Cubature formulas of S.L. Sobolev: evolution of the theory and applications

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Abstract

The paper contains the description of the theory of approximate calculation of integrals over arbitrary multi-dimensional domains. This research branch is developed in several research centers in Russia and, in particular, in the Ufa Mathematical Institute of the Russian Academy of Sciences. We consider the best approximations of linear functionals on a certain semi-Banach space B by linear combinations of the Dirac functions with supports in the nodes of a certain lattice:

$$(l_N, f) \equiv \int\limits_\Omega f(x)dx - \sum_{\substack{k\in\mathbb{Z}^n,\\ H_N k\in\Omega}} c_k f(H_N k), \tag{1}$$

where H_N is an $n \times n$ matrix, such that $\det H_N \neq 0$ and $\det H_N \to 0$ as $N \to \infty$ and $f:\ \mathbb{R}^n \to \mathbb{C}$, $f \in B \subset C(\mathbb{R}^n)$.

This setting of the problem was given by academician Sergei L'vovich Sobolev in the middle of the last century.

Introduction

For the sake of brevity we give a simplified setting of the problem.

The Sobolev cubature formulas $K_h f = h^n \sum\limits_{hHk\in\Omega} c_k f(hHk)$ allow approximate calculation of integrals $If = \int\limits_\Omega f(x)dx$, where Ω is a domain in $\mathbb{R}^n$ and $\{hHk|\ k \in \mathbb{Z}^n\}$ is a lattice of nodes. Here H is an $n \times n$ matrix and h is a small parameter. The sharpness of the approximation is determined by the norm of the error functional $I - K_h:\ B \to \mathbb{C}$, where $B \subset C(\mathbb{R}^n)$ is a certain semi-Banach space. A cubature formula K_h^{opt} is called optimal if

$$\|I - K_h^{opt}\|_{B^*} = \min_{c_k} \|I - K_h\|_{B^*}. \tag{2}$$

[1]Supported by the Russian Foundation for Basic Research (project 09-01-00349-a) and by the programme "Support of young scientists" of the Presidium of the Russian Academy of Sciences.

Moreover, a cubature formula K_h^{as} is called asymptotically optimal or optimal by order if

$$\nu(h) = \frac{\|I - K_h^{as}\|_{B^*}}{\|I - K_h^{opt}\|_{B^*}} \to 1 \text{ as } h \to 0, \tag{3}$$

$\overline{\lim\limits_{h\to 0}}\nu(h) < \infty$ respectively.

S.L. Sobolev gave an algorithm of calculation of the coefficients c_k of an asymptotically optimal formula for the case in which B is the space $L_2^{(m)}(\Omega)$ with the semi-norm

$$\|f\|_m = \left(\int\limits_\Omega \sum_{|\alpha|=m} \frac{m!}{\alpha!} |D^\alpha f(x)|^2 dx \right)^{\frac{1}{2}} \quad \text{where } m > \frac{n}{2}. \tag{4}$$

His main investigations were carried out in the sixties-seventies of the last century — see [1] and [2].

There is a well-known algebraic approach of constructing cubature formulas of high precision. The parameters of the cubature formula are chosen in such a way that the formula is exact for all polynomials of given degree. One of examples is the Gaussian quadrature formula. The main postulate of Sobolev's theory is the symbiosis of an algebraic and an analytical approaches. He constructed asymptotically optimal formulas as the sum of local formulas with the sizes of supports of order $O(h)$ and exact for all polynomials of degree m. He worked out and theoretically justified the algorithm of his "Sobolev formulas". Now these formulas are known as RBL (Regular Boundary Layer) formulas. The words "Regular Boundary Layer" mean that the coefficients are calculated only in $O(h)$-neighbourhood of the boundary Γ and for inner nodes all coefficients are equal to 1.

Let us give the exact description of the RBL algorithm. Let Ω be the domain of integration and $\{hk|\ k \in \mathbb{Z}^n\}$ be the lattice of nodes. We define the elementary mesh of the lattice as

$$h(Q + k) = \{x|\ x = hk + hy,\ y \in Q\}, \tag{5}$$

where $Q = [0, 1)^n$ is the unit cube. The elementary cubature formula is

$$\int\limits_{h(Q+k)} f(x)dx \approxeq h^n \sum_{\substack{s\in\mathbb{Z}^n,\\ |s|\le L}} a_{k,s} f(hk + hs). \tag{6}$$

The coefficients $a_{k,s}$ are determined by the conditions of exactness for all polynomials of degree M. If $\rho(x, \Gamma)$ is the distance of $x \in \Omega$ to Γ and $\rho(hk, \Gamma) > L_1 h$ for some $L_1 > 0$ independent of h and k, then it is assumed that $a_{k,s} = a_s$ are independent of k and h. Otherwise the coefficients $a_{k,s}$ essentially depend on k, s and h.

Later on it was proved that these RBL formulas possess the property of asymptotic optimality not only for $B = L_2^{(m)}(\Omega)$ but also for some other spaces B.

Sobolev's algorithm was modified in order to make it applicable for designing practical programmes for calculation of integrals, and new BBL (Bounded Boundary Layer) cubature formulas were constructed. The programmes using BBL formulas work for domains of arbitrary shape and for dimensions from 2 to 10.

They are designed for multiprocessor computing systems and have high efficiency of using processors — around 70–90%. These programmes were, in particular, applied for solving integral equations on domains of arbitrary shapes.

Let us give the definition of BBL formulas.

We say that a cubature formula

$$K_N f \equiv \det H_N \sum_{\substack{k\in\mathbb{Z}^n,\\ H_N k\in\Omega}} c_{k,N} f(H_N k), \quad N = \frac{|\Omega|}{\det H_N} \tag{7}$$

is a BBL formula if for some $L_2 > 0$

$$\forall k \quad |c_{k,N}| \le L_2; \quad \rho(H_N k, \Gamma) > L_2 h \Rightarrow c_{k,N} = 1. \tag{8}$$

Among these formulas there exist some with bad approximation properties. For example, if all coefficients $c_{k,N}$ are equal to 1, the approximation is only of degree 1, $O(h)$. So it is important to find algorithms giving high-precision BBL formulas. We describe one of such algorithms. It produces asymptotically optimal formulas.

For the sake of simplicity we discuss only the case of cubic lattices of nodes

$$\{hk|\ k \in \mathbb{Z}^n\}. \tag{9}$$

Let Ω be an n–dimensional bounded domain:

$$\Omega = \{x|\ x \in \mathbb{R}^n,\ \Phi(x) > 0,\ \Phi \in C^M,\ D\Phi(x) \ne 0 \text{ if } \Phi(x) = 0\}. \tag{10}$$

The boundary $\Gamma = \{x|\ x \in \mathbb{R}^n,\ \Phi(x) = 0\}$ is smooth and could be locally represented by graphs of some functions, i.e. $\forall \hat{x} \in \Gamma\ \exists \varepsilon(\hat{x}) > 0$ such that, if $U_{\hat{x}} = \{x|\ |x - \hat{x}| < \varepsilon(\hat{x})\}$, then

$$\Gamma \cap U_{\hat{x}} = \{x|\ x \in U_{\hat{x}},\ \exists j = j(\hat{x}),\ x_j = \psi_j(x_1, \dots, x_{j-1}, x_{j+1}, \dots, x_n),\ \psi_j \in C^M\}. \tag{11}$$

Consider a finite covering of Γ with some $U_{\hat{x}(\alpha)},\ \alpha = 1, \dots, k$. Let $\Omega_0 = \Omega \setminus \bigcup\limits_{\alpha=1}^{k} U_{\hat{x}(\alpha)}$, then

$$\rho(\Omega_0, \Gamma) \equiv \varepsilon_0 > 0. \tag{12}$$

Let $U^0 = \{x|\ \rho(x, \Omega_0) < \varepsilon_0/2\}$, then the set $\{U^\alpha\}_{\alpha=0}^k$ is a finite covering of $\overline{\Omega}$. It is well known that there exists a partition of unity $\{\varphi_\alpha\}_{\alpha=0}^K$ subordinated to this covering

$$\forall \alpha \quad \varphi_\alpha \in C_0^M(\mathbb{R}^n), \ \text{ supp } \varphi_\alpha \subset U^\alpha, \quad \sum_{\alpha=0}^{K} \varphi_\alpha(x)|_{x\in\Omega} \equiv 1. \tag{13}$$

Our BBL formulas are sums of local formulas for the sets $\omega^\alpha \equiv U^\alpha \cap \Omega$:

$$h^n \sum_{\substack{k\in\mathbb{Z}^n,\\ hk\in\Omega}} c_k f(hk) = \sum_{\alpha=0}^{K} h^n \sum_{\substack{k\in\mathbb{Z}^n,\\ hk\in\omega_\alpha}} c_{k,\alpha}\varphi_\alpha(hk) f(hk). \tag{14}$$

For the inner domain ω^0 we put $c_{k,0} \equiv 1$.

The local formulas for all $\omega^\alpha,\ \alpha = \overline{1,k}$ are constructed in a similar way. So we describe one of them. Let

$$\omega^1 = \{x|\ x_n \ge \psi(x_1, \dots, x_{n-1}) \equiv \psi(x'),\ x' \in \sigma \Subset \mathbb{R}^{n-1},\ \psi \in C^M(\overline{\sigma})\}. \tag{15}$$

Let us change variables: $y' = x',\ y_n = x_n - \psi(x')$. In the variables y the boundary $\Gamma \cap \omega^1$ is a part of the coordinate hyperplane $\{y|\ y_n = 0,\ y' \in \overline{\sigma}\}$. We construct auxiliary RBL cubature formulas as sums of elementary formulas (6) with the coefficients $a_{k,s} \equiv a_s$ independent of k and h and with the additional property $a_s = 0$ for $s_n < 0$. The inverse change of variables from y to x produces the cubature formula with the curved lattice of nodes

$$\{x^{(k)}|\ x_j^{(k)} = hk_j \text{ for } j = \overline{1, n-1},\ x_n^{(k)} = hk_n + \psi(hk'),\ k \in \mathbb{Z}^n\}. \tag{16}$$

This lattice of nodes does not coincide with the cubic lattice of nodes (9). Moreover, the distance from a node of the curved lattice to the nearest node of the cubic lattice is the same for all nodes on any ray $\{x|\ x = (hk', t), t \geq 0\}$. Furthermore

$$x_n^{(k)} = hk_n + h\frac{\psi(hk')}{h} = h\left(k_n + \left[\frac{\psi(hk')}{h}\right]\right) + h\left\{\frac{\psi(hk')}{h}\right\}, \tag{17}$$

where $[\alpha]$ and $\{\alpha\}$ denote the integer and the fractional parts of the number α. Keeping this in mind we can substitute every value of the given function on the nodes of the curved lattice by finite linear combinations of the values of this function on neighbouring nodes of the cubic lattice, i.e.

$$\begin{aligned} &f\left(hk', h\left(k_n + \left[\tfrac{\psi(hk')}{h}\right]\right) + h\left\{\tfrac{\psi(hk')}{h}\right\}\right) \approx \\ &\approx \sum_{s=0}^{S} b_s(hk') f\left(hk', h\left(k_n + \left[\tfrac{\psi(hk')}{h}\right]\right) + s\right). \end{aligned} \tag{18}$$

The coefficients $b_s(hk')$ are determined by the condition of exactness of this formula for any polynomial of degree M. The corresponding algebraic system has a solution if $S \geq M + 1$. After changing all values $f(hk', hk_n + \psi(hk'))$ by

$$\sum_{s=1}^{S} b_s(hk') f\left(hk', h\left(k_n + \left[\frac{\psi(hk')}{h}\right]\right)\right)$$

we get the desired cubature formula. This formula has the BBL-property and is asymptotically optimal on any space $\widetilde{W}_p^m(\Omega)$, $1 < p < \infty$, with $m \in \left(\frac{n}{p}, M\right)$ and on several other spaces which are often used in numerical mathematics.

We must note that we use some special norms of the spaces W_p^m. For example, if $\overline{\Omega} \subset Q = [0,1)^n$, then

$$\|f\|_{\widetilde{W}_p^m(\Omega)} \equiv \inf_{g|_\Omega = f} \|g\|_{\widetilde{W}_p^m(Q)}, \tag{19}$$

where $g(x) = \sum\limits_{k \in \mathbb{Z}^n} g_k e^{2\pi i k x}$ and

$$\|g\|_{\widetilde{W}_p^m(Q)} = \left[\int_Q \left|\sum_{k \in \mathbb{Z}^n} g_k (1 + |k|^2)^{m/2} e^{2\pi i k x}\right|^p\right]^{\frac{1}{p}}. \tag{20}$$

Next we present the results of numerical experiments for the programme which uses the BBL cubature formulas described above. This programme was designed in the Institute of Mathematics, Ufa, by Dr. D.Y. Rakhmatullin [3].

Programme "CubaInt"

The programme "CubaInt" is designed for calculating integrals on multi-dimensional convex bounded domains with smooth boundaries. It was tested for the following parameters:

- dimension n from 2 to 10;
- integrand $f(x) = \sum_{i=1}^{n} a_i x_i^{b_i}$;
- parameter of smoothness M from 2 to 6;
- lattice step h from 10^{-1} to 10^{-5};
- domain $\Omega = \{x : \Phi(x) > 0,\ \Phi(x) = 1 - \sum_{i=1}^{n} c_i (x_i - 0.5)^{d_i}\}$;
- number of processors P from 1 to 7000.

For the parameters listed below we compared calculations with the theoretical evaluations:
$a = (2, 1, 2, 1, ..., 2, 1),\ b = (2, 4, 2, 4..., 2, 4)$,
$c = (6.25, 39.0625, ..., 6.25, 39.0625),\ d = (2, 4, 2, 4..., 2, 4)$,
We conducted a number of tests with decreasing values of the parameter h. Thus we had a sequence of the parameters:

$$h_1,\ h_2,\ h_3,\ \ldots \text{ with } h_1 > h_2 > h_3 > \ldots$$

For them we computed the appropriate values of cubature formulas:

$$K_{h_1},\ K_{h_2},\ K_{h_3},\ \ldots$$

The absolute error of the calculations at the k -th step was computed as the absolute value of the difference of the value of the cubature formulas with two sequential values of the parameter h :

$$\triangle_k = |K_{h_k} - K_{h_{k+1}}|.$$

The theoretical error we considered is the value of the error functional $\|l_h\|_*$.

We represent every absolute error in the form $w_1 \cdot 10^{-w_2}$ with integer number w_2 and $1 \leqslant w_1 < 10$.

Tables 1–3 demonstrate the degrees of absolute errors (numbers w_2) of the results obtained by computer calculations and by theoretical approximations.

For example, we have the same accuracy 10^{-15} for $\widetilde{N} = 3200,\ \widetilde{N} := 1/h = N^{1/n}$ and $M = 4$ in both the left and the right sides of Table 1.

The calculations are bad in two cases. First, when rounding errors are significant (we use long double type). Secondly, when h and M are not sufficiently small which prevents including $2M$ lattice nodes along the corresponding rays to the boundary layer.

We must note that the convexity of a given domain is not necessarily required. Let us take for example the domain (Fig. 1).

$$\Omega = \left\{x : 1 - 6.25(x_1 - 0.5)^2 - 6.25(x_2 - 0.6 + 3(x_1 - 0.5)^2)^2 > 0\right\}.$$

Numerical results are shown in the Table 4.

$\widetilde{N}\backslash M$	2	3	4	5	6	$\widetilde{N}\backslash M$	2	3	4	5	6
50	3	3	2	2	1	50	4	6	7	9	11
100	4	4	3	2	2	100	4	6	8	10	12
200	7	5	4	5	3	200	5	7	10	12	14
400	8	9	11	7	7	400	6	8	11	14	16
800	8	10	12	13	14	800	6	9	12	15	18
1600	9	11	13	15	16	1600	7	10	13	17	20
3200	10	12	15	16	17	3200	8	11	15	18	22
6400	11	14	16	18	18	6400	8	12	16	20	23
12800	12	15	17	18	17	12800	9	13	17	21	25

Table 1: Experimental (left) and theoretical (right) degrees of the absolute errors, n=2

$\widetilde{N}\backslash M$	2	3	4	5	6	$\widetilde{N}\backslash M$	2	3	4	5	6
50	3	3	2	2	1	50	4	6	7	9	11
100	4	4	3	3	3	100	4	6	8	10	12
200	6	7	5	4	4	200	5	7	10	12	14
400	8	8	8	8	7	400	6	8	11	14	16
800	9	9	10	9	9	800	6	9	12	15	18
1600	9	10	11	11	10	1600	7	10	13	17	20

Table 2: Experimental (left) and theoretical (right) degrees of the absolute errors, n=3

$\widetilde{N}\backslash M$	2	3	4	5	6	$\widetilde{N}\backslash M$	2	3	4	5	6
25	4	3	3	3	2	25	3	5	6	7	9
50	4	4	4	3	3	50	4	6	7	9	11
75	5	4	4	3	2	75	4	6	8	10	12
100	5	4	4	4	3	100	4	6	8	10	12
125	6	5	4	4	4	125	5	7	9	11	13
150	7	6	5	5	4	150	5	7	9	11	14
175	7	7	5	5	4	175	5	7	9	12	14
200	7	7	6	5	5	200	5	7	10	12	14

Table 3: Experimental (left) and theoretical (right) degrees of the absolute errors, n=5

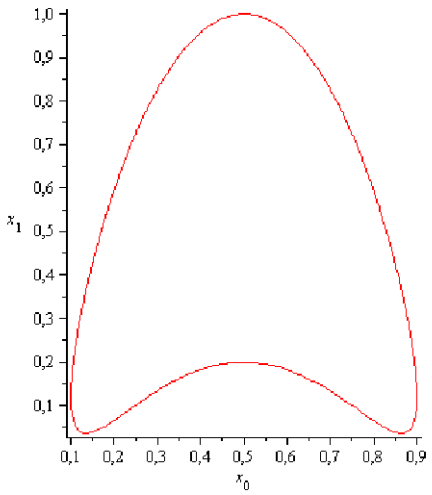

Figure 1: Non-convex domain

$\widetilde{N} \backslash M$	2	3	4	5	6
50	4	3	3	3	2
100	4	4	4	4	4
200	4	4	4	4	4
400	5	5	5	5	5

Table 4: Numeric results for non-convex domain

Let us analyze the programme speed and the quality of its parallelization. We use the speedup and efficiency parameters:

$$S_P = \frac{T_1}{T_P}, \qquad E_P = \frac{S_P}{P},$$

where T_P is time of calculating on P processors.

Figure 2 demonstrates both experimental (dark line) and theoretical (bright line) speedups.

For example, when $n = 2,\ M = 3,\ \widetilde{N} = 3200,\ P = 1200$, the efficiency is 0.83.

The efficiency is gradually decreasing with the growth of the number of processors. It is caused by different computational complexity in different lattice nodes and therefore non-uniform processors load.

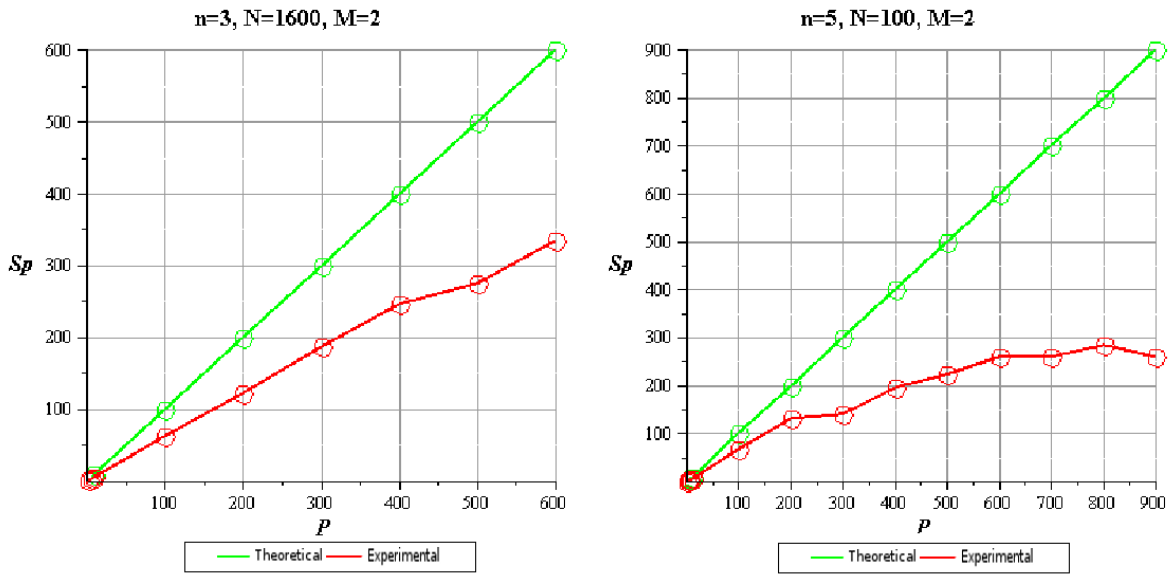

Figure 2: Experimental and theoretical speedups

Programme for solving integral equations

The programme was designed in the Institute of Mathematics, Ufa, by the post-graduate E.L. Bannikova [4].

Algorithm

We consider the integral equation

$$u(x) - \int\limits_{\Omega} K(x,y)u(y)dy = f(x), \quad x \in \Omega \subset \mathbb{R}^2. \tag{21}$$

Here Ω is a two-dimensional bounded closed domain with smooth boundary, $K \in C^M(\Omega \times \Omega)$ and $f \in C^M(\Omega)$.

Assume that $||K||_{C(\Omega\times\Omega)} = \theta < 1$. This is sufficient for convergence of successive approximations

$$u_0(x) = f(x),\ u_{s+1}(x) = f(x) + \int\limits_{\Omega} K(x,y)u_s(y)dy,\ s = 1, 2, \ldots.$$

The functions $u_s(x)$ are approximations of the solution u of equation (21) in the norm of the space $C(\Omega)$.

Numerical realization of this method uses BBL cubature formulas on every iteration.

Programme

The program for numerical solution of integral equations was designed on the base the above algorithms.

Calculations stop when the condition $||u_s - u_{s-1}|| \leq \varepsilon$ becomes true, where $||u|| = \max\limits_{x\in\Omega} |u(x)|$, ε — given accuracy.

Input data of the programme:

1. Integration domain Ω defined implicitly: $\Omega = \{x|\Phi(x) \geqslant 0\}$, $\Phi(x) \in C^M(Q)$, $|\Phi(x)| + |D\Phi(x)| \neq 0$, $\Omega \subset [0,1)^2$.
2. Function $K(x_1, y_1, x_2, y_2)$, $\max\limits_{x,y} |K(x,y)| < 1$.
3. Function $f(x_1, x_2)$.
4. Cubic lattice step $h < 0.01$.
5. Smoothness parameter M.
6. Number of processors P.

The approximate solution of the integral equation is the function u_s obtained on the last iteration. It is displayed as the table of the values of the function $u_s(x)$ in the nodes $x = hk \in \Omega$.

Programme tests

The programme was tested on the supercomputer "MVS-100k" of the Joint Supercomputer Center, Russian Academy of Sciences, Moscow.

We demonstrate some experimental results of solving integral equations. In order to compare experimental results with the precise solution, we took a function that must be a solution and calculated $f(x)$. Numerical experiment was done with that $f(x)$.

Here is the input data.

The precise solution was the function $u(x) = (x_1 - x_2)^5$.

1. $K(x,y) = (0.1x_1y_1 + 0.5x_2y_2)^3$.
2. $f(x) = (x_1 - x_2)^5 - (3.5 \cdot 10^{-6}x_2x_1^2 + 9.9 \cdot 10^{-7}x_1^3 - 1.4 \cdot 10^{-5}x_2^2x_1 - 1.1 \cdot 10^{-4}x_2^3)$.
3. Integration domains $\Omega_1 = \{x|\Phi_1(x) \geqslant 0\}$,
   $\Phi_1(x) = 1 - ((x_1 - 0.5)/0.4)^2 - ((x_2 - 0.5)/0.4)^4$ — this is a convex domain (see Fig. 3).
4. Cubic lattice step $h = 1/200,\ 1/300$.
5. Smoothness parameter $M = 2, 3$.
6. Number of processors 10–1000 .

The expected accuracy of the calculations is $O((h/\varepsilon_0)^M)$, where ε_0 estimates the thickness of the boundary layer (see (12)).

The calculation accuracy was obtained in two ways — by comparing with the precise solution and by the Runge rule, i.e. by stability of decimal digits in the results with decreasing h.

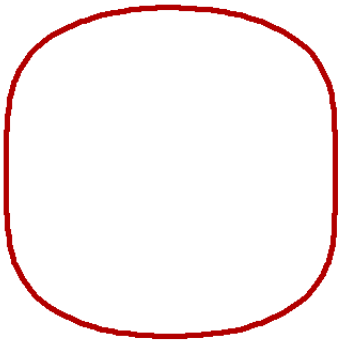

Figure 3: The domain Ω_1

s — iteration number	$h = 1/200,\ \|u_s - u_{s-1}\|$	$h = 1/300,\ \|u_s - u_{s-1}\|$
1	0.0987	0.0988
2	0.000776	0.000778
3	0.0000268	0.0000269
4	0.00000310	0.00000310

Table 5: The achieved accuracy for the smoothness parameter $M = 2$

Table 5 contains iterative process data with given parameters.

In Table 5 the accuracy was calculated by decimal digits stability. After comparing last iteration result with the exact solution $u(x) = (x_1 - x_2)^5$ we have got the coincidence of $5 - 6$ decimal digits.

Here the influence of $1/\varepsilon_0$ when $\varepsilon = 0.1$ is insignificant, because the domain Ω_1 is convex and the number of regions of the partition of the unity is small.

Table 6 shows the programme working time for different numbers of processors quantity.

P	**10**	**20**	**30**	**40**	**50**	**100**
T_P	230	127	93	75	63	36
S_P	10	18	24.7	36	42.6	63.8
E_P	1	0.9	0.8	0.8	0.72	0.63

Table 6: Running time with different numbers of processors P, $h = 1/200,\ M = 2$. The achieved accuracy is 10^{-5}.

The next numerical experiment was conducted with the non-convex domain

$$\Omega_2 = \{x|\Phi_2(x) \geqslant 0\},\ \Phi_2(x) = 1 - \sqrt{8(x_1 - 0.5)^2 + 8(x_2 - 0.5)^2} +$$
$$+(2(x_1-0.5)(x_2-0.5)\sin(1)+\cos(1)((x_1-0.5)^2-(x_2-0.5)^2)^2)/((x_1-0.5)^2+(x_2-0.5)^2)^2,$$

which is shown on Fig. 4.

In view of the fact that in this test the exact solution of the equation is not known, the accuracy of calculations is evaluated according to the stability of decimal digits. We have got that the theoretically expected accuracy of the numerical solution of the integral

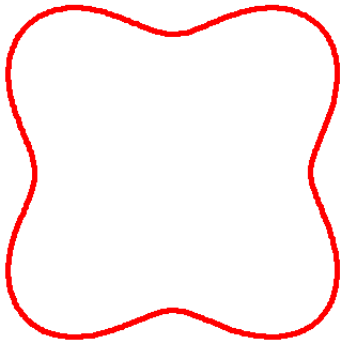

Figure 4: The domain Ω_2

equation $(h/\varepsilon_0)^M$ of order 10^{-5} is achieved on the fifth iteration with $h = 0.004, M = 3, \varepsilon_0 = 0.1$.

Yet another test with the following data:

1. $K(x,s) = (0.1x_1s_1 + 0.5x_2s_2)^3$.
2. $f(x) = (x_1 + x_2)^3$.
3. The integration domain (disconnected)

$$\Omega_3 = \{x | \Phi_3(x) \geqslant 0\},$$
$$\Phi_3(x) = -(1 - 9(x_1 - 0.5)^2 - 100((x_2 - 0.3) - 2(x_1 - 0.5)^2)^2)\cdot$$
$$\cdot(1/144 - ((x_1 - 0.5) - 1/10)^2 - (((x_2 - 0.5) + 0.2) - 0.3)^2),$$

see Fig. 5.

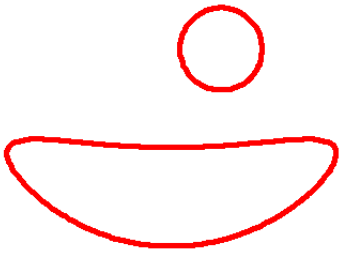

Figure 5: The domain Ω_3

The accuracy was calculated by decimal digits stability. Table 7 shows that the theoretically expected accuracy $(h/\varepsilon_0)^M$ of order 10^{-5} is achieved on the fourth iteration with $h = 0.005, M = 3, \varepsilon_0 = 0.3$.

s—iteration number	1	2	3	4
$\|\|u_s - u_{s-1}\|\|$	0.01	0.0001	0.00001	0.000001

Table 7: The achieved accuracy

The programme running time for 1000 processors was 12 seconds.

Thus, using of BBL-formulas is good for numerical solution of integral equations. The application of the iteration method in combination with BBL lattice cubature formulas allows to achieve accuracy 10^{-5} by 5–6 iterations.

We must note that this algorithm of solving integral equations allows to parallelize well the computing programme for use on the multiprocessor computing systems.

Conclusions

Here are the most important results of the theory of BBL formulas. In fact the asymptotic optimality and the optimality by order are very close concepts for cubature formulas with the bounded boundary layer. For simplicity, we assume that the domain Ω belongs to the unit cube $Q = [0,1)^n$ and has the boundary $\Gamma \subset C^M$. Let the lattice of nodes be cubical, $\{h \cdot k \mid k \in \mathbb{Z}^n\}$, with lattice step h, i.e., $h = 1/\widetilde{N},\ \widetilde{N} \in \mathbb{Z}_+$. The spaces $\widetilde{W}_p^m(\Omega)$ are defined by the norms (19), (20) where $m \in (n/p,\ M)$ with some natural $M > \frac{n}{p}$.

Let

$$K_h :\ f \to h^n \sum_{hk \in \Omega} c_k(h) f(hk) \tag{22}$$

be any sequence of BBL formulas.

THEOREM. *Let $1 < p_1 < p_2 < \infty$ and $\frac{n}{p_1} < m_1 < m_2 < M$. Then the sequence $\{K_h\}$ is asymptotically optimal in every space of the family $\{\widetilde{W}_p^m(\Omega)\}_{\substack{m \in (m_1, m_2) \\ p \in (p_1, p_2)}}$, if and only if it is optimal by order in every of these spaces.*

Remark. The number M is involved in the design of our BBL formulas described above, ensuring optimality by order for each of the spaces $\widetilde{W}_p^m(\Omega)$ with $m < M$. Therefore, these formulas are universally asymptotically optimal for every $m \in (n/p,\ M)$.

This is very important for the success of the programme for approximate integration of functions with various smoothness. We name algorithms with this property as conditionally unsaturated algorithms, trying to follow the terminology proposed in his time by K.I. Babenko [5].

The same sequence of cubature formulas remains asymptotically optimal on some spaces with anisotropic smoothness. Namely, let the space $\widetilde{W}_2^\mu(\Omega)$ be defined with the help of the norm

$$\begin{aligned} \|f\|_{\widetilde{W}_2^\mu(\Omega)} &= \inf_{g|_\Omega = f} \|g\|_{\widetilde{W}_2^\mu(Q)}, \quad g(x) = \sum_{k \in \mathbb{Z}^n} g_k e^{2\pi i k x}, \\ \|g\|_{\widetilde{W}_2^\mu(\Omega)} &= \left(\int_Q \sum_{k \in \mathbb{Z}^n} |g_k \mu(2\pi i k)|^2 \, dx \right)^{\frac{1}{2}}. \end{aligned} \tag{23}$$

We assume that $|\mu(\xi)| \le C(1+|\xi|)^m$ with some $m < M$ and the function μ (that describes smoothness) satisfies the estimate

$$\forall \alpha \in \mathbb{Z}_+^n \quad \frac{|D^\alpha \mu(\xi)|}{|\mu(\xi)|} \le C_\alpha (1+|\xi|)^{-\rho|\alpha|} \quad \text{with some } \rho > 0. \tag{24}$$

This is exactly the conditions of the hypoellipticity of pseudodifferential operator

$$g(x) \to \sum_{k \in \mathbb{Z}^n} g_k \mu(2\pi i k) e^{2\pi i k x}. \tag{25}$$

See [6]–[12] for details.

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