1}$, $f_{1,0}$, and $g_{0,2}$ are excluded using constraints (3.15),

$$f_{1,1}g_{1,1} = f_{1,0}g_{0,1}, \qquad f_{1,0} + g_{0,1} + \nu_1 = f + g,$$
$$fg = f_{1,1}g_{0,2}, \qquad f_{1,1} + g_{0,2} + \nu_2 = f_{0,1} + g_{0,1}.$$

The variables $f_{1,1}$ and $g_{0,1}$ can be excluded using the first integrals

$$f + f_{1,1} - g_{0,1} = c, \qquad gg_{0,1} = Ef_{1,1}, \qquad E' = -\varepsilon E, \qquad \varepsilon = \nu + \nu_1 + \nu_2,$$

which results in the system

$$f' = (g - E)f - \frac{E}{g - E}(c - f)(c - f + \nu_2),$$
$$g' = -2fg - g^2 + g(E + c - \nu) + (c - \nu_1)E.$$

Excluding f and substituting $g(x) = Ey(E)/\big(y(E) - 1\big)$, we obtain Eq. ($P_5$) with the parameters

$$\alpha = \frac{\nu_2^2}{2\varepsilon^2}, \qquad \beta = -\frac{(c - \nu_1)^2}{2\varepsilon^2}, \qquad \gamma = \frac{c - \nu}{\varepsilon^2}, \qquad \delta = -\frac{1}{2\varepsilon^2}.$$

The discrete evolution is governed by the system

$$g_{0,2} = \frac{fg}{f_{1,1}}, \qquad f_{0,1} = f_{1,1} + g_{0,2} - g_{0,1} + \nu_2,$$

while

$$T_n(E) = E, \qquad T_n(c) = c + \nu_2, \qquad T_n^2(c) = c + \nu_2 + \nu, \qquad T_n^3(c) = c + \varepsilon,$$

whence

$$c_n = c_0 + \varepsilon k + \begin{cases} 0, & n = 3k, \\ \nu_2, & n = 3k + 1, \\ \nu + \nu_2, & n = 3k + 2. \end{cases}$$

Using the first integrals to exclude the variables f, we obtain

$$g_{n+1} + g_{n-1} = E\left(1 + \frac{c_{n-1}}{g_n}\right)$$

for the first equation in the system (the index m can be dropped). The form of this equation is similar to equation (dP_1), but the former is distinguished because its set of the parameters c_n comprises three arithmetic progressions, not just one. This equation was first obtained in [23], where its equivalence to Eq. (dP_4) was also found.

Equation (P_6). In the most advanced case $f = f_{2,2}$ and $g = g_{2,2}$, we can conveniently write the composition of Darboux transformations (3.14) and (3.16) in the form

$$
\begin{aligned}
&f' - g'_{0,1} = fg - f_{1,1}g_{1,1}, \qquad \frac{f'_{1,0}}{f_{1,0}} - \frac{g'_{1,1}}{g_{1,1}} = f_{1,1} + g_{1,1} - f - g + \nu_1 + \nu_2,\\
&f_{1,0} + g_{0,1} + \nu_1 = f + g, \qquad f_{1,1}g_{1,1} = f_{1,0}g_{0,1},
\end{aligned}
$$

which suggests introducing the new variables $u = f_{1,0}/g_{1,1}$ and $v = f - g_{0,1}$. Using the first integrals

$$v + v_{1,1} = c, \qquad uu_{1,1} = E, \qquad E' = \varepsilon E, \qquad \varepsilon = \nu + \nu_1 + \nu_2 + \nu_3$$

and using the formulas

$$f = \frac{1}{E-1}(u_{1,1}v_{1,1} + Ev), \qquad g = \frac{1}{E-1}\big(u(v_{1,1} - \nu_3) + v - \nu_1\big)$$

to exclude the variables f and g, we obtain the system

$$
\begin{aligned}
u' &= \frac{1}{E-1}\big((2v - c)(u-1)(u-E) + \nu_3 u^2 + (\nu_1+\nu_2)Eu - (\nu_2+\nu_3)u - \nu_1 E\big),\\
v' &= \frac{1}{E-1}uv(c - \nu_3 - v) + \frac{E}{u(E-1)}(v-c)(v-\nu_1),
\end{aligned}
$$

which can be reduced to Eq. (P_6) by substituting $u(x) = y(E)$ with arbitrary values of the parameters,

$$\alpha = \frac{(\nu_3 - c)^2}{2\varepsilon^2}, \qquad \beta = -\frac{(\nu_1 - c)^2}{2\varepsilon^2}, \qquad \gamma = \frac{\nu_2^2}{2\varepsilon^2}, \qquad \delta = \frac{1}{2} - \frac{\nu^2}{2\varepsilon^2}.$$

The evolution in n is described by the formulas

$$
\begin{aligned}
&T_n(E) = E, \qquad T_n(c) = c + \nu + \nu_2, \qquad T_n^2(c) = c + \varepsilon,\\
&u_{0,1} = \frac{uv + E(c-v)}{uv + c - v}, \qquad v_{0,1} = \nu_2 + \frac{u-1}{(E-1)u}\big(uv + E(c-v)\big).
\end{aligned}
$$

Excluding v and setting $u = y + 1$ and $E = X + 1$, we obtain the difference equation

$$c_n \frac{X - y_{n+1}}{y_{n+1}y_n + X} + c_{n-1}\frac{X - y_{n-1}}{y_{n-1}y_n + X} = \nu_1 + c_{n-1},$$

where $c_{2k} = c_0 + \varepsilon k$ and $c_{2k+1} = c_0 + \varepsilon k + \nu + \nu_2$.

3.4. Spectral properties.

Arithmetic progression. We now discuss the spectral properties of potentials related to the Painlevé transcendents [1, 25]. All examples in Sec. 3.3 are algebraically similar: a quasi-periodic closing of the chain of Darboux transformations results in an operator algebra that generalizes the creation and annihilation algebra of the harmonic oscillator. This means that in terms of the Painlevé transcendents, we can explicitly construct wave functions ψ for a discrete set of values of λ_n comprising several arithmetic progressions

$$\mu_1 + m\varepsilon, \qquad \ldots, \qquad \mu_N + m\varepsilon, \quad m \in \mathbb{Z},$$

in the corresponding spectral problem. In the case of the Schrödinger operator for example, we obtain the sequence of operators

$$L_n = u_n - D_x^2, \qquad A_n = f_n - D_x, \qquad A_n^+ = f_n + D_x,$$

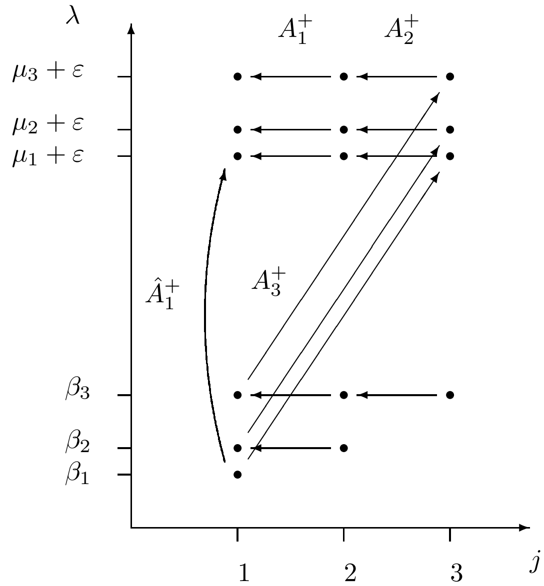


Fig. 2. The beginning of constructing eigenfunctions.

related by the chain of Darboux transformations

$$L_n = A_n^+ A_n + \mu_n \quad \longmapsto \quad L_{n+1} = A_n A_n^+ + \mu_n = A_{n+1}^+ A_{n+1} + \mu_{n+1}.$$

The function f_n then satisfies chain (3.7), and the wave functions $\varphi_{n,k}$ of the potentials u_n, which correspond to the values $\lambda = \beta_k$, are constructed recursively,

$$\varphi_{k,k} = \exp\biggl(\int f_k\,dx\biggr), \qquad \varphi_{n,k} = A_n^+ \varphi_{n+1,k}, \quad n < k. \tag{3.22}$$

We impose the periodicity condition, apply the operators A_n^+, and, after N steps, obtain a solution of the equation with the potential shifted by ε. As a result, we obtain a spectrum comprising N arithmetic progressions. For $N = 3$, this process is schematically depicted in Fig. 2. Therefore, as in the harmonic oscillator case ($N = 1$), we construct eigenfunctions of the operator L_n using mutually conjugate creation and annihilation operators of the Nth order,

$$\hat{A}_n^+ = A_n^+ \cdots A_{n+N-1}^+, \qquad \hat{A}_n = A_{n+N-1} \cdots A_n.$$

The following operator relations are an analogue of the harmonic oscillator algebra:

$$\begin{aligned}
&\hat{A}_n^+ \hat{A}_n = P(L_n), \qquad \hat{A}_n \hat{A}_n^+ = P(L_n + \varepsilon), \qquad P(\lambda) = (\lambda - \mu_n) \cdots (\lambda - \mu_{n+N-1}),\\
&[L_n, \hat{A}_n^+] = \varepsilon \hat{A}_n^+, \qquad [L_n, \hat{A}_n] = -\varepsilon \hat{A}_n.
\end{aligned}$$

A more complicated question is what are the conditions under which functions (3.22) become eigenfunctions. We encounter analytic difficulties here: we must select solutions of the Painlevé equation that are regular in the given interval and have the proper asymptotic behavior. For example, we showed above that solutions of system (3.7), (3.8) for $N = 3, 4$ are expressed through the respective fourth and fifth Painlevé transcendents. Little is known about solutions with $N \geq 5$, although they presumably have the Painlevé property, i.e., their general solution has no movable essential singularities. Qualitative information about the solution behavior at infinity and about singularities on the real axis is crucial for the spectral theory. The relation $2\sum f_n = \varepsilon x$ implies that functions f_n grow linearly "in the average," which suggests the asymptotic formula

$$f_n = -\frac{\varepsilon}{2N}x + O(1), \qquad u_n = \frac{\varepsilon^2}{4N^2}x^2 + O(x), \quad x \to \pm\infty.$$

Moreover, if the functions f_n are regular for real x, then the functions $\varphi_{n,k}$ constructed by formulas (3.22) are eigenfunctions. To justify the construction, we must verify these a priori assumptions about the solution. Here, we mostly rely on numerical simulations, which demonstrate that solutions with the required properties exist in a large domain of the space of parameters and initial conditions of the system with odd N. Potentials for even N have a singularity at zero, and we must pose the problem on the half-axis (see Fig. 3). The number of arithmetic progressions constituting the spectrum becomes less than N.

Automodel reduction. The above ideas were first applied to a slightly different variant of quasi-periodic closing in [26], where instead of the spectral shift $\lambda \to \lambda + \varepsilon$, the dilation

$$\hat{\psi}(x,\lambda) = \psi(qx, q^k\lambda), \quad 0 < q < 1,$$

which is also admitted by spectral problem (2.3), was used, where $k = 1$ for the second-order potential and $k = 2$ for the linear potential. Already in the simplest case of chain (3.7) closed on the first step, we obtain the differential equation

$$f' + \hat{f}' = f^2 - \hat{f}^2 - 1 + q^2, \qquad \hat{f}(x) = qf(qx),$$

with the shifted argument. Its general solution cannot be expressed through elementary nor known special functions. This equation, however, has the Painlevé property [27–29] and admits a countable family of rational solutions [29]. For this, we construct a solution fixing the initial value $f(0)$, after which the Taylor coefficients are unambiguously determined and the Taylor series converges in a circle $|x| < a$. The function $\hat{f}(x) = qf(qx)$ is then determined in a larger circle $|x| < a/q$, and we can hence define $f(x)$ in this circle as the solution of the Riccati equation. Repeating this process, we consequently define the function f in the whole complex plane. When solving the Riccati equation, singularities may be accumulated, but these singularities can only be poles.

Numerical experiment shows that for $|f(0)| < 1$, real solutions are determined in the whole axis and have the asymptotic behavior

$$f'(x) = O\left(\frac{1}{x^3}\right), \qquad f(x) = \mp 1 + O\left(\frac{1}{x^2}\right), \quad x \to \pm\infty.$$

Character graphs for the function $f(x)$ are depicted in Fig. 3. Following the scheme in the previous section, we can show that the potentials $u(x)$ corresponding to such solutions are reflectionless and have an infinite set of eigenvalues $\lambda_j = -q^{2j}$, $j = 1, 2, \dots$, comprising a convergent geometric progression. For the operator

$$L = u - D_x^2 = L_f - 1, \qquad L_f = (f + D_x)(f - D_x),$$

we have the inequality

$$\langle L_f\varphi, \varphi\rangle = \int (f\varphi - \varphi')^2\,dx \geq 0,$$

and the ground state for L_f is determined by the function $\varphi = \exp\left(\int f\,dx\right)$.

To conclude our examples, we formulate the following plausible statement.

The principle of quasi-periodic closing. The discrete spectrum of problem (2.3) invariant w.r.t. the composition of a Darboux transformation and a classical symmetry can be found explicitly.

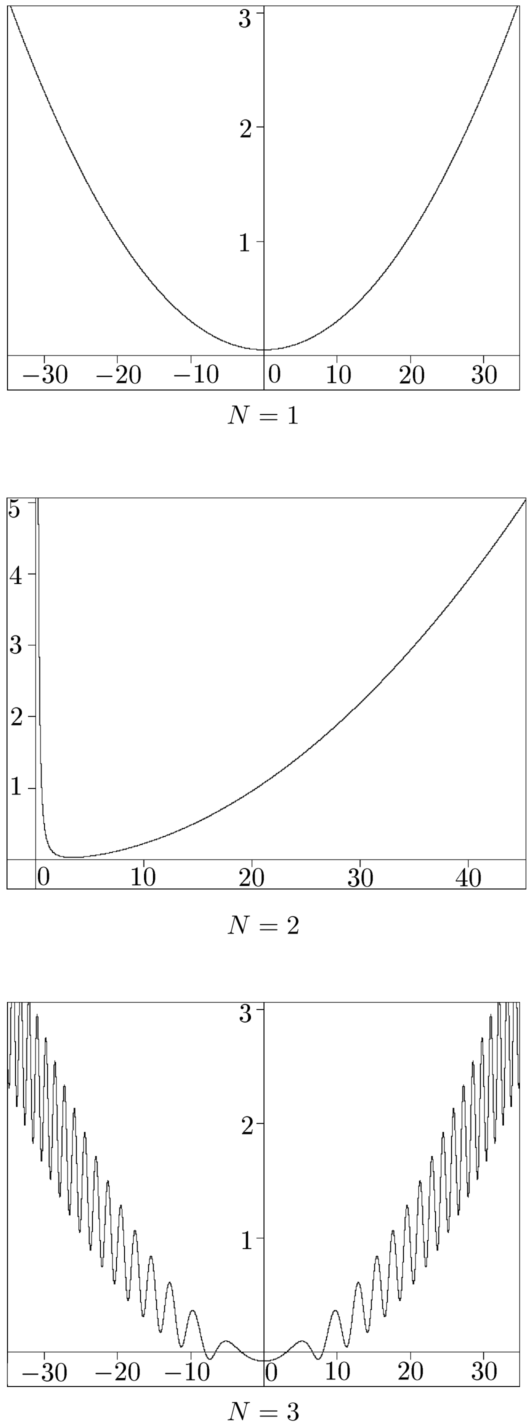
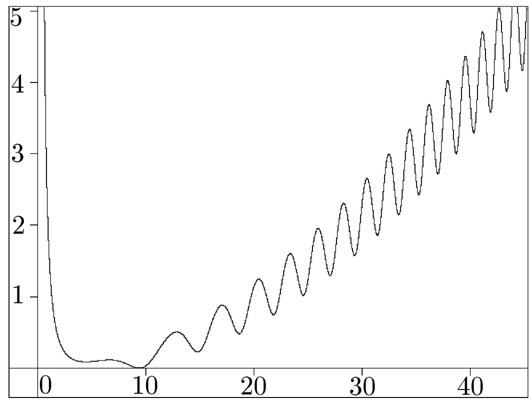
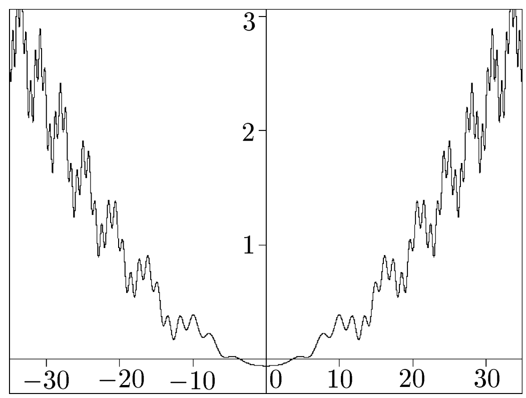


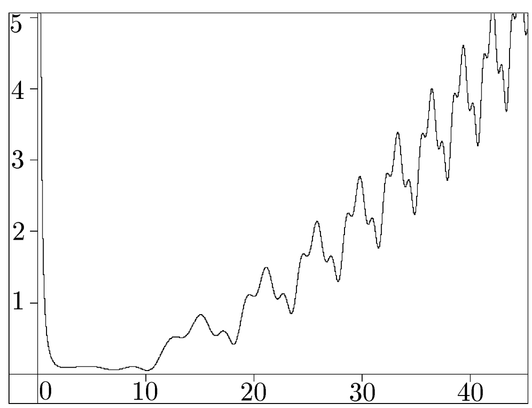
Fig. 3. (Continued on the following page.)



$N = 4$



$N = 5$



$N = 6$

Fig. 3. Potentials u_1: the initial conditions for odd N are $f_j(0) = 0$, which ensures that the potential is even.

4. Integrable Lagrangians

We now describe a wide class of systems of equations that is interesting from the applications standpoint (see [10, 30]),

$$\mathbf{u}_t = A(\mathbf{u})\mathbf{u}_{xx} + B(\mathbf{u}, \mathbf{u}_x), \qquad \det A \neq 0, \qquad \mathbf{u} = (u_1, u_2), \tag{4.1}$$

which has local higher conservation laws of the form

$$g_t = f_x. \tag{4.2}$$

The term *local* means that the density g and the current f of conservation law (4.2) are functions of a finite number of variables $\mathbf{u}, \mathbf{u}_x, \mathbf{u}_{xx}, \dots$. We call necessary conditions for the existence of local higher conservation laws the *integrability conditions*. The first two of these conditions are that $\operatorname{tr} A = 0$ and that the function $(\det A)^{-1/4}$ is the density of the conservation law. With these conditions, we can prove that with the change of variables

$$t' = ct, \qquad \mathbf{u}' = \mathbf{u}'(\mathbf{u}) = \bigl(u_1'(u_1, u_2), u_2'(u_1, u_2)\bigr), \qquad dx' = g\,dx, \qquad g = (\det A)^{-1/4},$$

we can reduce system (4.1) to the form

$$iu_t = u_{xx} + F(u, v, u_x, v_x), \qquad iv_t = -v_{xx} + G(u, v, u_x, v_x). \tag{4.3}$$

The simplest integrability conditions for system (4.3) are formulated in Appendix 2. In particular, for a nonlinear nondecomposable system of the form

$$iu_t = u_{xx} + F(u, v), \qquad iv_t = -v_{xx} + G(u, v),$$

the integrability conditions are satisfied in two cases: for the NS equation

$$iu_t = u_{xx} + 2u^2 v, \qquad -iv_t = v_{xx} + 2v^2 u$$

and for the Boussinesq equation

$$iu_t = u_{xx} + (u + v)^2, \qquad -iv_t = v_{xx} + (u + v)^2.$$

The list of equations (4.1) satisfying the integrability conditions can be segregated into two parts. We are interested in the part that is related to auxiliary spectral problem (2.3) of the second order (the NS-type equation). The second part of the list pertains to spectral problems of the third order (the class of the Boussinesq equation). It contains interesting examples (see the appendices), but considering it is beyond the scope of our theory, which is a more elementary alternative approach that does not use the integrability conditions.

4.1. Systems of NS-type equations. We now write the key integrable systems of the NS-type equations: the NS equation itself, the Heisenberg magnet, and the Landau–Lifshitz model. We first discuss the divergent systems

$$ip_t = D_x\bigl(p_x + F(p, z)\bigr), \qquad iz_t = D_x\bigl(-z_x + G(p, z)\bigr),$$

for which we have the third conservation law with the density pz. We then have $F_p = G_z$, and with the redefinition $z = q_x$, the system becomes Lagrangian with the Lagrange function

$$L = L_0 + V(p, q_x), \qquad L_0 = iqp_t + p_x q_x, \qquad F = V_z, \qquad G = V_p. \tag{4.4}$$

All integrable divergent systems (see [10]) correspond to polynomials of the second degree in p and z,

$$V(p, z) = \varepsilon p^2 z^2 + \alpha p z^2 + \beta z^2 + \gamma p^2 z + \delta p^2 \tag{4.5}$$

(this is an arbitrary polynomial up to the terms pz, p, z, and 1, which we neglect). The corresponding systems are

$$ip_t = p_{xx} + \big(2P(p)z + \gamma p^2\big)_x, \qquad iz_t = -z_{xx} + \big(2Q(z)p + \alpha z^2\big)_x, \tag{4.6}$$

$$P(p) = \varepsilon p^2 + \alpha p + \beta, \qquad Q(z) = \varepsilon z^2 + \gamma z + \delta. \tag{4.7}$$

In particular, all of them have local master symmetries (see Sec. 6.2). Up to linear transformations of p and z and Galileo transformations, we have three cases:

$$V = \varepsilon p^2 z^2 + \beta z^2 + \delta p^2, \qquad V = pz^2 + p^2 z, \qquad V = pz^2 + p^2.$$

We can obtain the famous NS-type systems (the NS equation and the Heisenberg magnet) from system (4.6) by a change of variables of the form

$$p = h(u - v), \qquad z = v_x. \tag{4.8}$$

That the system in u and v has form (4.1) with $A = \operatorname{diag}(-i, i)$ automatically leads to the equation $h' = P(h)$, which gives

$$iu_t = u_{xx} + 2Q(u_x)h(u - v) + \alpha u_x^2, \qquad iv_t = -v_{xx} + 2Q(v_x)h(u - v) + \alpha v_x^2. \tag{4.9}$$

Modifications of these systems of equations are obtained from (4.6) by the differential substitution

$$p = f(u - v)u_x + g(u - v), \qquad z = v_x. \tag{4.10}$$

The form of functions f and g stems from the same condition as in (4.8), which in this case results in the ODEs

$$f' = 2\varepsilon fg + \alpha f - \gamma f^2, \qquad g' = P(g) - \delta f^2, \tag{4.11}$$

which admit the first integral

$$P(g) - \gamma fg + \delta f^2 = \mathrm{const} \cdot f \tag{4.12}$$

and can be solved in elementary functions.

The system in the variables u and v is

$$\begin{aligned} iu_t &= u_{xx} + 2Q(u_x)\big(f(u - v)v_x + g(u - v)\big) + \alpha u_x^2, \\ iv_t &= -v_{xx} + 2Q(v_x)\big(f(u - v)u_x + g(u - v)\big) + \alpha v_x^2. \end{aligned} \tag{4.13}$$

System (4.13) can obviously be reduced to (4.9) by the transformation

$$\tilde{u} = v + h^{-1}(fu_x + g), \qquad \tilde{v} = v.$$

This transformation becomes trivial for $f = 0$ because the function g satisfies the same equation that the function h does, i.e., system (4.13) becomes (4.9).

System (4.13) is also invariant w.r.t. the involution

$$u \leftrightarrow -v, \qquad x \leftrightarrow -x, \qquad t \leftrightarrow -t,$$

and there hence exists the additional transformation

$$p = f(u-v)v_x + g(u-v), \qquad q = u,$$

which relates (4.6) and (4.13).

In addition to (4.4) and (4.5), we consider the Lagrangian

$$L = L_0 + p^2\bigl(q_x^2 - R(q)\bigr) + \frac{1}{2}pR'(q) - \frac{1}{12}R''(q), \qquad R^V = 0, \tag{4.14}$$

which plays a special role in our classification. As in the previous case, this Lagrangian admits two parameterizations, which reduce the corresponding dynamic system to form (4.1) with $A = \mathrm{diag}(-i, i)$.

The first parameterization is $p = (v-u)^{-1}$, $q = v$ (cf. (4.8)) and leads to the system

$$iu_t = u_{xx} + 2\frac{R(u) - u_x^2}{u-v} - \frac{1}{2}R'(u), \qquad iv_t = -v_{xx} + 2\frac{R(v) - v_x^2}{u-v} + \frac{1}{2}R'(v), \tag{4.15}$$

where R is an arbitrary fourth-order polynomial.

The second parameterization is $p = (r_v - u_x)/2r$, $q = v$, where $r = r(u, v)$ is a symmetric polynomial of the second order in u and v such that $R(u)$ is its discriminant,

$$R(u) = r_v^2 - 2rr_{vv}, \qquad r(u,v) = r(v,u), \qquad r_{uuu} = 0. \tag{4.16}$$

This parameterization results in the system of equations

$$\begin{aligned} iu_t &= u_{xx} + \bigl(R(u) - u_x^2\bigr)\frac{v_x + r_u}{r} - \frac{1}{2}R'(u), \\ iv_t &= -v_{xx} + \bigl(R(v) - v_x^2\bigr)\frac{u_x - r_v}{r} + \frac{1}{2}R'(v). \end{aligned} \tag{4.17}$$

The transformation that reduces (4.17) to (4.15) is

$$\tilde{u} = v + \frac{2r}{u_x - r_v}, \qquad \tilde{v} = v.$$

We now present several examples. System (4.9), which corresponds to the Lagrangian

$$L = L_0 + pq_x^2 \pm p^2,$$

becomes the NS equation after the substitution $\tilde{u} = e^u$, $\tilde{v} = e^{-v}$ and the reduction $\tilde{u} = \psi$, $\tilde{v} = \bar{\psi}$:

$$i\psi_t = \psi_{xx} \pm 2|\psi|^2\psi.$$

Analogously, the Lagrangian $L = L_0 + p^2 q_x^2$ results in the system of equations

$$iu_t = u_{xx} - 2\frac{u_x^2}{u-v}, \qquad iv_t = -v_{xx} - 2\frac{v_x^2}{u-v},$$

which is related to the complexification of the Heisenberg model,

$$s_t = [s, s_{xx}], \qquad s \in \mathbb{C}^3, \qquad \langle s, s\rangle = 1,$$

upon the substitution

$$s = S(u, v) = \frac{1}{u-v}(1 - uv, i + iuv, u + v) \tag{4.18}$$

(for $v = -1/\bar{u}$, this substitution coincides with the stereographic projection). The same substitution relates system (4.15) to the Landau–Lifshitz equation,

$$s_t = [s, s_{xx} + Js], \qquad s \in \mathbb{C}^3, \qquad \langle s, s\rangle = 1, \tag{4.19}$$

where J is a symmetric matrix with complex entities in general. The polynomials R and r are determined through the matrix J,

$$R(u) = \frac{1}{4}(u-v)^4\langle s_v, Js_v\rangle, \qquad r(u,v) = \frac{i}{4}(u-v)^2\langle s, Ks\rangle, \tag{4.20}$$

where $s = S(u, v)$ and the matrices J and K are related by

$$J = CI - \det(K)K^{-1}, \tag{4.21}$$

where C is a constant. We call (4.15) the Landau–Lifshitz system and (4.17) the modified Landau–Lifshitz system. These systems of equations are the most general among systems of the NS type in the sense that other systems can be obtained from these two similarly to how all the Painlevé equations can be obtained from (P_6). However, in contrast to the Painlevé list, we use both limiting transitions and differential substitutions.

4.2. Generalizing the Toda chain. We now discuss the list of integrable chains corresponding to the variational problem for the functional

$$\mathcal{L} = \int dx \sum_n L(z_n, q_{n+1}, q_n), \qquad z_n \equiv q_{n,x}. \tag{4.22}$$

The Euler–Lagrange equation is then

$$D_x(L_z) = L_q + T_n^{-1}(L_{q_1}), \tag{4.23}$$

where T_n is the operator of the shift $q_n \to q_{n+1}$. Below, we consider Lagrangians of the special form

$$L(z, q_1, q) = L_0(z, q) + zV(q_1, q) + U(q_1, q), \tag{4.24}$$

where the first and the last terms determine the respective kinetic and potential energies and the middle term determines the magnetic field (or gyroscopic forces). The cases $V = 0$ and $V \neq 0$ are principally different and correspond to two different parameterizations of (4.8) and (4.10) (see below).

The selection criterion for integrable systems is the symmetry test for variational symmetries of the form

$$q_t = B(z_1, z, z_{-1}, q_1, q, q_{-1}). \tag{4.25}$$

By virtue of the Noether theorem, a variational symmetry generates the conservation law according to the formulas

$$D_t(L) \in \operatorname{Im} D_x + \operatorname{Im}(T_n - 1) \quad \Leftrightarrow \quad B\frac{\delta L}{\delta u} \in \operatorname{Im} D_x + \operatorname{Im}(T_n - 1)$$

and is also a higher symmetry in the usual sense, i.e., an equation compatible with chain (4.23). For example, the exponential Toda chain

$$q_{xx} = e^{q_1 - q} - e^{q - q_{-1}} \tag{4.26}$$

corresponds to the Lagrangian $L = z^2/2 - e^{q_1 - q}$ and has the variational symmetry

$$q_t = e^{q_1 - q} + e^{q - q_{-1}} + q_x^2.$$

We can segregate the chain systems into two classes: the shift-invariant chains and the elliptic-type chains related to the Landau–Lifshitz model. Any Lagrangian that has a classical symmetry $q_\tau = \psi(q)$ can be reduced to the form

$$L = L_0(z) + zV(y) + U(y), \qquad y \equiv q_1 - q, \tag{4.27}$$

which is invariant w.r.t. the shift $q_\tau = 1$ (corresponding to the momentum $\sum_n L_{z_n}$ conservation law). The corresponding chain (4.23) is

$$q_{xx} = Q(z)\big(z_1 f(y) - z_{-1} f(y_{-1}) + g(y) - g(y_{-1})\big), \tag{4.28}$$

where

$$Q(z) = \frac{1}{L_0''(z)}, \qquad f(y) = -V'(y), \qquad g(y) = -U'(y).$$

A direct and simple calculation results in the following theorem.

Theorem 2. *Lagrangians* (4.27) *admit the variational symmetry of the form*

$$q_t = Q(z)\big(z_1 f(y) + z_{-1} f(y_{-1}) + g(y) + g(y_{-1})\big) + s(z) \tag{4.29}$$

iff

$$\begin{aligned} &f' = 2\varepsilon f g + \alpha f - \gamma f^2, \qquad g' = P(g) - \delta f^2, \qquad s = \alpha z^2, \\ &P = \varepsilon g^2 + \alpha g + \beta, \qquad\quad Q = \varepsilon z^2 + \gamma z + \delta. \end{aligned} \tag{4.30}$$

We note that the definitions of the functions P, Q, f, and g exactly coincide with (4.7) and (4.11).

In the important particular case $f = 0$ (it is convenient to identify g with the function h in (4.9)), we obtain the class of integrable chains

$$q_{xx} = Q(z)\big(h(y) - h(y_{-1})\big), \qquad h' = P(h), \qquad Q'' = P'' = 2\varepsilon, \tag{4.31}$$

which contains Toda chain (4.26). We call the general case (4.28), (4.30) with $f \neq 0$ the class of the relativistic Toda chain named after the most famous representative of this class, chain (R_3) with $\nu = 0$ in the list at the end of this section.

As in Sec. 4.1, two Lagrangians of form (4.24), which do not admit the shift invariance, play a special role. The kinetic energy for these Lagrangians is determined by the term

$$L_0 = \frac{1}{2\sqrt{R}}\big[\big(\sqrt{R} + z\big)\log\big(\sqrt{R} + z\big) + \big(\sqrt{R} - z\big)\log\big(\sqrt{R} - z\big)\big],$$

and the functions U and V are given by the formulas

$$V = 0, \qquad U = -\log(q_1 - q), \tag{4.32}$$

$$V_{q_1} = -\frac{1}{2r(q_1, q)}, \qquad U = \frac{1}{2}\log r(q_1, q), \tag{4.33}$$

where in the both cases $R(z)$ is a polynomial of degree not higher than four, $r(u, v)$ is a symmetric polynomial of the degree not higher than two in each variable, and R is its discriminant (see (4.16)). The chains corresponding to these Lagrangians have the respective forms

$$q_{xx} = \big(R(q) - z^2\big)\left(\frac{1}{q_1 - q} - \frac{1}{q - q_{-1}}\right) + \frac{1}{2}R'(q), \tag{4.34}$$

$$q_{xx} = \big(z^2 - R(q)\big)\left(\frac{z_1 + \partial_q r(q_1, q)}{2r(q_1, q)} - \frac{z_{-1} - \partial_q r(q, q_{-1})}{2r(q, q_{-1})}\right) + \frac{1}{2}R'(q). \tag{4.35}$$

The replacement $q = \varphi(\tilde{q})$, $(\varphi')^2 = R(\varphi)$ allows setting $R = 1$ in the kinetic term L_0. In the case of multiple roots R (or, equivalently, the multiple reducibility of the polynomial r), this replacement and fractional-linear transformations reduce the Lagrangian to form (4.27). In the general case, this reducibility results in more involved chains expressed through elliptic functions. For example, chain (4.34) can be written in terms of the Weierstrass zeta function,

$$q_{xx} = (q_x^2 - 1)\big(\zeta(q + q_1) + \zeta(q - q_1) + \zeta(q + q_{-1}) + \zeta(q - q_{-1}) - 2\zeta(q)\big).$$

Statement 1. *The equations*

$$q_t = \big(R(q) - z^2\big)\left(\frac{1}{q_1 - q} + \frac{1}{q - q_{-1}}\right), \tag{4.36}$$

$$q_t = \big(z^2 - R(q)\big)\left(\frac{z_1 + \partial_q r(q_1, q)}{2r(q_1, q)} + \frac{z_{-1} - \partial_q r(q, q_{-1})}{2r(q, q_{-1})}\right) \tag{4.37}$$

determine variational symmetries of Lagrangian (4.24) *in the respective cases* (4.32) *and* (4.33)*.*

The functions P, Q, f, g, h, R, and r determine systems (4.9), (4.13), (4.15), and (4.17) of the NS type, which coincide with the respective chains (4.28), (4.31), (4.34), and (4.35). This can be explained as follows. Adding and subtracting chain (4.28) and its symmetry (4.29), we obtain system of equations (4.13) in the functions $u = q$ and $v = q_{-1}$ (up to the replacement $t \to it$). Analogously, chain (4.31) results in system (4.9), and chains (4.34) and (4.35) result in the respective systems (4.15) and (4.17). According to the results in [31], we have this amazing fact because the lattice equations determine Bäcklund autotransformations for systems of the NS type.

In order not to lose actual equations among the general formulas for chains and ODE systems, we present two lists of integrable chains. We note that these lists together with chains (4.34) and (4.35) (R and r are determined in (4.16)) completely reproduce the classification results in [32–35] excluding only one modification of the Toda chain presented in Appendix 2. An analogous detailed list of equations of the NS type was presented in [30]. Using the ideas in [31] on the correspondence between chain and continuous equations and taking Lagrangians of forms (4.4) and (4.24), we can therefore unify theories that seem absolutely different.

Lists (T) and (R) are obtained from (4.31) and (4.28) (with the determining functions (4.30)) using rather simple pointwise transformations, in particular, linear transformations depending on x and n: $\tilde{q}_n = q_n + ax + bn + c$. We recall that we use the notation $z = q_x$ and $y = q_1 - q$; μ and ν are arbitrary constants.

The class of the Toda chain comprises

$$z_x = e^y - e^{y_{-1}}, \tag{T$_1$}$$

$$z_x = z(y - y_{-1}), \tag{T$_2$}$$

$$z_x = z(e^y - e^{y_{-1}}), \tag{T$_3$}$$

$$z_x = (\mu - z^2)\left(\frac{1}{y} - \frac{1}{y_{-1}}\right), \tag{T$_4$}$$

$$z_x = (\mu - z^2)(\tanh y - \tanh y_{-1}). \tag{T$_5$}$$

The class of the relativistic Toda chain comprises

$$z_x = z_1 e^y - z_{-1} e^{y_{-1}} - e^{2y} + e^{2y_{-1}}, \tag{R$_1$}$$

$$z_x = z\left(\frac{z_1}{y} - \frac{z_{-1}}{y_{-1}} + y - y_{-1}\right), \tag{R$_2$}$$

$$z_x = z\left(\frac{z_1}{1 + \mu e^{-y}} - \frac{z_{-1}}{1 + \mu e^{-y_{-1}}} + \nu(e^y - e^{y_{-1}})\right), \tag{R$_3$}$$

$$z_x = z(z+1)\left(\frac{z_1}{y} - \frac{z_{-1}}{y_{-1}}\right), \tag{R$_4$}$$

$$z_x = z(z - \mu)\left(\frac{z_1}{\mu + e^y} - \frac{z_{-1}}{\mu + e^{y_{-1}}}\right), \tag{R$_5$}$$

$$z_x = (z^2 + \mu)\left(\frac{z_1 - y}{\mu + y^2} - \frac{z_{-1} - y_{-1}}{\mu + y_{-1}^2}\right), \tag{R$_6$}$$

$$z_x = \frac{1}{2}(z^2 + 1 - \mu^2)\left(\frac{z_1 - \sinh y}{\mu + \cosh y} - \frac{z_{-1} - \sinh y_{-1}}{\mu + \cosh y_{-1}}\right). \tag{R$_7$}$$

4.3. The Hamiltonian description. We now turn from the Lagrangian to the Hamiltonian description of chains (4.23). The Legendre transformation for functional (4.22),

$$p = \frac{\partial L}{\partial q_x}, \qquad H = pq_x - L,$$

results in Hamiltonian systems of the form

$$q_x = \frac{\delta H}{\delta p}, \qquad p_x = -\frac{\delta H}{\delta q}, \qquad H = H(p, q, q_1), \tag{4.38}$$

where the variational derivative is

$$\frac{\delta f}{\delta u} = \sum_k \frac{\partial}{\partial u} T_n^k(f).$$

The general transformation $q, q_x \to q, p$ is given by a formula $p = \varphi(q_x, q, q_1)$, i.e., it is an invertible triangular differential substitution.

We first consider Lagrangians (4.24) with $V \neq 0$. Integrable cases are selected because an additional cancellation occurs in these cases and the Hamiltonian can be presented in the form $H = A(q_1, p) + B(q, p)$. In actual calculations, it is convenient to improve the canonical Darboux brackets using a triangular pointwise replacement, which transforms systems (4.38) into a more general form:

$$u_x = r(u, v)\frac{\delta H}{\delta v}, \qquad v_x = -r(u, v)\frac{\delta H}{\delta u}. \tag{4.39}$$

For instance, relativistic Toda chain (R_3) with $\mu = 1$ and $\nu = 0$ can be conveniently written in the polynomial form

$$u_x = uv(u_1 + u), \qquad v_x = -uv(v + v_{-1}), \tag{4.40}$$

which corresponds to $r = uv$ and $H = u_1 v + uv$. The invertible change of variables $q, q_x \to u, v$ then has the form

$$u = e^q, \qquad v = \frac{q_x}{e^{q_1} + e^q}.$$

We note that we do not need the Hamiltonian property to perform the inverse transition from systems of the form

$$u_x = f(u_1, u, v), \qquad v_x = g(v_{-1}, v, u) \tag{4.41}$$

with $f_v \neq 0$ to relativistic chains; this property may be even absent. Introducing the variables $q = u$ and $z = f(u_1, u, v)$, we find that the replacement is invertible, $u = q$ and $v = \psi(q_1, q, z)$, and it is easy to verify that q does satisfy a chain of the form $q_x = z$, $z_x = \Phi(z_1, z, z_{-1}, q_1, q, q_{-1})$.

Using the list of relativistic Toda chains (R), we can thus obtain a list of integrable Hamiltonian chains. We present them together with the Hamiltonians and structure functions r, which are chosen such that the equations are polynomial or rational. The list of integrable Hamiltonian chains (4.39) is

$$\begin{aligned} &u_x = u_1 + \alpha u + u^2 v, \qquad -v_x = v_{-1} + \alpha v + v^2 u, && (\mathrm{H}_1)\\ &r = 1, \qquad H = u_1 v + \alpha uv + \frac{1}{2} u^2 v^2; \end{aligned}$$

$$\begin{aligned} &u_x = r(u_1 - u + \alpha r_v) + \beta r_v, \qquad -v_x = r(v_{-1} - v + \alpha r_u) + \beta r_u, && (\mathrm{H}_2)\\ &r = k_1 uv + k_2 u + k_3 v + k_4, \qquad r_u r_v \neq 0, \qquad H = (u_1 - u)v + \alpha r + \beta \log r; \end{aligned}$$

and

$$u_x = \frac{2r}{u_1 - v} + r_v + \alpha u + \beta, \qquad -v_x = \frac{2r}{v_{-1} - u} + r_u - \alpha v - \beta, \tag{H_3}$$

where the polynomial r is given by one of three formulas

$$\begin{aligned} &r = k_1(u - v)^2 + k_2(u - v) + k_3, && \alpha = 0,\\ &r = k_1 uv + k_2 u^2 + k_3 v^2, && \beta = 0,\\ &r = k_1 u^2 v^2 + k_2 uv(u + v) + k_3(u^2 + v^2) + k_4 uv + k_5(u + v) + k_6, && \alpha = \beta = 0, \end{aligned}$$

and the respective Hamiltonians are

$$\begin{aligned} &H = \log r - 2\log(u_1 - v) - s(u - v), && s' = \frac{\beta}{r},\\ &H = \log r - 2\log(u_1 - v) + s, && s_v = \frac{\alpha u}{r}, \qquad s_u = -\frac{\alpha v}{r},\\ &H = \log r - 2\log(u_1 - v). \end{aligned}$$

This list was obtained using the symmetry classification [36] before the equivalent list (the equivalence recently became known) of the relativistic Toda chains. Some of the equations in this list were discussed in [31, 37] (in the latter paper, zero-curvature representations and special autotransformations were obtained for several chains).

A nonrelativistic Lagrangian has form (4.24) with $V = 0$, and the replacement $q, q_x \to q, p$ is pointwise: $p = \varphi(q_x, q)$. We therefore choose

$$u = q, \qquad v = q_x$$

as the Hamiltonian variables. In shift-invariant case (4.27), chain (4.39) is

$$r = P(v), \qquad H = G(u_1 - u) + F(v), \qquad G' = g, \qquad F'(v) = \frac{v}{P(v)},$$

and we have

$$r = v^2 - R(u), \qquad H = \frac{1}{2}\log r - \log(u_1 - u)$$

for elliptic chain (4.34).

4.4. Chains with *n*-dependence. Some of the chains above admit integrable n-dependent generalizations. They appear if we treat a chain as Bäcklund transformation iterations at different values of the spectral parameter. In the simplest case (H_1), which corresponds to the NS equation, we can choose the parameter ε to be an arbitrary series in n. For the rest of the Hamiltonian chains in list (H), such generalizations were found in [37]. In the Lagrangian form, we obtain the relativistic Toda chains with coefficient functions described by the same system (4.11) as before, but with a variable value of the first integral (4.12) [38]:

$$\begin{aligned} &f_n' = 2\varepsilon f_n g_n + \alpha f_n - \gamma f_n^2, \qquad g_n' = P(g_n) - \delta f_n^2, \\ &P(g_n) - \gamma f_n g_n + \delta f_n^2 = c_n f_n. \end{aligned}$$

The next example is the chain of form (H_3),

$$u_{n,x} = \frac{2r_n}{u_{n+1} - v_n} + r_{n,v_n}, \qquad -v_{n,x} = \frac{2r_n}{v_{n-1} - u_n} + r_{n,u_n},$$

where $r_n = r_n(u_n, v_n)$ as before, is a polynomial of degree not higher than two in each variable, but with n-dependent coefficients, and is not necessarily symmetric. We let

$$R_n(u) = r_{n,v}^2 - 2r_n r_{n,vv}, \qquad \widetilde{R}_n(v) = r_{n,u}^2 - 2r_n r_{n,uu}$$

denote the discriminants of $r_n(u,v)$. This chain is integrable, i.e., has higher symmetries as soon as the dependence of r on n is such that

$$\widetilde{R}_n(u) = R_{n+1}(u).$$

The algebraic problem of describing such polynomials can be easily solved assuming that at least one of the discriminants R_n has no multiple roots, i.e., that the curve

$$z^2 = R_n(u) = a_n u^4 + 4b_n u^3 + 6c_n u^2 + 4d_n u + e_n$$

is elliptic. A direct calculation shows that the relative invariants

$$g_{2,n} = a_n e_n - 4b_n d_n + 3c_n^2, \qquad g_{3,n} = \det\begin{pmatrix} a_n & b_n & c_n \\ b_n & c_n & d_n \\ c_n & d_n & e_n \end{pmatrix}$$

of the polynomial R_n coincide with the relative invariants $\widetilde{R}_n$. Because $\widetilde{R}_n = R_{n+1}$, these invariants are n-independent, i.e., $g_{2,n} = g_2$ and $g_{3,n} = g_3$. We note that the polynomial R_n is changed under a fractional-linear transformation exactly as the related curve. Hence (see [39]), fractional-linear transformations reduce all the polynomials R_n to the canonical Weierstrass form. The parameter r_n can then be reconstructed up to a single arbitrary parameter,

$$R_n(u) = R(u) = 4u^3 - g_2 u - g_3, \qquad r_n(u,v) = \frac{1}{\sqrt{R(\lambda_n)}} h(u, v, \lambda_n),$$

where

$$h(u,v,w) = \left(uv + uw + vw + \frac{g_2}{4}\right)^2 - (u+v+w)(4uvw - g_3). \tag{4.42}$$

The answer can be formulated in the invariant form: the polynomials R and r_n are defined by formulas (4.20) where J and K are symmetric matrices, J is n-independent, and $K = K_n$ is determined by formula (4.21) with arbitrary constants C_n.

In the degenerate case, we present two Heisenberg chains with a partial anisotropy for which the n-dependence of r is

$$r_n = -\beta_n(u_n - v_n)^2 - \frac{\varepsilon}{\beta_n}$$

and

$$r_n = \gamma_n(u_n^2 + v_n^2) + \beta_n u_n v_n, \qquad \gamma_n^2 - \beta_n^2 = \delta.$$

5. Additional lists of equations

5.1. Hyperbolic equations. At present, the problem of classifying integrable hyperbolic equations

$$u_{xy} = f(x,y,u,u_x,u_y) \tag{5.1}$$

is far from solved. The answer is known in the particular case $f = f(u)$: the corresponding equations are the Liouville, sine-Gordon, and Tzitzeika equations,

$$u_{xy} = e^u, \qquad u_{xy} = \sin u, \qquad u_{xy} = e^u + e^{-2u}.$$

In the general case, the known lists are long and are constantly being extended. As an example, we note the equation

$$u_{xy} = \mathrm{sn}(u)\sqrt{(1-u_x^2)(1-u_y^2)},$$

recently found in [41, 42].

Substantial progress has been achieved in the problem of equations with two pseudoconstants, i.e., with x and y integrals $D_x(A) = D_y(B) = 0$, where the function A depends on $y, u, u_y, u_{yy}, \ldots$ and B depends on $x, u, u_x, u_{xx}, \ldots$ [43]. It was proved that for multicomponent exponential systems

$$u^i_{xy} = \exp\left(\sum_{j=1}^{N} a_i u^i\right), \quad i = 1, \ldots, N,$$

the requirement that $2N$ independent pseudoconstants exist results in Cartan matrices of semisimple Lie algebras [44, 45].

Here, we prove that a list of two-component integrable systems of hyperbolic equations can be easily obtained using symmetries of chains from lists (R) and (H). For example, the relativistic Toda chain

$$q_{xx} = \frac{q_{1,x}q_x}{1+e^{q-q_1}} - \frac{q_{-1,x}q_x}{1+e^{q_{-1}-q}}$$

admits both the higher symmetry

$$q_t = \frac{q_{1,x}q_x}{1+e^{q-q_1}} + \frac{q_{-1,x}q_x}{1+e^{q_{-1}-q}}$$

in Theorem 2 and the symmetry

$$q_y = \frac{1}{q_x}(1+e^{q_1-q})(1+e^{q-q_{-1}}).$$

All chains in list (R) admit such symmetries. For Hamiltonian chains (4.41), these symmetries become

$$u_y = \tilde{f}(u_{-1}, u, v), \qquad \tilde{v}_y = g(v_1, v, u). \tag{5.2}$$

Calculating the mixed derivatives u_{xy} and v_{xy} and using (5.2) and (4.41) to exclude the variables u_1, v_1, and u_{-1}, v_{-1}, we can express the result through u and v and their first derivatives [31]. As a result, we obtain a list of systems of the form

$$u_{xy} = F(u_x, u_y, u, v), \qquad v_{xy} = G(v_x, v_y, v, u).$$

This list is by no means complete and only provides examples of equations with the special structure of the Bäcklund transformations (see the remark after Theorem 3 below).

The examples of integrable hyperbolic systems are

$$u_{xy} = 2uvu_y - u, \qquad v_{xy} = -2uvv_y - v; \tag{h$_1$}$$

$$u_{xy} = \frac{u_x u_y}{r} + r(1 - u_y), \qquad v_{xy} = \frac{v_x v_y}{r} + r(1 + v_y), \qquad r = u + v; \tag{h$_2$}$$

$$u_{xy} = \frac{u_x}{r}(vu_y - 1) + ru_y, \qquad v_{xy} = \frac{v_x}{r}(uv_y + 1) - rv_y, \qquad r = uv + \delta; \tag{h$_3$}$$

$$u_{xy} = \frac{vu_x u_y}{r} - ur, \qquad v_{xy} = \frac{uv_x v_y}{r} - vr, \qquad r = uv - 1; \tag{h$_4$}$$

and

$$u_{xy} = \frac{1}{r}\big(r_u u_x u_y + \tilde{r}(u_x + u_y) + \tilde{r}_v r - \tilde{r} r_v\big), \qquad v_{xy} = \frac{1}{r}\left(r_v v_x v_y - \tilde{r}(v_x + v_y) + \tilde{r}_u r - \tilde{r} r_u\right), \tag{h$_5$}$$
$$\tilde{r} = rr_{uv} - r_u r_v, \qquad r(u, v) = r(v, u), \qquad r_{uuu} = 0.$$

System (h$_4$) is the well-known Pohlmeyer–Lund–Regge model (see [46, 47]), and system (h$_5$) was obtained in [48]. All systems from this list are Lagrangian. For example, we have

$$L = \iint \frac{1}{r}(u_x v_y + r_u u_x - r_v v_y + \tilde{r})\, dx\, dy$$

for (h$_5$). All these systems also admit a complex reduction. For example, (h$_1$) becomes

$$u_{xy} = u - 2i|u|^2 u_y$$

with the substitution $\partial_x \to i\partial_x$ and $\partial_y \to i\partial_y$ under the condition $v = \bar{u}$.

In system (h$_5$), the function $\tilde{r}$, similar to r, satisfies the conditions $\tilde{r}(u, v) = \tilde{r}(v, u)$ and $\tilde{r}_{uuu} = 0$; moreover, $\tilde{r}\tilde{r}_{uv} - \tilde{r}_u \tilde{r}_v = \mathrm{const} \cdot r$. In particular, the equations become simplified under the condition $r = \mathrm{const} \cdot (u - v)^2$ (which corresponds to the isotropic case by virtue of (4.20)),

$$u_{xy} = \frac{2u_x u_y}{u - v} - i(u_x + u_y), \qquad v_{xy} = \frac{2v_x v_y}{v - u} + i(v_x + v_y). \tag{5.3}$$

In the following theorem, we present formulas establishing the exact correspondence between hyperbolic systems (h) and Hamiltonian chains (H). All chains in this theorem can be reduced to the chains in list (H) using simple pointwise replacements.

Theorem 3. *Hyperbolic systems* (h_1)–(h_5) *can be obtained by excluding the shifts from the respective compatible pairs of chains*:

$$u_x = u_1 + u^2 v, \qquad -v_x = v_{-1} + v^2 u, \tag{X$_1$}$$

$$u_y = \frac{u_{-1}}{vu_{-1} - 1}, \qquad -v_y = \frac{v_1}{uv_1 - 1}; \tag{Y$_1$}$$

$$u_x = (u+v)(u_1 - u), \qquad -v_x = (u+v)(v_{-1} - v), \tag{X$_2$}$$

$$u_y = \frac{u+v}{v+u_{-1}}, \qquad -v_y = \frac{u+v}{u+v_1}; \tag{Y$_2$}$$

$$u_x = (uv+\delta)(u_1 + u), \qquad -v_x = (uv+\delta)(v_{-1} + v), \tag{X$_3$}$$

$$u_y = \frac{u+u_{-1}}{vu_{-1} - \delta}, \qquad -v_y = \frac{v+v_1}{uv_1 - \delta}; \tag{Y$_3$}$$

$$u_{x_\pm} = (uv-1)u_{\pm 1}, \qquad -v_{x_\pm} = (uv-1)v_{\mp 1}; \tag{X$_4^\pm$}$$

$$u_{x_\pm} = \frac{2r}{u_{\pm 1} - v} + r_v, \qquad v_{x_\pm} = \frac{2r}{u - v_{\mp 1}} - r_u. \tag{X$_5^\pm$}$$

In the last two formulas, we use the concise notation $x = x_+$ *and* $y = x_-$.

The reduced chains determine the explicit Bäcklund transformations for systems (h). For example, for system (h_1), we have the transformations $(u,v) \to (u,v)_{\pm 1}$,

$$u_1 = u_x - u^2 v, \qquad v_1 = \frac{v_y}{uv_y + 1},$$
$$u_{-1} = \frac{u_y}{vu_y - 1}, \qquad v_{-1} = -v_x - v^2 u,$$

which transform solutions of this system into new solutions. These transformations are inverse to each other by virtue of (h_1).

In conclusion, we note that as in the case of equations of form (5.1), the hierarchy of symmetries for the systems from list (X,Y) can be segregated into two parts, each containing only x or y derivatives and comprising systems of the NS type and their higher symmetries. This hierarchy can be obtained by excluding shifts from the higher symmetries of chains (4.41) and (5.2). For example, the simplest representatives of this hierarchy for Eq. (h_1) are

$$u_{t_1} = u_{xx} - 2(u^2 v_x + u^3 v^2), \qquad -v_{t_1} = v_{xx} + 2(v^2 u_x - v^3 u^2),$$
$$u_{t_{-1}} = u_{yy} + 2u_y^2 v_y, \qquad -v_{t_{-1}} = v_{yy} - 2v_y^2 u_y.$$

5.2. The Sklyanin chain and one more class of integrable equations on a lattice. Because the pairs of chains from Theorem 3 belong to the same hierarchy of integrable equations, a linear combination of the corresponding currents, $D_t = aD_x + bD_y$, is also integrable. Obviously, it is Hamiltonian with the same structure (4.39) and the Hamiltonian $aH_+ + bH_-$. We thus obtain examples of systems of the type

$$u_t = F(u_1, u_{-1}, u, v), \qquad v_t = G(v_1, v_{-1}, v, u). \tag{5.4}$$

These chains have a form that is more symmetric w.r.t. the shifts, and they can therefore be interesting as integrable approximations of the NS-type equations.

In particular, the Ablowitz–Ladik [49] chain,

$$\begin{aligned} u_t &= u_1 - 2u + u_{-1} - uv(u_1 + u_{-1}), \\ -v_t &= v_1 - 2v + v_{-1} - uv(v_1 + v_{-1}), \end{aligned} \tag{5.5}$$

which appears as a linear combination $D_t = -D_x - D_y + 2D_z$ of Eqs. ($\mathrm{X}_4^{\pm}$) and an obvious dilation $u_z = u$, $v_z = -v$, belongs to this class. Another linear combination $D_t = D_x - D_y$ results in the chain

$$u_t = (uv-1)(u_1 - u_{-1}), \qquad v_t = (uv-1)(v_1 - v_{-1}),$$

which can be reduced to the modified Volterra equation

$$u_t = (u^2-1)(u_1 - u_{-1})$$

by setting $v = u$. Therefore, a single pair of chains ($\mathrm{X}_4^{\pm}$) generates as many as three known equations: Pohlmeyer–Lund–Regge equation (h_4) and the Ablowitz–Ladik and Volterra chains.

We now consider an example related to the pair of chains ($\mathrm{X}_5^{\pm}$), which are governed by the Poisson bracket (we hereafter indicate only nonzero brackets)

$$\{u_n, v_n\} = 2r(u_n, v_n) \tag{5.6}$$

and by the Hamiltonians in involution

$$H_{\pm} = \sum_n \left(\frac{1}{2}\log r(u_n, v_n) - \log(u_{n\pm1} - v_n)\right).$$

This is an interesting example because it is closely connected (see [48]) with the known Sklyanin chain [50]. We recall that this chain is determined by the Poisson brackets

$$\{\sigma_n^a, \sigma_n^0\} = (J_b - J_c)\sigma_n^b\sigma_n^c, \qquad \{\sigma_n^a, \sigma_n^b\} = -\sigma_n^0\sigma_n^c \tag{5.7}$$

(the indices a, b, and c indicate a cyclic permutation of 1, 2, and 3) and by the Hamiltonian

$$H = \sum_n \log\left(\sigma_n^0\sigma_{n+1}^0 + \sum_{a=1}^{3}\left(\frac{c_1}{c_0} - J_a\right)\sigma_n^a\sigma_{n+1}^a\right),$$

where c_0 and c_1 are the values of the Casimir functions (which coincide at all lattice sites),

$$c_0 = \sum_{a=1}^{3}(\sigma_n^a)^2, \qquad c_1 = (\sigma_n^0)^2 + \sum_{a=1}^{3} J_a(\sigma_n^a)^2.$$

Landau–Lifshitz equation (4.19) can be obtained from the equations $\sigma_{n,t}^a = (1/\varepsilon)\{H, \sigma_n^a\}$ in the continuous limit $\sigma_n^0 \to 1$, $\sigma_n^a \to (\varepsilon/2)s_n^a(x)$, and $x = i\varepsilon n$ for $\varepsilon \to 0$.

The following statement relates the Sklyanin chain to the current $D_x + D_y$, where $x = x_+$ and $y = x_-$ are times that correspond to the relative Hamiltonians H_+ and H_-.

Statement 2. *Let $s_n = S(u_n, v_n)$ be vector* (4.18) *and the polynomial r be related to the matrix J by formulas* (4.20) *and* (4.21):

$$r(u,v) = \frac{i}{4}(u-v)^2\big\langle S(u,v), KS(u,v)\big\rangle, \qquad J = CI - \det(K)K^{-1}$$

(*i.e.*, $K = \operatorname{diag}(K_1, K_2, K_3)$ *and* $J_a = C - K_bK_c$). *The variables*

$$\sigma_n^0 = \rho\sqrt{\det K}\,\langle s_n, Ks_n\rangle^{-1/2}, \qquad \sigma_n = -\rho\langle s_n, Ks_n\rangle^{-1/2}K^{1/2}s_n$$

then satisfy brackets (5.7)*, and the values of the functions and the Hamiltonian are* $c_0 = \rho^2$, $c_1 = C\rho^2$, *and* $H = -H_+ - H_- + \text{const}$.

An arbitrary symmetric polynomial r of second order in each variable can be reduced to the form above by a fractional-linear replacement $u \to v$, which corresponds to the orthogonal transformation diagonalizing the matrices J and K.

We also present the Hamiltonian structure in terms of the spin variables s_n,

$$\{s_n^a, s_n^b\} = \langle s_n, Ks_n\rangle s_n^c, \qquad H = -\sum_n \log \frac{\langle s_n, Ks_n\rangle}{1 + \langle s_n, s_{n+1}\rangle}.$$

The chains corresponding to $H_\pm$ can be written in the compact form

$$2is_{x_\pm} = \langle s, Ks\rangle\big(S(u_{\pm1}, v) - S(u, v_{\mp1})\big) - 2i[s, Ks]$$

if we use the elementary property of mapping (4.18)

$$\frac{i\big[S(u,v), S(p,q)\big] + S(u,v) + S(p,q)}{1 + \big\langle S(u,v), S(p,q)\big\rangle} = S(p,v).$$

These chains individually are not consistent with the reality condition for the vector s. Their linear combinations $D_x + D_y$ and $i(D_x - D_y)$ already have this property, and we obtain a chain in the unit sphere ($s \in \mathbb{R}^3$, $|s| = 1$),

$$s_t = a\langle s, Ks\rangle \left(\frac{[s, s_1]}{1 + \langle s, s_1\rangle} + \frac{[s, s_{-1}]}{1 + \langle s, s_{-1}\rangle}\right) - 2a[s, Ks] + b\langle s, Ks\rangle \left(\frac{s + s_1}{1 + \langle s, s_1\rangle} - \frac{s + s_{-1}}{1 + \langle s, s_{-1}\rangle}\right), \qquad (5.8)$$

where a and b are arbitrary real constants. The case $b = 0$ corresponds to the Sklyanin model, and the chain corresponding to $a = 0$ is its symmetry.

The variables σ and s coincide at $K = I$ and $\rho = -1$, and we obtain the Heisenberg chain

$$s_t = a\left(\frac{[s, s_1]}{1 + \langle s, s_1\rangle} + \frac{[s, s_{-1}]}{1 + \langle s, s_{-1}\rangle}\right) + b\left(\frac{s + s_1}{1 + \langle s, s_1\rangle} - \frac{s + s_{-1}}{1 + \langle s, s_{-1}\rangle}\right). \qquad (5.9)$$

This chain was introduced by Ragnisco and Santini [51] for arbitrary a and b.

We can also use the spin variables s to rewrite other equations related to chains ($\mathrm{X}_5^\pm$). For example, system (h_5) (where r is determined by formulas (4.20) and (4.21)) becomes

$$\begin{aligned} s_{xy} = p[s, s_x + s_y] + \frac{\langle s, Ks\rangle}{2\big\langle s, [s_x, s_y]\big\rangle}\big(p_y[s, s_x] - p_x[s, s_y]\big) + {} \\ + \frac{\langle s_x, Ks\rangle\langle s_y, s_y\rangle[s, s_x] - \langle s_y, Ks\rangle\langle s_x, s_x\rangle[s, s_y]}{\langle s, Ks\rangle\big\langle s, [s_x, s_y]\big\rangle} - \langle s_x, s_y\rangle s, \end{aligned}$$

where $p = i\tilde{r}/r = \langle Ks, Ks\rangle/\langle s, Ks\rangle - (\operatorname{tr} K)/2$. In particular, system (5.3) becomes

$$s_{xy} = [s, s_x + s_y] - \langle s_x, s_y\rangle s$$

in the isotropic case $K = -2I$.

5.3. Chains of the Volterra equation type. There are four lists of integrable chains obtained within the classification in the framework of the symmetry approach. Three of the lists, the Toda chain class (4.31) and (4.34), the relativistic Toda chain class (4.28), (4.30), and (4.35), and the Hamiltonian relativistic chain class (see list (H) in Sec. 4.3) are in the above scheme. The fourth list, the Volterra equation class, fits this scheme only partially. This class is named after the simplest chain

$$u_x = u(u_1 - u_{-1}). \tag{5.10}$$

The Volterra equation class comprises

$$u_x = P(u)(u_1 - u_{-1}), \tag{V$_1$}$$

$$u_x = P(u^2)\left(\frac{1}{u_1+u} - \frac{1}{u+u_{-1}}\right), \tag{V$_2$}$$

$$u_x = Q(u)\left(\frac{1}{u_1-u} + \frac{1}{u-u_{-1}}\right), \tag{V$_3$}$$

$$u_x = \frac{F(u_1,u,u_{-1}) + \nu\big(F(u_1,u,u_1)\big)^{1/2}\big(F(u_{-1},u,u_{-1})\big)^{1/2}}{u_1-u_{-1}}, \tag{V$_4$}$$

$$u_x = f(u_1-u) + f(u-u_{-1}), \qquad f' = P(f), \tag{V$_5$}$$

$$u_x = f(u_1-u)f(u-u_{-1}) + \mu, \qquad f' = \frac{P(f)}{f}, \tag{V$_6$}$$

$$u_x = \big(f(u_1-u) + f(u-u_{-1})\big)^{-1} + \mu, \qquad f' = P(f^2), \tag{V$_7$}$$

$$u_x = \big(f(u_1+u) - f(u+u_{-1})\big)^{-1}, \qquad f' = Q(f), \tag{V$_8$}$$

$$u_x = \frac{f(u_1+u) - f(u+u_{-1})}{f(u_1+u) + f(u+u_{-1})}, \qquad f' = \frac{P(f^2)}{f}, \tag{V$_9$}$$

$$u_x = \frac{f(u_1+u) + f(u+u_{-1})}{f(u_1+u) - f(u+u_{-1})}, \qquad f' = \frac{Q(f)}{f}, \tag{V$_{10}$}$$

$$u_x = \frac{\big(1-f(u_1-u)\big)\big(1-f(u-u_{-1})\big)}{f(u_1-u) + f(u-u_{-1})} + \mu, \qquad f' = \frac{P(f^2)}{1-f^2}, \tag{V$_{11}$}$$

where $\nu \in \{0, \pm 1\}$, $P''' = Q^V = 0$,

$$F(z_1,z_2,z_3) = (\alpha z_2^2 + 2\beta z_2 + \gamma)z_1 z_3 + (\beta z_2^2 + \lambda z_2 + \delta)(z_1+z_3) + \gamma z_2^2 + 2\delta z_2 + \varepsilon,$$

and μ and the coefficients of the polynomials P, Q, and F are arbitrary constants. The complete list is presented only in [52] (see also [10, 53]).

All the equations except (V$_4$) with $\nu = 0$ are integrable at least because they can be reduced by discrete substitutions to Volterra equation (5.10) or to the polynomial form of the Toda chain

$$u_x = u(v_1 - v), \qquad v_x = u - u_{-1}. \tag{5.11}$$

It is easy to see that one of the two transformations

$$\tilde{u} = f(u_1+u), \qquad \tilde{u} = f(u_1-u)$$

reduces chains (V$_5$) and (V$_6$) to (V$_1$), chains (V$_7$), (V$_9$), and (V$_{11}$) to (V$_2$), and chains (V$_8$) and (V$_{10}$) to (V$_3$). Equations (V$_1$), (V$_2$), (V$_3$), and (V$_4$) with $\nu \neq 0$ are more involved modifications of Eqs. (5.10) and (5.11), which are related to these equations by Miura transformations (see [52, 54]). For example, the change

$$\tilde{u} = \frac{bu_1u + (a-b)(u_1-u)/2 + a}{u_1+u}, \qquad c = \frac{a+b}{2}$$

reduces Eq. (V_2) with $P = (u^2-1)(a^2-b^2u^2)$ to the modified Volterra equation (V_1) with $P = u^2 - c^2$, which in turn can be reduced to Volterra equation (5.10) by the variable change

$$\tilde{u} = (u+c)(u_1-c).$$

Chain (V_4) with $\nu = 0$ becomes the Krichever–Novikov equation in the continuous limit (see Eq. (A.4) below); it occupies a special place in the list because it cannot be reduced to Eqs. (5.10) and (5.11) by discrete substitutions. Nevertheless, this chain has an infinite hierarchy of higher symmetries and conservation laws because it has the local master symmetry

$$u_{n,y} = (c+n)u_{n,x}. \tag{5.12}$$

The dependence of the coefficients of the polynomial F on the time y is introduced as in Sec. 6.3 (see Eq. (6.16) and the comment below), the initial data being a symmetric polynomial $h(z_1,z_2) = F(z_1,z_2,z_1)$ of the second order in each variable.

Chain (V_4) with $\nu = 0$ admits an integrable generalization with coefficients that are not constants but are periodic in n [55]. The polynomial F_n for such a generalization is

$$F_n(z_1,z_2,z_3) = (\alpha z_2^2 + 2\beta_n z_2 + \gamma_n)z_1z_3 + (\beta_{n+1}z_2^2 + \lambda z_2 + \delta_n)(z_1+z_3) + \gamma_{n+1}z_2^2 + 2\delta_{n+1}z_2 + \varepsilon, \tag{5.13}$$

where the n-dependent coefficients are doubly periodic in n,

$$\beta_{n+2} = \beta_n, \qquad \gamma_{n+2} = \gamma_n, \qquad \delta_{n+2} = \delta_n.$$

The master symmetry is as above, and the y dependence is introduced analogously. However, the corresponding polynomial $h_n(z_1,z_2) = F_n(z_1,z_2,z_1)$ is n dependent and nonsymmetric. Local master symmetries for some of the other equations in list (V) can be found in [56].

Some of the equations from the above list are closely related to equations in other sections. For example, chains (V_1) written in the variables

$$\tilde{u}_n = u_{2n}, \qquad \tilde{v}_n = u_{2n-1} \tag{5.14}$$

become chains (6.8) and are therefore related to generalizations of Toda chain (4.31). Transition (5.14) transforms Eq. (V_4) with $\nu = 0$ and with a polynomial of general form (5.13) into a system of two equations with constant coefficients,

$$u_x = \frac{2r}{v_1 - v} + r_v, \qquad v_x = \frac{2r}{u - u_{-1}} - r_u, \qquad r = r(u,v), \qquad r_{uuu} = r_{vvv} = 0 \tag{5.15}$$

(the polynomial r is constructed from h_n, $r(\tilde{u}_n,\tilde{v}_n) = h_{2n-1}(u_{2n},u_{2n-1})$). Differentiating the first equation w.r.t. x, we can exclude the variables v_i and obtain Eq. (4.34) in the variable $q = u$ with the polynomial R, which is the discriminant of r, $R(u) = r_v^2 - 2rr_{vv}$.

System (5.15) and therefore chain (V_4) with $\nu = 0$ are related to Eq. (4.34). Master symmetry (5.12) can be rewritten as the master symmetry of system (5.15),

$$u_{n,y} = (c+2n)u_{n,x}, \qquad v_{n,y} = (c+2n-1)v_{n,x}.$$

The dependence of the coefficients of the polynomial r on the time y can be introduced using Eq. (6.16) with the initial condition $\tilde{r}(0,u,v) = r(u,v)$.

5.4. Other integrable equations on a lattice. To complete the picture, we present a few more interesting examples of integrable equations on a lattice (one- and two-dimensional).

In [55], we meet an interesting generalization of Toda chain (4.26) with variable coefficients,

$$\frac{q_{n,xx}}{c_{n+1}c_n} = \exp\left(\frac{q_{n+1}-q_n}{c_{n+1}}\right) - \exp\left(\frac{q_n-q_{n-1}}{c_n}\right), \qquad c_n = an+b,$$

where a and b are arbitrary constants. This equation can be reduced to chain (A.13) with $\mu = 0$ (see below) and therefore to the Toda chain by the transformation

$$\frac{q_n}{c_{n+1}c_n} = (T_n - 1)\left(\frac{\tilde{q}_n}{c_n} + \lambda_n\right),$$

where $\tilde{q}_n$ is a solution of (A.13) and λ_n is determined from the equation

$$c_{n+1}(\lambda_{n+1}-\lambda_n) - c_{n-1}(\lambda_n - \lambda_{n-1}) = \log c_n.$$

The Bogoyavlenskii chain,

$$u_{n,x} = u_n\left(\sum_{i=1}^{M} u_{n+i} - \sum_{i=1}^{M} u_{n-i}\right),$$

generalizes Volterra equation (5.10) and is integrable for any $M \geq 1$. Some modifications of this chain were found in [57].

We now briefly discuss hyperbolic chains of the type

$$F_n(u_{n,x}, u_{n+1,x}, u_n, u_{n+1}) = 0,$$

among which the Bäcklund transformation

$$u_{n,x} + u_{n+1,x} = \alpha_n \sin(u_{n+1} - u_n) \tag{5.16}$$

for the sine-Gordon equation is the best known. The Miura transformation relates this chain to dressing chain (3.7) (see Sec. 3.2). A general classification of equations of this type is still lacking although there are numerous examples (see, e.g., [58]). To the best of our knowledge, all these examples except the chain

$$u_{n,x}u_{n+1,x} = r(u_n, u_{n+1}), \qquad r(u,v) = r(v,u), \qquad r_{uuu} = 0$$

are related to Eq. (5.16) by a sequence of differential substitutions. It was shown in [59] that the above chain determines the Bäcklund autotransformation for Krichever–Novikov equation (A.4) (see below); R is then the discriminant of r (4.16).

To conclude this section, we present the list in [60] of two-dimensional equations on the lattice, which comprises integrable two-dimensional generalizations of all chains in list (T) (see Sec. 4.2) and all chains of form (6.8) (see Sec. 6.2 below), among them the well-known two-dimensional generalizations of the Toda chain and the Volterra equation.

The two-dimensional chains are

$$q_{xy} = e^{q_1-q} - e^{q-q_{-1}}, \tag{2D$_1$}$$

$$q_{xy} = q_x(q_1 - 2q + q_{-1}), \tag{2D$_2$}$$

$$q_{xy} = q_x(e^{q_1-q} - e^{q-q_{-1}}), \tag{2D$_3$}$$

$$q_{xy} = (q_x + a)(q_y - a)\big(f(q_1-q) - f(q-q_{-1})\big), \qquad f' = f^2 - b^2, \tag{2D$_4$}$$

$$u_y = u(v_1 - v), \qquad v_x = u - u_{-1}, \tag{2D$_5$}$$

$$u_y = u(v_1 - v), \qquad v_x = v(u - u_{-1}), \tag{2D$_6$}$$

$$u_x = (u^2 - b^2)(v_1 - v), \quad u_y = (u^2 - b^2)(w_1 - w), \quad v_y = w_x = (v+a)(w-a)(u - u_{-1}), \tag{2D$_7$}$$

where a and b are arbitrary constants. The chains from this list determine the Bäcklund autotransformations for (2+1)-dimensional systems of equations analogous to the Davey–Stewartson equation. All of them are mutually related and can be eventually reduced to chain ($2D_5$). Some of the variable changes are rather simple, and the others are two-dimensional discrete analogues of the Miura transformation. The list of transformations is

$$
\begin{array}{lll}
(2\mathrm{D}_1) \to (2\mathrm{D}_5)\colon & u = e^{q_1 - q}, & v = q_y, \\
(2\mathrm{D}_2) \to (2\mathrm{D}_5)\colon & u = q_x, & v = q - q_{-1}, \\
(2\mathrm{D}_3) \to (2\mathrm{D}_6)\colon & u = q_x, & v = e^{q - q_{-1}}, \\
(2\mathrm{D}_4) \to (2\mathrm{D}_7)\colon & u = f(q_1 - q), & v = q_x, \qquad w = q_y, \\
(2\mathrm{D}_7) \to (2\mathrm{D}_6)\colon & \tilde{u} = (u + b)(v_1 + a), & \tilde{v} = (u - b)(w - a), \\
(2\mathrm{D}_6) \to (2\mathrm{D}_5)\colon & \tilde{u} = uv_1, & \tilde{v} = v + D_x^{-1} u_y.
\end{array}
$$

6. Local master symmetries

We obtain the so-called integrability conditions by requiring that an evolution equation from the given class have higher local symmetries and conservation laws. These conditions can be conveniently written as a series of conservation laws with the densities ρ_j $(j = 1, 2, \dots)$, which can be calculated directly from the right-hand side of the equation

$$\mathbf{u}_t = \mathbf{G}(x, \mathbf{u}, \mathbf{u}_x, \dots, \partial^n \mathbf{u}/\partial x^n), \quad n \geq 2 \tag{6.1}$$

(see Appendix 1). Clearly, these necessary conditions do not ensure the existence of higher symmetries and conservation laws. This difficulty can sometimes be overcome using master symmetries. If $u_\tau = H$ is a master symmetry, then the higher conservation law densities are $D_\tau^k(\rho)$ $(k = 1, 2, \dots)$, where ρ is a nontrivial density from the necessary conditions.

The master symmetry was introduced by Fokas and Fuchssteiner [61]. The function H, which determines the master symmetry, is nonlocal for many equations (e.g., for the KdV, NS, and Toda chain equations). However, Fuchssteiner [8] found a local master symmetry for the Landau–Lifshitz model, which is the universal equation for the classification of the NS-type equations. In this section, we present local master symmetries (which are much more convenient to work with) for universal equations from the lists of generalizations of the Toda chain and for other examples of integrable equations.

6.1. Locality conditions. We first discuss the general necessary conditions for the locality of master symmetries for evolution equations (6.1). We recall the derivation of integrability conditions for scalar evolution equations (see, e.g., [9, 10] for the details). The determining equation for higher symmetries $u_{t'} = A$ of scalar equation (6.1) (the function A depends on x derivatives of u up to the order m) can be written in the commutator form: $[A, G] = 0$. The commutator of two functions F_1 and F_2 is

$$[F_1, F_2] = D_1(F_2) - D_2(F_1), \qquad D_j(u) = F_j.$$

Linearizing the determining equation, we obtain

$$[A_*, G_*] = D_{t'}(G_*) - D_t(A_*),$$

where A_* and G_* are the linearizing differential operators of the respective orders m and n,

$$A_* = \sum_{k=0}^{m} \frac{\partial A}{\partial u_k} D_x^k, \qquad u_k = \frac{\partial^k u}{\partial x^k}$$

for example. Because $m \gg n$, we obtain the equation

$$L_t = [G_*, L], \tag{6.2}$$

where L is a pseudodifferential operator of the first order, $L = a_1 D_x + a_0 + a_{-1} D_x^{-1} + \dots$. The residue of such an operator is the coefficient a_{-1} in the scalar case ($\operatorname{tr} a_{-1}$ in the general case). Integrability conditions can be expressed through residues of powers of the operator L,

$$\rho_j = \operatorname{res} L^j, \quad j = -1, 0, 1, 2, \dots,$$

and are local conservation laws,

$$(\rho_j)_t = (\sigma_j)_x$$

(the so-called canonical conservation laws).

The master symmetry $u_\tau = H$ can be found from the equation

$$\big[[H, G], G\big] = 0,$$

which is analogous to the symmetry-determining equation. We know the formula for the higher symmetries $u_{t_k} = A_k$,

$$A_k = \operatorname{ad}_H^k(G), \qquad \operatorname{ad}_H(G) = [H, G]. \tag{6.3}$$

Linearizing the relation $[H, A_k] = A_{k+1}$, we obtain

$$\big[H_*, (A_k)_*\big] = D_\tau\big((A_k)_*\big) - D_{t_k}(H_*) + (A_{k+1})_*,$$

and Eq. (6.2) is therefore replaced by the operator equation

$$L_\tau = [H_*, L] + \tilde{L}, \tag{6.4}$$

where $\tilde{L}$ is another solution of Eq. (6.2).

We now show how operator equation (6.4) results in a recursive formula for the canonical densities $\rho_j = \operatorname{res} L^j$. For this, we consider the operator

$$\tilde{L} = b_1 L^{r+1} + b_0 L^r + b_{-1} L^{r-1} + \dots, \quad r \geq 1. \tag{6.5}$$

For $b_1 \neq 0$, this case corresponds to a nontrivial master symmetry of weight r (i.e., such that formula (6.3) results in higher symmetries of orders $n + kr$). Rewriting operator equation (6.4) in the form

$$L^j_\tau = [H_*, L^j] + j L^{j-1} \tilde{L} \tag{6.6}$$

and evaluating the residues, we find that

$$(\rho_j)_\tau \sim j(b_1 \rho_{j+r} + b_0 \rho_{j+r-1} + b_{-1} \rho_{j+r-2} + \cdots) \tag{6.7}$$

for any integer j (the equivalence means the equality up to total derivatives w.r.t. x).

Substituting $j = -r - 1$ in (6.7), we obtain the first condition for the existence of a local master symmetry,

$$\rho_{-1} \sim 0 \quad \Leftrightarrow \quad \rho_{-1} \in \operatorname{Im} D_x$$

for any order $n \geq 2$ of an equation and for any weight $r \geq 1$ of a master symmetry. This condition implies that the canonical density ρ_{-1} must be trivial. For example, for Burgers-type equations (B_2) and (B_3) (see

Appendix 1) we find that $\partial G/\partial u_{xx} = 1/u^2$, i.e., $\rho_{-1} = cu$ ($c \neq 0$). These equations therefore have no local master symmetries of positive weight. Burgers equation (B_1) with $a = 0$ does satisfy this condition because it has the local master symmetry $u_\tau = xu_t + u^2$.

The general symmetry approach theory implies that orders of an equation and its higher symmetries are odd for an equation with higher conservation laws of sufficiently high order (cf. [9]). The weight of a local master symmetry of such an equation is greater than or equal to 2. Using relation (6.7) with $j = 1-r$, we can show that (for $r \geq 2$)

$$\rho_1 \in \operatorname{Im} D_x.$$

For equations of form (A.2) in Appendix 1, we find

$$\rho_1 = c_1 \frac{\partial F}{\partial u_{n-2}} + c_2, \qquad c_1 \neq 0.$$

Hence, the (nonlinear) equations of the third and fifth orders from lists (K) and (K^5) in Appendix 1 admit no local master symmetries.

For equation systems of form (A.5) (see Appendix 1), we can analogously prove that if a system has a local master symmetry and the canonical density ρ_1 from (A.6) is trivial, then the densities ρ_2 and ρ_3 must be trivial as well. For NS equation (A.5) with $F = 2u^2v$ and $G = -2v^2u$, we obtain $\rho_1 = 0$, and $\rho_2 = 8uv + \mathrm{const}$ is therefore a nontrivial density, i.e., all the master symmetries of this system are also nonlocal.

More subtle considerations show that scalar evolution equations (6.1) may have a nontrivial local master symmetry only if they are a kind of Burgers equation (linearized equations). But this is not true for the NS-type systems and for equations on a lattice of the Toda and Volterra chain types. We demonstrate this in Secs. 6.2 and 6.3.

6.2. Local master symmetries for a class of integrable equations. We now write local master symmetries for divergent systems of equations (4.6) and (4.7) and (in the proper gauge) for the corresponding Toda chain generalizations (4.28), (4.31), and (4.30) (or, equivalently, for chains in lists (T) and (R) in Sec. 4.2).

Chains (4.28) and (4.31) can be conveniently written in the gauge related to divergent systems (4.6) (chains thus rewritten determine the Bäcklund autotransformations for divergent systems [31]). With the substitution

$$u_n = h(q_{n+1} - q_n), \qquad v_n = q_{n,x}$$

(cf. gauge (4.8)), chain (4.31) becomes

$$\begin{aligned} &u_{n,x} = P(u_n)(v_{n+1} - v_n), \\ &v_{n,x} = Q(v_n)(u_n - u_{n-1}), \\ &P(w) = \varepsilon w^2 + \alpha w + \beta, \qquad Q(w) = \varepsilon w^2 + \gamma w + \delta. \end{aligned} \tag{6.8}$$

Using the linear transformations of u, v, and x, we can segregate three cases in which the local master symmetries are

$$\begin{aligned} u_{n,y} &= P(u_n)\big[(2n+k+c)v_{n+1} - (2n+c)v_n\big] + \gamma u_n^2, \\ v_{n,y} &= Q(v_n)\big[(2n-1+k+c)u_n - (2n-1+c)u_{n-1}\big] + \alpha v_n^2, \end{aligned}$$

where c is a constant (responsible for adding $u_{n,x}$ and $v_{n,x}$) and the number k is given differently in each of the three cases:

$$\begin{aligned} &P = w, && Q = 1, && k = 4; \\ &P = w, && Q = w, && k = 3; \\ &P = w^2 + \beta, && Q = w^2 + \delta, && k = 2. \end{aligned} \tag{6.9}$$

These master symmetries were found in [62, 63].

Master symmetries of divergent systems (4.6) are obtained by excluding shifts by virtue of (6.8) (we assume that $p = u_n$ and $z = v_n$) from the chain corresponding to the differentiation $D_\tau = xiD_t + D_y$. These symmetries are

$$p_\tau = xip_t + (k+c)p_x + kP(p)z + \gamma p^2,$$
$$z_\tau = xiz_t + (c-1)z_x + kQ(z)p + \alpha z^2,$$

where we consider the same three cases for P and Q that are segregated by linear substitutions and k is given in (6.9). The master symmetry of the system corresponding to $P = Q = w^2$ was obtained in [64].

In chain (4.28), we substitute

$$u_n = q_{n,x}, \qquad v_n = q_{n,x} f(q_{n+1} - q_n) + g(q_{n+1} - q_n),$$

which results in the system of equations of the form

$$\begin{aligned} u_{n,x} &= A_n(u_n, v_n)(u_{n+1} - u_n) + P(u_n)(v_n - v_{n-1}), \\ v_{n,x} &= A_n(u_n, v_n)(v_n - v_{n-1}) + Q(v_n)(u_{n+1} - u_n). \end{aligned} \tag{6.10}$$

In the new notation, integrable chains are segregated by formulas (6.8) and by the equation in A_n

$$A_n^2(u,v) - (\lambda_n + \gamma u + \beta v + 2\varepsilon uv)A_n(u,v) + P(u)Q(v) = 0 \tag{6.11}$$

(we assume that $f \neq 0$). We note that chains (4.28) are transformed in (6.10) with λ_n (and hence A_n) having no explicit dependence on n. We consider chains of a more general type. An example corresponding to $P = Q = w$ was presented in [22], and the whole class was considered in [38].

The local master symmetries are

$$\begin{aligned} u_{n,y} &= (2n+k+c)A_n(u_n, v_n)(u_{n+1} - u_n) + P(u_n)\big((2n+k+c)v_n - (2n-k+c)v_{n-1}\big), \\ v_{n,y} &= (2n-k+c)A_n(u_n, v_n)(v_n - v_{n-1}) + Q(v_n)\big((2n+k+c)u_{n+1} - (2n-k+c)u_n\big), \end{aligned} \tag{6.12}$$

where P, Q, and k are given in (6.9) and the parameter λ_n is y dependent by virtue of the equation $\lambda_{n,y} = -\lambda_n^2$ in the first two cases and by virtue of the equation $\lambda_{n,y} = 4\beta\delta - \lambda_n^2$ in the last case. A local master symmetry for the relativistic case was found in [65], but in a different gauge.

6.3. Elliptic case. We present the local master symmetries for chains (4.34), (4.35) (r and R are from (4.16)), and (H_3) (see Sec. 4.3). We call these chains the elliptic-type equations because they are closely related to systems (4.15) and (4.17) and therefore to Landau–Lifshitz equation (4.19), which depends elliptically on the spectral parameter in the linear problem.

As is known, the master symmetry for Eq. (4.19) is [8]

$$s_\tau = \big[s, x(s_{xx} + Js) + s_x\big] = xs_t + [s, s_x].$$

We now show that the master symmetries for the constructed chains result in the same formula. We do not even need to change the gauge. For example, the master symmetries for chains (4.34) and (4.35) must be written in the variables q_n and $z_n = q_{n,x}$. However, we must deal with a complex dependence of the master symmetry on time.

In the simpler nonrelativistic case (4.34), the master symmetry is

$$z_{n,y} = (\lambda + 2n)z_{n,x} + q_{n,t}, \qquad q_{n,y} = (\lambda + 2n)q_{n,x}, \tag{6.13}$$

where t is the time of the highest symmetry (4.36) and λ is an arbitrary constant. All the master symmetries give examples of equations that are integrable in a sense and depend on time (y in the discrete case and τ in the continuous case) and always on the spatial variable (n and x correspondingly). For example, we can exclude the variable z_n from master symmetry (6.13) and obtain a nice integrable equation with an explicit (nonlinear) n dependence,

$$q_{n,yy} = \left((\lambda+2n)R(q_n) - \frac{q_{n,y}^2}{\lambda+2n}\right)\left(\frac{\lambda+2n+1}{q_{n+1}-q_n} - \frac{\lambda+2n-1}{q_n-q_{n-1}}\right) + \frac{(\lambda+2n)^2}{2}R'(q_n).$$

In the relativistic case, the formula for the master symmetry is also simple. It is

$$z_{n,y} = (\lambda+n)z_{n,x} + q_{n,t}, \qquad q_{n,y} = (\lambda+n)q_{n,x} \tag{6.14}$$

for chain (4.35) (t is the time of the highest symmetry (4.37)) and is

$$u_{n,y} = (\lambda+n)u_{n,x}, \qquad v_{n,y} = (\lambda+n-1)v_{n,x} \tag{6.15}$$

for chain (H_3), which also contains a Hamiltonian representation for (4.35). The essential difference from the nonrelativistic case is that the dependence of the master symmetry on the time y is introduced in coefficients of the polynomial r, which determines chains (4.35) and (H_3). This dependence is determined by the partial differential equation in $\tilde{r} = \tilde{r}(y,u,v)$

$$\tilde{r}_y = \tilde{r}\tilde{r}_{uv} - \tilde{r}_u\tilde{r}_v \tag{6.16}$$

($\tilde{r}$ is obtained from the polynomial r by substituting functions depending on y for the coefficients of the polynomial). The polynomial in the right-hand side of (6.16) has the same form as the polynomial $\tilde{r}$ but with different coefficients. Equation (6.16) is therefore equivalent to a closed system of ODEs in the coefficients of the polynomial $\tilde{r}$, which has a solution for arbitrary initial data $\tilde{r}(0,u,v) = r(u,v)$.

For example, if we want to construct a master symmetry for chain (H_3) with $r = u+v$, then solving Eq. (6.16) with this initial condition, we obtain the polynomial $\tilde{r} = u+v-y$. For Eq. (H_3) with the found polynomial $\tilde{r}$, instead of r, we have the master symmetry, which allows constructing higher symmetries and densities of conservation laws. These symmetries and densities persist for all values of the parameter y. For $y=0$, we obtain the set of higher symmetries and conserved densities for initial chain (H_3) with $r = u+v$. The master symmetries are thus constructed not for an equation itself but for its generalization depending on the additional parameter y. Such master symmetries generate higher symmetries and conservation laws for both this generalization and the initial equation.

We note that the same scheme for constructing master symmetries (and formulas (6.15) and (6.16)) is suitable not only for a nonsymmetric polynomial r with n-independent coefficients but also for a general n-dependent polynomial r_n (see Sec. 4.4).For polynomial (4.42) for example, Eq. (6.16) results in the y dependence of the parameter λ_n; this dependence is given by the equation

$$\lambda_{n,y} = \sqrt{R(\lambda_n)}\,.$$

As in Sec. 6.2, the chain master symmetries (6.13)–(6.15) can be rewritten in the form of master symmetries of systems (4.15) and (4.17),

$$u_\tau = xiu_t + u_x, \qquad v_\tau = xiv_t - v_x,$$
$$u_\tau = xiu_t + u_x, \qquad v_\tau = xiv_t.$$

In the second case, the dependence on the time τ is introduced in the coefficients of the polynomial r again using Eq. (6.16), in which the variable y is replaced by τ. Using transition formulas (4.18), (4.20), and (4.21), we can prove that the master symmetry in the first case coincides with the master symmetry of Landau–Lifshitz equation (4.19).

The Sklyanin chain, which was discussed in Sec. 5.2, must presumably have a local master symmetry because the chain is an elliptic-type equation. However, we presently know such a master symmetry only for the particular case corresponding to the Heisenberg magnet, namely, for chain (5.9) with $b = 0$ and $a = 1$:

$$s_{n,y} = \left[s_n, \frac{(\varepsilon + n)s_{n+1}}{1 + \langle s_n, s_{n+1} \rangle} + \frac{(\varepsilon + n - 1)\, s_{n-1}}{1 + \langle s_n, s_{n-1} \rangle} \right].$$

7. Necessary condition for the integrability of two-dimensional equations

We now consider equations that can be transformed into their nontrivial master symmetries by proper invertible changes of variables. We call such equations B-integrable (by analogy with the S- and C-integrable Calogero equations). In the one-dimensional case, the Burgers equation

$$u_t = u_{xx} + 2uu_x$$

is B-integrable. Substituting

$$u \to x^{1/2} u, \qquad x \to 2x^{1/2}, \qquad t \to \tau,$$

it becomes the equation

$$u_\tau = xu_t + u^2 + \frac{3}{2} u_x,$$

which is its master symmetry. The B-integrable equations therefore satisfy the conditions

$$\big[[D_\tau, D_t], D_t \big] = 0, \qquad D_\tau \sim D_t, \tag{7.1}$$

where the respective evolution differentiations D_t and D_τ correspond to the equation and its master symmetry and the equivalence is up to invertible transformations. Moreover, a master symmetry must increase the order of a symmetry such that the sequence of recursively determined higher symmetries

$$[D_\tau, D_t] = D_{t_1}, \qquad [D_\tau, D_{t_k}] = D_{t_{k+1}}$$

(D_{t_k} corresponds to the kth higher symmetry) is infinite.

If an equation admits B-integrability, it is necessarily integrable. In the (1+1)-dimensional case, B-integrable equations are presumably just linearizable equations, while the situation is changed in the (2+1)-dimensional case. The class of two-dimensional B-integrable equations that are simultaneously S-integrable (i.e., integrable using the inverse scattering method) is rather wide. To demonstrate this, we present several examples pertaining to the Kadomtsev–Petviashvili (KP) and Davey–Stewartson (DS) equations. The definition of B-integrability is constructive, which feeds the hope that it can be used to classify integrable cases and to refine the integrability test. An analogous example (the pointwise relation between the KP equations and the cylindric KP equation) was given in [66].

7.1. The Kadomtsev–Petviashvili equation. We discuss the examples pertaining to the KP equation. In [67], the KP equation

$$u_{t_3} = u_{xxx} - 6uu_x + 3v_{yy}, \qquad v_x = u \tag{7.2}$$

was endowed with the master symmetry

$$u_{\tau_1} = yu_{t_3} - 2u_x + 2xu_y + 4v_y, \tag{7.3}$$

which in turn can be obtained from the KP equation by a pointwise transformation of the form

$$\tilde{x} = \xi(x, y), \qquad \tilde{y} = \eta(y), \qquad \tilde{u} = \alpha(x, y)u + \beta(x, y).$$

In what follows, we write such changes of variables concisely in the form

$$x \to \xi(x, y), \qquad y \to \eta(y), \qquad u \to \alpha(x, y)u + \beta(x, y), \tag{7.4}$$

assuming that the variables $\tilde{x}$, $\tilde{y}$, and $\tilde{u}$ correspond to the initial equation and the variables x, y, and u correspond to its master symmetry. We also often replace the formulas for transformations of x and y with the formulas for D_x and D_y, which are more convenient for calculations.

In the KP equation, the replacement is

$$\begin{aligned} &D_x \to y^{1/3} D_x, \qquad &&D_y \to \frac{1}{3} xy^{-1/3} D_x + y^{2/3} D_y, \\ &u \to y^{2/3}(u + a_x), \qquad &&v \to y^{1/3}(v + a), \qquad a = \frac{x^3}{54y^2} + \frac{x}{3y} \end{aligned} \tag{7.5}$$

(the variables x and y then are transformed as $x \to xy^{-1/3}$ and $y \to 3y^{1/3}$).

The modified KP equation and its master symmetry,

$$u_t = u_{xxx} - 6u^2 u_x + 6u_x v_y + 3v_{yy}, \qquad v_x = u, \tag{7.6}$$

$$u_\tau = yu_t - u_x + 2xu_y - u^2 + 3v_y \tag{7.7}$$

are related to (7.2) and (7.3) by the Miura transformation

$$\tilde{u} = u_x + u^2 - v_y \tag{7.8}$$

($\tilde{u}$ satisfies the KP equation). The conjugation between change (7.5) and the Miura transformation is a relation that is much more involved than a pointwise transformation. Nevertheless, we also have a pointwise change of variables between (7.6) and (7.7) (this change is of form (7.4)). The variables x and y are changed as in (7.5), and the dependent variables u and v are transformed by the formulas

$$u \to y^{1/3}(u + a_x), \qquad v \to v + a, \qquad a = \frac{x^2}{12y} - \frac{1}{6}\log y. \tag{7.9}$$

A higher symmetry of the KP equation is

$$u_{t_4} = u_{xxy} - 4uu_y - 2u_x v_y + w_{yyy}, \qquad w_x = v. \tag{7.10}$$

We now consider the integrable equation

$$u_t = u_{t_4} + \varepsilon u_{t_3}, \tag{7.11}$$

where ε is arbitrary, and show that different nontrivial master symmetries can be obtained by different changes of variables.

The first change of variables,

$$D_x \to y^{1/4} D_x, \qquad D_y \to \left(\frac{1}{4} x y^{-1/2} - \varepsilon y^{1/4}\right) D_x + y^{1/2} D_y,$$
$$u \to y^{1/2}(u + a_{xx}), \qquad v \to y^{1/4}(v + a_x), \qquad w \to w + a,$$
$$a = \frac{x^4}{256 y^2} + \left(\frac{3}{16} y^{-1} - \frac{3\varepsilon^2}{8} y^{-1/2}\right) x^2 + 2\varepsilon^3 y^{1/4} x - \frac{3}{8} \log y,$$

transforms Eq. (7.11) into its master symmetry

$$u_{\tau_2} = y u_{t_4} + \frac{x}{4} u_{t_3} + u_{xx} - \frac{3}{2} u_y + \frac{3}{2} w_{yy} - \frac{1}{2} v u_x - 2u^2. \tag{7.12}$$

The second, simpler change transforms (7.11) into the linear combination $u_\tau = u_{t_4} + k u_{\tau_1}$ of Eq. (7.10) and lower master symmetry (7.3), which is also its nontrivial master symmetry,

$$D_x \to D_x, \qquad D_y \to (ky - \varepsilon) D_x + D_y,$$
$$u \to u + a_{xx}, \qquad v \to v + a_x, \qquad w \to w + a,$$

where

$$a = -\frac{k}{12} x^3 + \frac{3}{8}(k^2 y^2 - \varepsilon^2) x^2 - \left(\frac{7}{16} k^3 y^3 - \frac{3}{8} \varepsilon^2 k y - k - \varepsilon^3\right) y x + $$
$$+ \frac{k}{96}(15 k^3 y^3 - 9 k \varepsilon^2 y - 64 k - 48 \varepsilon^3) y^3.$$

This change is analogous to the simpler transformation

$$D_x \to D_x, \qquad D_y \to 2cy D_x + D_y,$$
$$u \to u + a_x, \qquad v \to v + a, \qquad a = 2c^2 x y^2 - \frac{5}{3} c^3 y^4 - \frac{1}{2} c x^2,$$

which relates the KP equation to its deformation

$$u_\tau = u_{t_3} + 6c\big(x u_x + 2(yu)_y\big).$$

We note that the equation

$$u_{\tau_0} = x u_x + 2(yu)_y \tag{7.13}$$

is then a trivial master symmetry. Indeed, giving the variables x, y, and u the respective weights -1, -2, and 2 (i.e., D_x and D_y have the respective weights 1 and 2) and the differentiations D_{t_n} and D_{τ_n} the weights n, Eqs. (7.2) and (7.10) become homogeneous with the homogeneity powers 5 and 6, and master symmetries (7.3), (7.12), and (7.13) have the homogeneity powers 3, 4, and 2. The master symmetries D_{τ_n} increase the degrees of higher symmetries by n.

We showed in Sec. 2 that Painlevé equations (P_1)–(P_5) describe stationary solutions of deformations of the KdV and NS equations. The two-dimensional analogue of Eq. (2.2) is the linear combination

$$u_t = c_2 u_{\tau_2} + c_1 u_{\tau_1} + c_0 u_{\tau_0}. \tag{7.14}$$

However, this combination can be reduced to Eq. (7.10) (for $c_2 \neq 0$) or to (7.2) (for $c_2 = 0$, $c_1 \neq 0$) by a pointwise substitution, which leaves time the invariant variable, and the stationarity condition then does not lead to "two-dimensional" Painlevé equations. For example, for $c_2 = 0$, the stationary equation is pointwise equivalent to the Boussinesq equation

$$(u_{xxx} - 6uu_x)_x + 3u_{yy} = 0.$$

On the other hand, Eqs. (7.14) determine the two-dimensional analogue of the Painlevé equation therefore relating it not only to the Boussinesq equation itself (see, e.g., [68]) but also to its higher symmetry.

All the above changes of variables belong to the group of gauge transformations of the auxiliary linear problem for the KP equation,

$$\psi_y = \psi_{xx} - u\psi, \tag{7.15}$$

which is a two-dimensional analogue of the one-dimensional Schrödinger spectral problem (2.5). The gauge transformations are

$$\begin{aligned} &D_x \to aD_x, \qquad D_y \to 2a^2\sigma_x D_x + a^2 D_y, \qquad \psi \to e^{\sigma}\psi, \\ &u \to a^2(u + \sigma_{xx} - \sigma_x^2 - \sigma_y), \qquad \sigma = \frac{a'}{4a}x^2 + bx + c, \end{aligned}$$

where a, b, and c are arbitrary functions of y. In particular, replacement (7.5) together with the formula $\psi \to y^{-1/6}\exp(x^2/12y)\psi$ transforms the symmetry of linear problem (7.15)

$$\psi_t = A(y)\psi_{xxx} + \frac{1}{2}xA'\psi_{xx} - \frac{3}{2}Au\psi_x - \frac{1}{4}\big(3A(u_x + v_y) + A'(2xu + v)\big)\psi$$

with $A = 1$, which corresponds to the KP equation, into the symmetry with $A = y$, which corresponds to its master symmetry (7.3). We note that the function ψ_x/ψ satisfies modified KP equation (7.6) and its master symmetry (7.7). This formula determines the transformation inverse to Miura transformation (7.8).

To conclude this section, we recall the well-known matrix KP equation, which is also B-integrable. It is easy to verify that the equation

$$u_t = u_{xxx} - 3(uu_x + u_x u - v_{yy} + v_y u - uv_y), \qquad v_x = u,$$

where $u \in \mathrm{Mat}_n(\mathbb{R})$, and its master symmetry

$$u_\tau = yu_t - 2u_x + 2xu_y + 4v_y + uv - vu$$

are mutually related by the same change of variables (7.5) as in the scalar case.

7.2. The Davey–Stewartson equation. The class of equations discussed in this section differs from the previous one by simpler formulas for transformations of the independent variables x and y. These replacements belong to the group of gauge transformations

$$\begin{aligned} &D_x \to a(x)D_x, \qquad D_y \to b(y)D_y, \qquad \psi \to c(x)\psi, \\ &u \to a\left(u + \frac{c'}{c}\right), \qquad v \to abv \end{aligned} \tag{7.16}$$

of the spectral problem

$$\psi_{xy} = u\psi_y + v\psi. \tag{7.17}$$

Equation (7.17) is a two-dimensional analogue of spectral problem (2.6) with arbitrary parameters, and (7.15) is an analogue of reduced problem (2.5). We consider the basic examples of equations that are integrable through (7.17) and are hence close to the DS equation.

Example 1. Starting with the linear equation

$$\psi_t = A(x)\psi_{xx} - (2Aw_x + A_x w)\psi,$$

we obtain the systems

$$u_t = (u_x + u^2 - 2w_x)_x, \qquad v_t = (-v_x + 2uv)_x, \qquad w_y = v, \tag{7.18}$$

$$u_\tau = \big(x(u_x + u^2 - 2w_x) - w\big)_x, \qquad v_\tau = \big(x(-v_x + 2uv) - v\big)_x, \tag{7.19}$$

corresponding to $A = 1$ and $A = x$ respectively. The second system is the master symmetry of the first system. In particular, $[D_\tau, D_t] = 2D_{t'}$, where the differentiation w.r.t. t' determines the higher symmetry of system (7.18)

$$u_{t'} = (u_{xx} + 3uu_x + u^3 - 3uw_x - 3w_{1,x})_x,$$
$$v_{t'} = (v_{xx} - 3uv_x + 3u^2 v - 3vw_x)_x, \qquad w_{1,y} = uv.$$

Systems (7.18) and (7.19) are related by the pointwise substitution

$$\begin{aligned} &D_x \to x^{1/2} D_x, \qquad D_y \to D_y, \\ &u \to x^{1/2}u - \frac{1}{4}x^{-1/2}, \qquad v \to x^{1/2}v, \qquad w \to x^{1/2}w - \frac{3}{16}x^{-1/2}, \end{aligned} \tag{7.20}$$

i.e., system (7.18) is B-integrable.

The following three examples, as well as the example with the modified KP equation, demonstrate that a pointwise equivalence between an equation and its master symmetry can survive even noninvertible transformations and reductions.

Example 2. The following formulas determine the system and its master symmetry:

$$u_t = u_{xx} + 2uw_x, \qquad -v_t = v_{xx} + 2vw_x, \qquad w_y = uv, \tag{7.21}$$

$$u_\tau = xu_t + uw + (2k+1)u_x, \qquad v_\tau = xv_t - vw + (2k-1)v_x. \tag{7.22}$$

They are related to (7.18) and (7.19) by the transformation

$$\tilde u = \frac{u_x}{u}, \qquad \tilde v = -uv$$

(the tilde denotes variables in systems (7.18) and (7.19)). Strictly speaking, this transformation is consistent with master symmetry (7.22) at $k = -1/2$, but the parameter k can be arbitrary in general because it corresponds to adding a classical symmetry. For $k = \pm 1/4$, Eqs. (7.21) and (7.22) are related by the pointwise transformation given by formulas (7.20) and

$$u \to x^{1/4+k}u, \qquad v \to x^{1/4-k}v, \qquad w \to x^{1/2}w. \tag{7.23}$$

To pass to the DS equation, we relabel the evolution differentiations D_t and D_τ corresponding to (7.21) and (7.22) with D_{t_+} and D_{τ_+}. The involution

$$x \leftrightarrow y, \qquad D_{t_+} \leftrightarrow D_{t_-}, \qquad D_{\tau_+} \leftrightarrow D_{\tau_-}, \qquad w \leftrightarrow \widehat{w} \tag{7.24}$$

results in the system, the master symmetry, and the pointwise transformation relating them dual to (7.21)–(7.23). For example, the system of equations dual to (7.21) is

$$u_{t_-} = u_{yy} + 2u\widehat{w}_y, \qquad -v_{t_-} = v_{yy} + 2v\widehat{w}_y, \qquad \widehat{w}_x = uv. \tag{7.25}$$

The differentiations D_{t_+} and D_{t_-} commute, and their linear combination therefore results in the integrable system of equations (cf. Sec. 5.2)

$$u_t = \alpha u_{t_+} + \beta u_{t_-}, \qquad v_t = \alpha v_{t_+} + \beta v_{t_-}. \tag{7.26}$$

In particular, the condition $\beta = \pm\alpha$ corresponds to the DS equation. The composition of replacement (7.20), (7.23) and the dual replacement transforms system (7.26) into the master symmetry corresponding to the differentiation $D_\tau = \alpha D_{\tau_+} + \beta D_{\tau_-}$. System (7.26) with arbitrary α and β and therefore the DS equation are then B-integrable. We note that the differentiation D_{τ_+} commutes with D_{t_-}, which follows because replacement (7.20), (7.23) leaves the variable y invariant and does not change system (7.25). The differentiations D_{τ_-} and D_{t_+} commute for the same reason.

Example 3. The higher symmetry of system (7.21) constructed using (7.22) is

$$\begin{aligned} u_{t_3} &= u_{xxx} + 3u_x D_y^{-1}(uv)_x + 3uD_y^{-1}(u_x v)_x, \\ v_{t_3} &= v_{xxx} + 3v_x D_y^{-1}(uv)_x + 3vD_y^{-1}(uv_x)_x. \end{aligned} \tag{7.27}$$

It admits the reduction $v = 1$, which results in the equation

$$u_t = (u_{xx} + 3uw_x)_x, \qquad w_y = u. \tag{7.28}$$

The master symmetry and the replacement, which allows constructing this master symmetry, are

$$u_\tau = \big(x(u_{xx} + 3uw_x) + u_x + uw\big)_x, \tag{7.29}$$

$$\begin{aligned} &D_x \to x^{1/3} D_x, \qquad D_y \to D_y, \\ &u \to x^{1/3} u, \qquad\quad w \to x^{1/3} w - \frac{1}{18} x^{-2/3}. \end{aligned} \tag{7.30}$$

For example, this master symmetry generates the higher symmetry $[D_\tau, D_t] = 3D_{t_5}$,

$$u_{t_5} = \big(u_{xxxx} + 5(u_x w_x)_x + 5u(w_{xxx} + w_x^2 + w_{1,x})\big)_x,$$

where w_1 is nonlocal and such that $w_{1,y} = uw_x$.

Equations (7.28) and (7.29) arise from the compatibility condition for the two linear problems

$$\psi_{xy} + u\psi = 0, \qquad \psi_t = A(x)\psi_{xxx} + A'\psi_{xx} + C\psi_x, \qquad C = 3Aw_x + A'w$$

corresponding to the respective choices $A = 1$ and $A = x$. The form of the linear problem suggests that we deal with the Veselov–Novikov hierarchy [69–71]. Using involution (7.24) as in Example 2, we obtain the equation

$$u_{t'} = (u_{yy} + 3u\widehat{w}_y)_y, \qquad \widehat{w}_x = u \tag{7.31}$$

and can prove that the equations, which correspond to differentiations of the form $D_{\hat{t}} = \alpha D_t + \beta D_{t'}$ and hence the Veselov–Novikov equation, are B-integrable.

Example 4. The reduction $v = u$ of system (7.27) results in the equation

$$u_t = u_{xxx} + 3u_x w_x + \frac{3}{2} u w_{xx}, \qquad w_y = u^2.$$

This reduction has all the properties of the preceding two examples. In particular, the master symmetry

$$u_\tau = x u_t + \frac{3}{2} u_{xx} + 3u_x w + 2u w_x$$

can be obtained using the pointwise transformation

$$\begin{aligned} &D_x \to x^{1/3} D_x, && D_y \to D_y, \\ &u \to x^{1/6} u, && w \to x^{1/3} w - \frac{7}{72} x^{-2/3}. \end{aligned}$$

Using involution (7.24), we can prove the B-integrability of an arbitrary linear combination.

7.3. The two-dimensional Boussinesq equation. We now discuss the hierarchy of the system of equations

$$u_t = u_{xx} + 2v_x, \qquad -v_t = v_{xx} - 2u u_x + 2u_y, \tag{7.32}$$

which differs from those above from the standpoint of B-integrability and provides a negative example. System (7.32) generalizes (Bq_1) in the list of systems of the Boussinesq equation type in Appendix 1. We can exclude the variable v by differentiating the first equation w.r.t. t, which results in the equation

$$u_{tt} = D_x(u_{xxx} + 4u u_x - 4u_y). \tag{7.33}$$

Up to the redefinition $y \leftrightarrow t$ and scalings, it is KP equation (7.2). To be precise, the independent variables t_3 and y in the KP equation become the respective y and t variables in Eq. (7.33). We can easily rewrite higher symmetries in the new variables and, in particular, obtain the higher symmetry of system (7.32)

$$\begin{aligned} u_{t_4} &= D_x(u_{xxx} + 2u_y + 2v_{xx} + 2u u_x + 4uv) + 4v_y, \\ -v_{t_4} &= D_x\left(v_{xxx} + 2v_y + 2u_{xy} - 2u u_{xx} - u_x^2 + 2u v_x - 2v^2 - \frac{4}{3}u^3\right) + 4w_{yy} \end{aligned} \tag{7.34}$$

from (7.10), where $w_x = u$.

On the one hand, the master symmetry for system (7.32) cannot be related to the very equation by a pointwise transformation as was the case in the previous examples. Indeed, if such a system existed, a master symmetry would become local and therefore generate local higher symmetries, while higher symmetries of (7.32) are nonlocal. On the other hand, the pointwise transformation in (7.34),

$$\begin{aligned} &D_x \to y^{1/4} D_x, && D_y \to y^{3/4} D_y + \frac{1}{4} x y^{-1/4} D_x, \\ &u \to y^{1/2} u + \frac{1}{4} x y^{-1/2}, && v \to y^{3/4} v, \qquad w \to y^{1/4} w + \frac{1}{8} x^2 y^{-3/4}, \end{aligned}$$

corresponds to the master symmetry

$$\begin{aligned} u_\tau &= y u_{t_4} + x u_t + \frac{5}{2} u_x + 4v, \\ v_\tau &= y v_{t_4} + x v_t - u_{xx} - \frac{5}{2} v_x - 5w_y + u^2. \end{aligned}$$

In particular, letting D_{t_1} and D_{t_3} denote the differentiations corresponding to the respective equations $u_{t_1} = u_x$ and $u_{t_3} = u_y$, we obtain

$$\mathrm{ad}_{D_\tau} : D_{t_1} \to D_t \to D_{t_3} \to D_{t_4}.$$

We therefore have a mismatch on the very first step; all other hierarchy members presumably remain B-integrable.

Similar examples exist for (1+1)-dimensional equations, where the integrability within the symmetry approach naturally means the existence of infinite hierarchies of local higher symmetries and conservation laws. This integrability can sometimes be broken by a simple transformation analogous to $\tilde{u} = u_x$. Such examples nevertheless do not distort the picture, and we can use this definition for classifying and testing. One such example pertains to system of equations (Bq_1) in Appendix 1, whose two-dimensional generalization is system (7.32). The transformation $\tilde{u} = u$, $\tilde{v} = v_x - u^2/2$ results in the system

$$u_t = u_{xx} + p, \qquad v_t = -v_{xx} + u_x^2 - up, \qquad p = 2v + u^2. \tag{7.35}$$

Equations (A.6) imply that $\rho_2 = 4u + \mathrm{const}$, but this function is not a conserved density. System (7.35), in contrast to (Bq_1), therefore cannot have local conservation laws and higher symmetries.

7.4. The calculation technique. In fact, we must not use the auxiliary linear problem to verify conditions (7.1). We now describe this procedure with the example of the equation

$$u_{yt} + \alpha u_x u_{xy} + \beta u_y u_{xx} + u_{xxxy} = 0, \qquad \alpha\beta \neq 0. \tag{7.36}$$

The known test indicates that it is integrable only for $\alpha = \beta$ (see [72, 73]).

The first problem is the choice of the general form of the change of variables. In the scalar equation case, we must a priori admit arbitrary contact transformations (for systems, they are exhausted by pointwise transformations according to the classic Bäcklund result, but we must also include invertible differential substitutions). Nevertheless, in the most of the significant examples, it suffices to consider pointwise substitutions of the form

$$x \to \xi(x, y), \qquad y \to \eta(x, y), \qquad u \to \varphi(u, x, y). \tag{7.37}$$

Roughly speaking, this corresponds to equations with constant coefficients in the principal linear part. Nonlocal terms of an equation provide corrections to the form of the replacement. Indeed, in practice, the structures of singularities of an equation and of its master symmetry coincide. Because the operators D_x and D_y under replacement (7.37) are changed as

$$D_x \to \frac{1}{\Delta}(\eta_y D_x - \eta_x D_y), \qquad D_y \to \frac{1}{\Delta}(\xi_x D_y - \xi_y D_x) \tag{7.38}$$

(where $\Delta = \xi_x \eta_y - \xi_y \eta_x$), which implies that if, for instance, the equation contains the operator D_y^{-1} (as is the case for Eq. (7.28)), then $\xi_y = 0$. Indeed, supposing the opposite, we find that the operator D_y^{-1} transforms into the operator that is inverse to the operator in the right-hand side of (7.38), which drives the master symmetry out of the limits of the class under consideration. Analogously, if an equation contains the operator D_x^{-1} (as is the case for KP equation (7.2)), then $\eta_x = 0$. Eventually, for equations containing nonlocalities of both types in the right-hand side, we have $\xi_y = \eta_x = 0$, which means that we deal with the simplest change of variables. The Veselov–Novikov equation, which is a linear combination of Eqs. (7.28) and (7.31), has such a structure.

In all the examples above, the replacement was linear in u. This is because a nonlocality of the type $D_x^{-1}(u)$ must be expressed through a nonlocality of the same type. Of course, if we "distort" an equation with such a nonlocality by a transformation $u = f(\tilde{u})$, then the linearity of the substitution is broken.

The above reasoning is by no means rigorous; it only clarifies how the form of an equation determines the form of the substitution. However, we can easily prove it for a concrete equation by analyzing determining relation (7.1).

The second problem is the problem of nonlocalities, which is general for the theory of two-dimensional integrable equations. We do not propose new tools for solving it and only briefly discuss the specific properties of our problem.

In the simplest cases, we can work with local equations at the price of abandoning the evolution property. For example, an equation of the form (which is true for Eq. (7.36))

$$u_{y,t} = F(x, y, u, u_x, u_y, u_{xx}, u_{xy}, u_{yy}, \dots)$$

can be transformed by replacement (7.37) with $\xi_y = 0$ into an equation of the form

$$u_{y,\tau} = G(x, y, u, u_x, u_y, u_{xx}, u_{xy}, u_{yy}, \dots).$$

Defining

$$u_{y,t'} = H = D_\tau(F) - D_t(G),$$

we can in principle verify the equality $R = D_t(H) - D_{t'}(F) = 0$. The difficulty is that this equality now contains not only derivatives w.r.t. x and y but also w.r.t. t, t', and τ. We can eliminate these derivatives using a multiple differentiation w.r.t. y, which also produces the condition for the equation $R = 0$ to be consistent with the initial equation.

It is more convenient to work with the evolutionary type of equations. Replacing $t \to -t$ and $\alpha u_y \to 3u$, we can reduce Eq. (7.36) to the form

$$u_t = u_{xxx} + 3uD_y^{-1}u_{xx} + 3\lambda u_x D_y^{-1}u_x, \tag{7.39}$$

which coincides with (7.28) for $\lambda = 1$ (which corresponds to $\alpha = \beta$). When verifying determining relation (7.1), we encounter the need to further replace the changes around nonlocalities. We must then take the integration constants into account; for example, the nonlocality of $D_y^{-1}u$ is transformed as

$$D_y^{-1}u \to D_y^{-1}\big(\eta_y\varphi(u, x, y)\big) + \mu(x).$$

If we have several nonlocalities, we must introduce several functions μ.

In conclusion, we note that the operator language is convenient for actual calculations where the equality of functions is replaced by the condition for equality of the corresponding pseudodifferential operator of the form

$$\sum_{i,j<\infty} a_{ij} D_x^i D_y^j = \sum_{i<\infty} A_i D_x^i,$$

obtained by the linearization. Determining equation (7.1) is then replaced by the relations

$$[F_*, H_*] + D_{t'}(F_*) - D_t(H_*) = 0, \qquad H_* = [F_*, G_*] + D_\tau(F_*) - D_t(G_*), \tag{7.40}$$

where F_*, G_*, and H_* are the linearization operators for the right-hand sides of the equation, its master symmetry, and their commutator, i.e.,

$$u_t = F, \qquad u_\tau = G, \qquad u_{t'} = H = D_\tau(F) - D_t(G).$$

We analyze relation (7.40) step by step, subsequently fitting the replacement form. For example, to verify the linearity of a replacement, we can verify it only for the first two terms in F_* and in G_*. We have

$$F_* = D_x^3 + uD_y^{-1}D_x^2 + \dots .$$

Because the nonlocality is related to D_y^{-1}, we have $\xi = \xi(x)$ and

$$D_x \to A = a(x)D_x + b(x,y)D_y, \qquad D_y \to c(x,y)D_y, \qquad ac \neq 0.$$

The transformed equation is

$$u_\tau = G = \frac{1}{\varphi_u}\big(A^3(\varphi) + 3\varphi A^2(\psi) + 3A(\varphi)A(\psi)\big), \qquad \psi = D_y^{-1}\left(\frac{\varphi}{c}\right) + \mu(x),$$

whence

$$\begin{aligned}
&G_* = g_3 D_x^3 + 3g_2 D_x^2 + \dots\,, \\
&g_3 = a^2 b, \qquad\qquad g_2 = g_{2,1}D_y + g_{2,0} + \dots\,, \\
&g_{2,1} = a^2 b, \qquad\quad\; g_{2,0} = \frac{\varphi_{uu}}{\varphi_u}a^2(au_x + bu_y) + d(u,x,y).
\end{aligned}$$

Substituting these expressions in (7.40) and zeroing the coefficients of D_x^7 and D_x^6, we obtain

$$D_x^2(g_3) = D_x^2(g_{2,1}) = D_x^2(g_{2,0}) = 0,$$

which proves the linearity of the replacement: $\varphi_{uu} = 0$. Clarifying its form further and analyzing conditions (7.40) in several steps, we obtain either a contradiction if $\lambda \neq 1$ or formula (7.30) if $\lambda = 1$.

Appendix 1: Canonical densities

We now briefly discuss the integrability conditions and lists of integrable equations for the classes of KdV, Burgers, NS, and Boussinesq equations (see, e.g., [10, 12]). We first consider the scalar evolution equations

$$u_t = G(x, u, u_1, u_2, \dots, u_n), \quad n \geq 2, \tag{A.1}$$

where $u_1 = u_x$, $u_2 = u_{xx}$, etc. In what follows, we use the notation $G_0 = \partial G/\partial u$ and $G_k = \partial G/\partial u_k$ for brevity.

For Eq. (A.1) of any order n, we can write the simplest integrability conditions (the necessary conditions for the existence of higher symmetries and conservation laws), which are local conservation laws $\rho_t = \sigma_x$, where ρ is the special density (the so-called canonical density) constructed by the right-hand side of the equation. This condition states that there exists such a function σ of a finite number of variables $x, u, u_1, u_2, \dots$, and this condition can be easily verified for any given equation.

Statement 3. *If Eq.* (A.1) *has a symmetry of order higher than n, then the function $\rho = G_n^{-1/n}$ is a canonical density. If the equation*

$$u_t = u_n + F(x, u, u_1, u_2, \dots, u_m), \quad F_m \neq 0, \quad m \leq n-2, \tag{A.2}$$

has a symmetry of order higher than $n+m$, then the function $\rho = F_m$ is also a canonical density.

As shown in Sec. 6.1, canonical densities for Eq. (A.1) are defined as the residues of powers of the first-order solution of Eq. (6.2): $\rho_k = \operatorname{res} L^k$. If an equation is integrable, the operator L coincides in principal with $G_*^{1/n}$, i.e., the first among the canonical densities is ρ_{-1}. Similar considerations show that the n subsequent densities ρ_k are

$$\rho_k = \operatorname{res}(G_*)^{k/n}, \quad k = 1, \dots, n-1, \quad \rho_n = \sigma_1,$$

where σ_1 is found from the relation $(\rho_1)_t = (\sigma_1)_x$. For equations of form (A.2), we present the first four such canonical densities:

$$\begin{aligned} &\rho_1 = F_{n-2}, && \rho_2 = F_{n-3}, \\ &\rho_3 = F_{n-4} + \frac{3-n}{2n} F_{n-2}^2, && \rho_4 = F_{n-5} + \frac{4-n}{n} F_{n-3} F_{n-2}. \end{aligned} \tag{A.3}$$

For example, for the third-order equations

$$u_t = u_3 + F(x, u, u_1),$$

the canonical densities are

$$\rho_1 = F_1, \qquad \rho_2 = F_0, \qquad \rho_3 = \sigma_1.$$

Up to pointwise transformations (including the Galileo transformation) and potential transformations $u \to u_x$, the equations that have higher symmetries and conservation laws constitute the following list.

The list of equations of the KdV type is

$$u_t = u_3 + P(u) u_1, \qquad P''' = 0; \tag{K$_1$}$$

$$u_t = u_3 - \frac{1}{2} u_1^3 + (\alpha e^{2u} + \beta e^{-2u}) u_1. \tag{K$_2$}$$

In [12], the reader can find a wider list of integrable third-order equations from which we select the Krichever–Novikov equation [74]

$$u_t = u_3 - \frac{3}{2u_1} \big(u_2^2 + R(u)\big), \tag{A.4}$$

where R is an arbitrary polynomial of the fourth degree. This equation, akin to the Landau–Lifshitz model or to Eq. (P$_6$), is the universal equation in the corresponding class.

For the fifth-order equations

$$u_t = u_5 + F(x, u, u_1, u_2, u_3),$$

the canonical densities are

$$\rho_1 = F_3, \qquad \rho_2 = F_2, \qquad \rho_3 = F_1 - \frac{1}{5} F_3^2, \qquad \rho_4 = F_0 - \frac{1}{5} F_2 F_3, \qquad \rho_5 = \sigma_1.$$

In this case, the list of integrable equations (up to the same transformations as in the previous example) comprises five equations. (The higher symmetries of third-order equations are excluded.)

The list of equations of the fifth order is

$$u_t = u_5 + 5uu_3 + 5u_1u_2 + 5u^2u_1, \tag{K$_1^5$}$$

$$u_t = u_5 + 5uu_3 + \frac{25}{2} u_1u_2 + 5u^2u_1, \tag{K$_2^5$}$$

$$u_t = u_5 + 5(u_1 - u^2)u_3 + 5u_2^2 - 20uu_1u_2 - 5u_1^3 + 5u^4u_1, \tag{K$_3^5$}$$

$$\begin{aligned} u_t = u_5 &+ 5(u_2 - u_1^2 + ae^{2u} - be^{-4u})u_3 + 15(ae^{2u} + 4be^{-4u})u_1u_2 - \\ &- 5u_1u_2^2 + u_1^5 - 90be^{-4u}u_1^3 + 5(ae^{2u} - be^{-4u})^2 u_1, \end{aligned} \tag{K$_4^5$}$$

$$\begin{aligned} u_t = u_5 &+ 5(u_2 - u_1^2 - ae^{2u} + be^{-u})u_3 - 5u_1u_2^2 - \\ &- 15ae^{2u}u_1u_2 + u_1^5 + 5(ae^{2u} - be^{-u})^2 u_1. \end{aligned} \tag{K$_5^5$}$$

There is a hypothesis that starting with order six, all nonlinear integrable equations of form (A.2) are exhausted by higher symmetries of equations of orders 2, 3, 4, and 5.

For the second-order equations, the classification problem was completely solved for equations of general form (A.1). Up to contact transformations, the list of nonlinear equations is as follows.

The list of equations of the Burgers type is

$$u_t = D_x(u_1 + u^2 + a), \tag{B$_1$}$$

$$u_t = D_x\left(\frac{u_1}{u^2} + \alpha x u + \beta u\right), \tag{B$_2$}$$

$$u_t = D_x\left(\frac{u_1}{u^2} - 2x\right), \tag{B$_3$}$$

where α and β are arbitrary constants and $a(x)$ is an arbitrary function. This classification result was obtained in [75].

Eventually, for the systems of equations

$$u_t = u_2 + F(u, v, u_1, v_1), \qquad v_t = -v_2 + G(u, v, u_1, v_1) \tag{A.5}$$

(comparing with (4.3) we scale the time, which does not change the symmetry properties of the equations), the three lowest canonical densities are

$$\begin{aligned} \rho_1 &= \frac{1}{2}(F_{u_1} + G_{v_1}), \\ \rho_2 &= \sigma_1 - \frac{1}{4}(F_{u_1}^2 + G_{v_1}^2) + F_{v_1} G_{u_1} + F_u - G_v, \\ \rho_3 &= \sigma_2, \end{aligned} \tag{A.6}$$

where σ_i must be found from the relations $(\rho_i)_t = (\sigma_i)_x$. The lists of NS-type systems having conservation laws with such densities are in Sec. 4.1 (the complete lists of such integrable systems were found in [10, 30]), and we here present only those systems from the Boussinesq equation class that are related to spectral problems of the third order.

The list of systems of the Boussinesq type is

$$u_t = u_2 + 2v_1, \qquad v_t = -v_2 + 2uu_1, \tag{Bq$_1$}$$

$$u_t = u_2 + v_1, \qquad v_t = -v_2 + u_1^2 + \left(v + \frac{u^2}{2}\right) u_1, \tag{Bq$_2$}$$

$$u_t = u_2 + (u+v)^2, \qquad -v_t = v_2 + (u+v)^2, \tag{Bq$_3$}$$

$$u_t = u_2 + (u+v)v_1 - \frac{(u+v)^3}{6}, \qquad v_t = -v_2 + (u+v)u_1 + \frac{(u+v)^3}{6}, \tag{Bq$_4$}$$

$$u_t = u_2 + v_1^2 + W_u v_1 + Z_v, \qquad -v_t = v_2 + u_1^2 - W_v u_1 + Z_u, \tag{Bq$_5$}$$

where

$$W = \sum_{k=1}^{3} \alpha_k e^{-w_k}, \qquad Z = \sum_{k=1}^{3} (\beta_k e^{w_k} + \gamma_k e^{-2w_k}), \qquad w_k = \varepsilon^k u + \varepsilon^{-k} v,$$

and $\varepsilon = \exp(2\pi i/3)$ is the cubic root of unity: $\varepsilon^3 = 1$.

Appendix 2: Integrability conditions and classification results for differential–difference equations

The classification results and integrability conditions for equations on the lattice, which we discuss in this section, are presented in the literature not as well as for the continuous equations. There are four classification results obtained for the lattice equations with one discrete and one continuous variable within the symmetry approach. These results pertain to the classes of equations

$$u_x = f(u_1, u, u_{-1}), \qquad f_{u_1} f_{u_{-1}} \neq 0, \tag{A.7}$$

$$q_{xx} = f(q_x, q_1, q, q_{-1}), \qquad f_{q_1} f_{q_{-1}} \neq 0, \tag{A.8}$$

$$q_{xx} = f(q_1, q, q_{1,x}, q_x) - g(q, q_{-1}, q_x, q_{-1,x}), \qquad f_{q_{1,x}} g_{q_{-1,x}} \neq 0, \tag{A.9}$$

$$u_x = f(u_1, u, v), \qquad v_x = g(v_{-1}, v, u), \qquad f_{u_1} f_v g_{v_{-1}} g_u \neq 0. \tag{A.10}$$

Elementary representatives of these classes are the respective Volterra equation (5.10), Toda chain (4.26), relativistic Toda chain (R_3) with $\mu = 1$ and $\nu = 0$ (see Sec. 4.2), and its Hamiltonian form (4.40). The classification results explain, in particular, why the three lists presented in Sec. 4 using several Lagrangians are indeed three *complete* lists of integrable chains. The detailed derivation of integrability conditions and the scheme of the symmetry approach as applied to lattice equations was given in [9, 55].

For Eqs. (A.7) (see [52, 53]), several lower integrability conditions are

$$\begin{gathered} \rho^i_x = (T-1)\sigma^i, \quad i = 1, 2, 3, \\ \rho^1 = \log f_{u_1}, \qquad \rho^2 = \sigma^1 + f_u, \qquad \rho^3 = \sigma^2 + \frac{1}{2}(\rho^2)^2 + f_{u_1} T f_{u_{-1}} \end{gathered} \tag{A.11}$$

and

$$\begin{gathered} \omega^i = (T-1)s^i, \quad i = 1, 2, \\ \omega^1 = \log\left(-\frac{f_{u_1}}{f_{u_{-1}}}\right), \qquad \omega^2 = s^1_x + 2 f_u. \end{gathered} \tag{A.12}$$

Theorem 4. *If an equation of form* (A.7) *has two local conservation laws of orders* $N_1 > N_2 > 4$ *then it satisfies conditions* (A.11) *and* (A.12).

We recall that the functions ρ and σ, which determine the local conservation law $\rho_x = (T-1)\sigma$, depend on a finite number of variables $u, u_{\pm 1}, u_{\pm 2}, \dots$ in the case of chains (A.7). The formal variational derivative $\delta\rho/\delta u$ is

$$\frac{\delta \rho}{\delta u} = \sum_k \frac{\partial}{\partial u} T^k \rho = \hat{\rho}(u_N, u_{N-1}, \dots, u_{-N}),$$

and the number N is the *order* of the conservation law.

The presented integrability conditions do not depend (as is the case in the classical scheme of the symmetry approach) on the forms and orders of conservation laws. We can use a formal variational derivative when verifying integrability conditions because

$$\frac{\delta h}{\delta u} = 0 \quad \Leftrightarrow \quad h \in \mathrm{const} + \mathrm{Im}(T-1).$$

Integrability conditions (A.11) and (A.12) are sufficiently effective for us to obtain the complete list of equations satisfying these conditions. An extra analysis of equations from this list shows that these conditions are simultaneously sufficient conditions for the existence of symmetries and conservation laws of arbitrarily high orders. The integrability conditions, which we present below for other classes of chains, are both necessary and sufficient conditions. The list of equations satisfying conditions (A.11) and (A.12) coincides (up to the transformations $\tilde{u}_n = \nu(u_n)$, $\tilde{x} = cx$) with list (V) in Sec. 5.3.

We also note that first-type conditions (A.11) allow constructing simplest local conservation laws for the equations in list (V). Both conditions (A.11) and conditions (A.12) can be easily verified for any given equation, i.e., provide a convenient integrability test.

We now discuss the chains of form (A.8). The integrability conditions are then determined by the functions (see [32])

$$\rho^1 = \log f_{q_1}, \qquad \rho^2 = 2\sigma^1 + f_{q_x}, \qquad \rho^3 = 2\sigma^2 - \frac{1}{2}(f_{q_x})_x + \frac{1}{4}\big(f_{q_x}^2 + (\rho^2)^2\big) + f_q,$$

$$\omega^1 = \log\left(\frac{f_{q_1}}{f_{q_{-1}}}\right), \qquad \omega^2 = s_x^1 + f_{q_x}.$$

Theorem 5. *The conditions with the above functions ρ^i and ω^i are satisfied if we have at least two conservation laws of orders $N_1 \geq 8$ and $N_2 > 2N_1 - 3$.*

The functions ρ and σ, which determine a local conservation law, now depend on the variables $q, q_x, q_{\pm 1}, q_{\pm 1,x}, \ldots$. The variational derivatives $\delta\rho/\delta q$ and $\delta\rho/\delta q_x$ are

$$\frac{\delta\rho}{\delta q} = \hat{\rho}(q_N, q_{N,x}, \ldots, q_{-N}, q_{-N,x}), \qquad \frac{\delta\rho}{\delta q_x} = \tilde{\rho}(q_N, q_{N,x}, \ldots, q_{-N}, q_{-N,x})$$

(we assume that $\hat{\rho}$ or $\tilde{\rho}$ depends on at least one of the variables $q_N, q_{N,x}, q_{-N}, q_{-N,x}$), and the number N of higher variables is called the order of the conservation law. Verifying the integrability conditions, we must investigate double variational derivatives because

$$\frac{\delta h}{\delta q} = \frac{\delta h}{\delta q_x} = 0 \quad \Leftrightarrow \quad h \in \mathrm{const} + \mathrm{Im}(T - 1).$$

The list of equations satisfying the integrability conditions comprises (up to the transformations $\tilde{q}_n = \nu(q_n)$, $\tilde{x} = cx$, and $\tilde{q}_n = (-1)^n q_n$) chains (4.31), (4.34), and

$$q_{xx} = \exp(q_1 - 2q + q_{-1}) + \mu. \tag{A.13}$$

Chain (A.13) is an obvious modification of Toda chain (4.26); the transformation is $\tilde{q} = q_1 - q$ ($\tilde{q}$ is a solution of (4.26)). Using the integrability conditions, we can construct not only simplest conservation laws but also higher symmetries for chains (4.31) and (4.34) because these chains are Hamiltonian.

The class of chains (A.10), in contrast to (A.7) and (A.8), is less convenient from the standpoint of the symmetry approach because of the presence of "nonstandard" higher symmetries and integrability conditions. The list of integrability conditions and the classification result, which we present below, are from [36].

We consider systems that have Hamiltonian structure (4.39). We can then demand only the presence of higher symmetries. It is of no purpose to involve local conservation laws because they cannot not lead to additional integrability conditions analogous to (A.12) (see, e.g., [36, 55]). The higher symmetries of chains (A.10) are segregated into two absolutely different kinds.

If we consider a symmetry of order N,

$$u_t = F(u_N, v_N, u_{N-1}, v_{N-1}, \dots), \qquad v_t = G(u_N, v_N, u_{N-1}, v_{N-1}, \dots) \tag{A.14}$$

and investigate the condition for the compatibility of (A.14) and (A.10), we find that $F_{v_N} = G_{u_N} = 0$ and derive the first integrability condition. In the case $F_{u_N} \neq 0$, we obtain the condition

$$(\log f_{u_1})_x = (T-1)\sigma, \tag{A.15}$$

which is a standard integrability condition analogous to (A.11). The developed technique for investigating these conditions shows that they are sufficiently effective to classify integrable equations.

In the case $G_{v_N} \neq 0$, we obtain a condition of another type,

$$\rho_x = (T^N - 1)g_v. \tag{A.16}$$

It is again a local conservation law because

$$T^N - 1 = (T-1)(T^{N-1} + T^{N-2} + \cdots + T + 1),$$

but it now depends on the symmetry order N and has no definite density ρ. We do not know how to deal with such conditions. To implement a classification, we introduce the so-called special symmetries, which allow avoiding nonstandard conditions (A.16). The special symmetry of order N is

$$u_t = F(u_N, u_{N-1}, v_{N-1}, \dots), \qquad v_t = G(u_{N-1}, u_{N-2}, v_{N-2}, \dots), \quad F_{u_N} \neq 0.$$

We therefore consider chains of form (A.10) with Hamiltonian structure (4.39), which have special symmetries of sufficiently high orders. The classification is up to pointwise transformations of the form

$$\tilde{u}_n = \nu(u_n), \qquad \tilde{v}_n = \eta(v_n), \qquad \tilde{x} = cx, \tag{A.17}$$

which preserve the given structure of chains and their symmetries.

Theorem 6. *If system* (A.10) *admits a special symmetry of order* $N \geq 4$*, then it satisfies the conditions of form* (A.11) *with the functions*

$$\rho^1 = \log f_{u_1}, \qquad \rho^2 = \sigma^1 + f_u, \qquad \rho^3 = \sigma^2 + \frac{1}{2}(\rho^2)^2 + f_v g_u.$$

The chain of form (A.10) with Hamiltonian structure (4.39) satisfies the given conditions iff change of variables (A.17) can be reduced to one of the chains in list (H) in Sec. 4.3. Because the systems in list (H) are Hamiltonian, their integrability conditions provide both the simplest conservation laws and the higher symmetries for these systems. When verifying integrability conditions for the given system, it is convenient to use the statement

$$\frac{\delta h}{\delta u} = \frac{\delta h}{\delta v} = 0 \quad \Leftrightarrow \quad h \in \mathrm{const} + \mathrm{Im}(T-1).$$

We therefore see that the last classification result, albeit weaker than the two previous results, provides us with a number of integrable chains having numerous applications.

From the standpoint of the general symmetry approach, the class of chains (A.9) is analogous to that of (A.10). Here, we again have two types of higher symmetries, which result in two different types of integrability conditions. The corresponding classification result was obtained in [33], where integrability conditions were not written and a simplified symmetry approach scheme was used. In [33], chains (A.9), having symmetries of the form

$$q_t = f(q_1, q, q_{1,x}, q_x) + g(q, q_{-1}, q_x, q_{-1,x}), \tag{A.18}$$

were considered. Such a symmetry can obviously be written in the variables $u = q_{n+1}$ and $v = q_n$ as a system of the NS type (we can rewrite the symmetry by virtue of (A.9)),

$$u_t = u_{xx} + 2g(u, v, u_x, v_x), \qquad v_t = -v_{xx} + 2f(u, v, u_x, v_x). \tag{A.19}$$

It is known (see, e.g., [10]) that a system of form (A.19) that admits conservation laws of sufficiently high order must satisfy the condition $g_{u_x} - f_{v_x} \in \operatorname{Im} D_x$. This condition gives a relation of the form

$$g_{u_x} - f_{v_x} = \big(s(u, v)\big)_x = s_u u_x + s_v v_x.$$

Returning to the variables q_i, we obtain the representation

$$T(g_{q_x}) - f_{q_x} = s_{q_1} q_{1,x} + s_q q_x \tag{A.20}$$

in terms of the right-hand side of the chain and a function of two variables $s(q_1, q)$. Relation (A.20) can be treated as the second condition (we also need a function s satisfying (A.20)).

Up to transformations of the form $\tilde{q}_n = \nu(q_n)$, chains (A.9) that have a symmetry of form (A.18) and satisfy condition (A.20) are exhausted by the list of equations comprising (4.28), (4.30), and (4.35).

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