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RIESZ BASES IN WEIGHTED SPACES

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Abstract. The article deals with weighted Hilbert spaces with convex weights. Let h be a convex function on a bounded interval I of the real axis. We denote a space of locally integrable functions on I, such that

$$||f|| := \sqrt{\int_I |f(t)|^2 e^{-2h(t)}\,dt} < \infty$$

by $L_2(I, h)$.

If $I = (-\pi; \pi)$, $h(t) \equiv 1$, the space $L_2(I, h)$ coincides with the classical space $L_2(-\pi; \pi)$ and the Fourier trigonometric system is a Riesz basis in this space. As it has been shown by B.J. Levin, nonharmonic Riesz bases in $L_2(-\pi; \pi)$ can be constructed using a system of zeros of entire functions of a sine type. In this paper, we prove that if a Riesz basis of exponentials exists in the space $L_2(I, h)$, this space is isomorphic (as a normed space) to the classical space $L_2(I)$. Thus, the existence of Riesz bases of exponentials is the exclusive property of the classical space $L_2(-\pi; \pi)$.

Keywords: Riesz basis, weighted Hilbert spaces, reproducing kernel, Fourier-Laplace transform, functions of sine type.

Let us assume that I ia a bounded interval of a real axis, $h(t)$ is a convex function on this interval, and $L^2(I, h)$ is a space of locally integrable functions on I satisfying the condition

$$||f|| := \sqrt{\int_I |f(t)|^2 e^{-2h(t)}\,dt} < \infty.$$

It is a Hilbert space with a scalar product

$$(f, g) = \int_I f(t)\overline{g}(t) e^{-2h(t)}\,dt.$$

The present paper is devoted to existence of the Riesz exponential bases in spaces $L_2(I, h)$. In the classical case, the Fourier system $e^{\pi n i}$ constructs an orthonormal basis when $I = (-\pi; \pi)$, $h(t) \equiv 1$. Evidently, there can be no orthonormal exponential bases in the spaces $L_2(I, h)$ in other cases. The notion of the Riesz basis is introduced in [10] by N.K. Bary and indicates the image of an orthonormal basis when the operator is bounded and invertible. Investigation of nonorthonormal exponential bases in the space $L_2(-\pi; \pi)$ has a long history and is vital nowadays as well. This topic is referred to as the nonharmonic Fourier analysis in the literature. Initially, coefficients of the exponential basis $(e^{\lambda_n t})$, $n = 1, 2, ...$, were considered as a perturbation of integers, i.e. in the form $\lambda_n = n + \alpha_n$. B.Ya. Levin suggested for the first time in [11] that a sequence of coefficients should be characterized as a set of zeroes of an entire function with various properties. These entire functions were later termed as the entire

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sine-type functions in [12]. Namely, an entire sine-type function is a term for entire exponential functions that satisfy the estimate

$$0 < c \leq |L(z)|e^{-\pi \mathrm{Re}\, z} \leq C < \infty$$

outside a certain vertical strip. It is demonstrated in [11], that a system of exponentials, whose sequence of coefficients coincides with a set of all zeroes of an entire sine-type function generates a generalized basis in $L_2(-\pi;\pi)$. Soon, V.D. Golovin demonstrated in [13] that if zeroes of a sine-type function have the separation property, i.e.

$$\inf_{n\neq m} |\lambda_n - \lambda_m| > 0,$$

then the corresponding system of exponentials constructs the Riesz basis in the space $L_2(-\pi;\pi)$.

The present paper demonstrates that exponential Riesz bases do not exist in nonclassical cases.

The basic tool of the present investigation is the Fourier-Laplace transform of functionals. A function

$$\widehat{S}(\lambda) = S(e^{\lambda t}),\ \lambda \in \mathbb{C}$$

is called a Fourier-Laplace transform of the functional S in the space $L_2(I,h)$. If the functional S is generated by an element $g \in L_2(I,h)$, then

$$\widehat{S}(\lambda) = \int_I e^{\lambda t - 2h(t)}\overline{g(t)}dt,\ \lambda \in \mathbb{C}.$$

Manifestly, the mapping $\mathcal{L} : S \longrightarrow \widehat{S}$ embeds the conjugate space $L_2^*(I,h)$ into a space of entire functions. For the sake of brevity, the space $L_2(I,h)$ is denoted by H. The image of mapping is designated by $\widehat{H} = \widehat{L}_2(I,h)$. The mapping $\mathcal{L} : H^* \longrightarrow \widehat{H}$ is one-to-one, because the system of all exponentials $\{e^{\lambda t}\},\ \lambda \in \mathbb{C}$ is complete. One can introduce an induced structure of the Hilbert space in the space $\widehat{H}$ by the formula

$$(\widehat{S}_1, \widehat{S}_2)_{\widehat{H}} = (S_1, S_2)_{H^*},\ S_1, S_2 \in H^*.$$

The mapping $\mathcal{L}$ is an isomorphism of spaces $\widehat{L}_2(I,h)$ and $L_2^*(I,h)$. Making use of the standard identification of linear continuous functionals in the Hilbert space with an element of the space, one obtains a conjugate linear isomorphism of spaces H and $\widehat{H}$ by the formula

$$f \longrightarrow (e^{\lambda t}, f)_H.$$

Such mapping is denoted by $\mathcal{L}$ as well and the image $\mathcal{L}(f)$ for $f \in L_2(I,h)$ is denoted by $\widehat{f}$. Thus,

$$\widehat{f}(\lambda) = \int_I \overline{f(t)}e^{\lambda t - 2h(t)}dt,\ f \in L_2(I,h),$$

while

$$(\widehat{f}, \widehat{g})_{\widehat{L}_2(I,h)} = (g, f)_{L_2(I,h)}.$$

Lemma 1. *For $w \in \mathbb{C}$, let us indicate the functional generated by the function e^{wt} by E_w. Then, the function $K(\lambda, w) = \widehat{E}_w(\lambda)$ is a reproducing kernel (see [3]) in the space $\widehat{L}_2(I,h)$, i.e. for any function $F \in \widehat{L}_2(I,h)$*

$$F(w) = (F(\lambda), K(\lambda, w)).$$

Proof. Indeed, if $F = \widehat{f}$, then

$$(F(\lambda), K(\lambda, w)) = (\widehat{f}, \widehat{E}_w) = (e^{wt}, f)_{L_2(I,h)} = \widehat{f}(w) = F(w).$$

Writing it directly, one obtains

$$K(\lambda, w) = (e^{\lambda t}, e^{wt}) = \int_I e^{\lambda t + \overline{w}t - 2h(t)}dt.$$

The system of elements $e_k,\ k=1,2,...$, in a Hilbert space is called an unconditional basis (see [1]), if it is complete and there are numbers $c, C>0$ such that the correlation

$$c\sum_{j=1}^{n}|c_k|^2||e_k||^2 \le ||\sum_{j=1}^{n}c_k e_k||^2 \le C\sum_{j=1}^{n}|c_k|^2||e_k||^2$$

holds for any set of numbers $c_1, c_2, ..., c_n$. It is known (see [2],[4]), that if the system $e_k,\ k=1,2,...$ is an unconditional basis, then any element of the space H is represented uniquely in the form of the series

$$x=\sum_{k=1}^{\infty}x_k e_k,$$

and

$$c\sum_{k=1}^{\infty}|x_k|^2||e_k||^2 \le ||x||^2 \le C\sum_{k=1}^{\infty}|x_k|^2||e_k||^2.$$

An unconditional basis $e_k,\ k=1,2,...$ in a Hilbert space is called the Riesz basis if $||e_k|| \asymp 1$ (see [4]).

Lemma 2. *The system of exponentials* $\{e^{\lambda_k t}\},\ k=1,2,...$ *is an unconditional basis in the space* $L_2(I,h)$ *if and only if the system* $\{K(\lambda,\lambda_k)\},\ k=1,2,...$ *constructs an unconditional basis in the space* $\widehat{L}_2(I,h)$.

Proof. Indeed, if a system of exponentials generates an unconditional basis, then any element $f\in L_2(I,h)$ is expanded into the series

$$f(t)=\sum_{k=1}^{\infty}c_k e^{\lambda_k t}, \tag{1}$$

and

$$||f||^2 \asymp \sum_{k=1}^{\infty}|c_k|^2||e^{\lambda_k t}||^2=\sum_{k=1}^{\infty}|c_k|^2K(\lambda_k,\lambda_k)=\sum_{k=1}^{\infty}|c_k|^2||K(\lambda,\lambda_k)||^2. \tag{2}$$

Upon scalar multiplication of (1) by $e^{\lambda t}$, one obtains

$$\widehat{f}(\lambda)=\sum_{k=1}^{\infty}\overline{c}_k K(\lambda,\lambda_k),$$

and the relation (2) means that

$$||\widehat{f}||^2 \asymp \sum_{k=1}^{\infty}|\overline{c}_k|^2||K(\lambda,\lambda_k)||^2.$$

The following properties of unconditional bases are proved in [4] (p. 374).

B1. The system $e_k,\ k=1,2,...$ is an unconditional basis in the space H if and only if the biorthogonal system $h_n,\ n=1,2...$ is an unconditional basis in the space H.

B2. If $e_k,\ k=1,2,...$, is an unconditional basis in the space H, and $h_n,\ n=1,2...$ is a biorthogonal system, then $||e_k||\cdot||h_k|| \asymp 1$.

Suppose that $K(z,z)=K(z)$.

Lemma 3. *If a system of exponents* $\{e^{\lambda_k t}\},\ k=1,2,...$, *is an unconditional basis in the space* $L_2(I,h)$, *then there are such constants* $c, C>0$ *that the following estimates hold for any function* $F\in\widehat{L}_2(I,h)$:

$$c||F||^2 \le \sum_{k=1}^{\infty}\frac{|F(\lambda_k)|^2}{K(\lambda_k)} \le C||F||^2. \tag{3}$$

Proof. Let the system of exponents $\{e^{\lambda_k t}\}$, $k = 1, 2, ...$, be an unconditional basis in the space $L_2(I, h)$. According to Lemma 2, the system $\{K(\lambda, \lambda_k)\}$, $k = 1, 2, ...$ is an unconditional basis in the space $\widehat{L}_2(I, h)$. Let $E_n(\lambda)$, $n = 1, 2, ...$ be an biorthogonal basis in $\widehat{L}_2(I, h)$. Due to the property B1, the system $E_n(\lambda)$, $n = 1, 2, ...$ is an unconditional basis, i.e. every element $F \in \widehat{L}_2(I, h)$ is represented by the series

$$F(\lambda) = \sum_{n=1}^{\infty} F_n E_n(\lambda),$$

and

$$||F||^2 \asymp \sum_{n=1}^{\infty} |F_n|^2 ||E_n(\lambda)||^2.$$

Due to biorthogonality, one has $F_n = (F(\lambda), K(\lambda, \lambda_n)) = F(\lambda_n)$ and by the property B2 $||E_n||K(\lambda_n) \asymp 1$. Thus, the relation (3) holds.

The works [7],[8] contain description of the space $\widehat{L}_2(I, h)$.

Theroem A. *The space $\widehat{L}_2(I, h)$ is isomorphic (as a Banach space) to the space of entire functions F, satisfying the conditions*

$$|F(z)| \leq C_F \sqrt{K(z)}, \ z \in \mathbb{C},$$

$$||F||^2 = \int_{\mathbb{R}} \int_{\mathbb{R}} \frac{|F(x + iy)|^2}{K(x)} d\widetilde{h}'(x) dy < \infty,$$

where

$$K(z) = \int_I |e^{2zt}| e^{-2h(t)} dt, \ \widetilde{h}(x) = \sup_{t \in I}(xt - h(t)).$$

Theorem 1. *If there is an exponential Riesz basis in the space $L_2(I, h)$, then $e^{h(t)} \asymp 1$, i.e. the space $L_2(I, h)$ is isomorphic (as a Banach space) to the classical space $L_2(I)$.*

Proof. If the system $\{e^{\lambda_k t}\}$ is the Riesz basis in $L_2(I, h)$, then the coefficients λ_k belong to a vertical strip, i.e. $|\text{Re } \lambda_k| \leq d$ for some $d > 0$.

As it was shown in [5] (see also [6]), coefficients of the unconditional basis $\{e^{\lambda_k t}\}$ satisfy the separation condition, i.e. $|\lambda_k - \lambda_m| \geq \delta$ for some $\delta > 0$ when $m \neq k$. Let us take a number $T > 0$ from the conditions $\widetilde{h}'(T) - \widetilde{h}'(-T) \geq \frac{|I|}{2}$ and $T \geq d + \delta$. Such number can be found because $\widetilde{h}'(\infty) - \widetilde{h}'(-\infty) = |I|$.

Since the function $K(z) = K(\text{Re } z)$ is bounded from zero and infinity in the strip $|\text{Re } z| \leq d$ then the relation

$$c||F||^2 \leq \sum_{k=1}^{\infty} |F(\lambda_k)|^2 \leq C||F||^2, \ F = \widehat{f} \in \widehat{L}_2(I, h) \tag{4}$$

holds due to Lemma 3. The left-hand side of the equality provides

$$\sum_{k=1}^{\infty} |F(\lambda_k)|^2 \geq c \int_{|x| \leq T} \int_{\mathbb{R}} |F(x + iy)|^2 dy d\widetilde{h}'(x). \tag{5}$$

The equality

$$F(x + iy) = \int_I e^{iyt} e^{xt - 2h(t)} \overline{f}(t) dt, \forall x \in \mathbb{R},$$

provides

$$\int_{\mathbb{R}} |F(x + iy)|^2 dy = \int_I |f(t)|^2 e^{2xt - 4h(t)} dt$$

by the Plancherel formula. Whence, if $max\{|t|, t \in I\} = a$, the estimate

$$e^{-2aT} \int_I |f(t)|^2 e^{-4h(t)} dt \leq \int_{\mathbb{R}} |F(x+iy)|^2 dy \leq e^{2aT} \int_I |f(t)|^2 e^{-4h(t)} dt \tag{6}$$

holds for all $x \in [-T; T]$. The left-hand inequality together with (5) indicates that

$$\sum_{k=1}^{\infty} |F(\lambda_k)|^2 \geq \frac{|I|c}{2} e^{-2aT} \int_I |f(t)|^2 e^{-4h(t)} dt. \tag{7}$$

The properties of subharmonic functions provide the estimate

$$|F(\lambda_k)|^2 \leq \frac{1}{\pi\delta^2} \int_{B(\lambda_k,\delta)} |F(z)|^2 dm(z).$$

Since the coefficients are separable, this provides

$$\sum_{k=1}^{\infty} |F(\lambda_k)|^2 \leq \frac{1}{\pi\delta^2} \int_{\bigcup_k B(\lambda_k,\delta)} |F(z)|^2 dm(z) \leq \frac{1}{\pi\delta^2} \int_{|x|\leq T} \int_{\mathbb{R}} |F(x+iy)|^2 dy dx.$$

Invoking the right-hand inequality in (6), one has

$$\sum_{k=1}^{\infty} |F(\lambda_k)|^2 \leq \frac{2T}{\pi\delta^2} e^{2aT} \int_I |f(t)|^2 e^{-4h(t)} dt.$$

This estimate, together with (4) and (7) means that the relations

$$b \int_I |f(t)|^2 e^{-4h(t)} dt \leq \int_I |f(t)|^2 e^{-2h(t)} dt \leq B \int_I |f(t)|^2 e^{-4h(t)} dt$$

hold for some constants b, B and for all $f \in L_2(I, h)$. Whence, one can conclude by means of standard methods with the use of "cap"type functions that $b \leq e^{-2h(t)} \leq B$.

Theorem 1 is proved.

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