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ESTIMATION OF THE BIFURCATION PARAMETER IN SPECTRAL PROBLEMS FOR EQUATIONS WITH DISCONTINUOUS OPERATORS

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Abstract. We consider the existence of solutions of the eigenvalue problem for nonlinear equations with discontinuous operators in the reflexive Banach space. Coercivity of the corresponding mapping is not supposed. An upper bound for the value of the bifurcation parameter is obtained by the variational method. This result confirms that the upper bound for the value of the bifurcation parameter obtained earlier in spectral problems for elliptic equations with discontinuous nonlinearities is true.

Keywords: eigenvalues, spectral problems, discontinuous operator, variational method, upper bound, bifurcation parameter.

In the works [1, 2], the problem of existence of nonzero solutions for the equation

$$Au = \lambda Tu \tag{1}$$

was considered in a real reflexive Banach space E depending on the parameter λ. Here A is a linear selfadjoint operator from E to E^* (E^* is a space conjugate to E), $T : E \to E^*$ is a discontinuous, compact, or an antimonotone mapping bounded on E. Theorems about existence of a ray of positive eigenvalues for equations of the form (1) and of existence of an eigenvector, which is a point of radial continuity of the operator T, for every such value are obtained in Theorem 2 of [1] and Theorem 2 of [2].

In the present paper, the upper estimate of the bifurcation parameter in eigenvalue problems for equations with discontinuous operators is considered. I use the same notation and definitions as in [1, 2] without their repeated description.

It should be mentioned that investigation of Equation (1), having a great number of specific applications, is of importance, if for no other reason than the extraordinary generality of this class of equations. Namely, the general results obtained can be applied to investigation of spectral problems for elliptic-type equations with discontinuous nonlinearities. The latter are of great practical importance, e.g. in solving the M.A. Gol'dshtik problem of separating flows of an incompressible fluid [3], the problem of vortex rings in an ideal fluid L.E. Fraenkel, M.S. Berger [4], the H.J. Kuiper problem [5] on heating a conductor at a constant voltage and temperature on its surface with hopping conductivity of the material at certain temperatures, as well as in solving other particular elliptic boundary-value problems, leading to the corresponding mappings of A and T, where estimation of the bifurcation parameter is useful. Such estimation for the M.A. Gol'dshtik problem is written out in [6]. In the works [7, 8], the upper estimate of the bifurcation parameter in eigenvalue problems for elliptic-type equations with discontinuous nonlinearities is obtained in a wide class of such spectral problems.

Equation (1) is studied by means of the variational method as before [9]. In order to realize the variational approach to Equation (1), it is necessary that the operator T be quasipotential in addition Since the operator A is linear and selfadjoint in Equation (1), then it is potential.

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Its potential equals $\phi(u) = \frac{1}{2}(Au, u)$ [10]. Let us associate the functional $f^{\lambda}(u) = \phi(u) - \lambda f(u)$, where f is a quasipotential of the operator T, to Equation (1). Let us assume that $f(0) = 0$ without loss of generality.

The fundamental result of the present paper is the following theorem.

Theorem. *Suppose that*
1) A is a linear selfadjoint operator acting from a real reflexive Banach space E into the dual space E^*, the space E is represented in the form of a direct sum of closed subspaces E_1 and E_2, $E_1 = \ker A$, while there is such a constant $\alpha > 0$ that

$$(Au, u) \geq \alpha ||u||^2$$

for any $u \in E_2$;
2) the mapping T is compact or antimonotone, quasipotential (with a quasipotential f) and bounded on E, $f(0) = 0$ and for some $u_0 \in E$ the value $f(u_0) > 0$; if $E_1 \neq \{0\}$, it is supposed additionally that

$$\lim_{u \in E_1, ||u|| \to +\infty} f(u) = -\infty;$$

3) if the mapping T is compact, then it is additionally supposed that

$$\lim_{t \to +0} (T(u + th) - Tu, h) \geq 0$$

для всех $u, h \in E$;
4) if the mapping T is antimonotone, then it supposed additionally that any point of discontinuity of the operator T is regular for $F_{\lambda}u = Au - \lambda Tu$ when $\lambda > \lambda_0 > 0$ (beginning with the value λ_0 the eigenvalue problem is solvable).
Then, the following upper estimate of the bifurcation parameter in spectral problems for equations with discontinuous operators is true:

$$\lambda_0 \leqslant \inf_{u_0 \in U} \frac{1}{2}(Au_0, u_0) \left(\int_0^1 (T(tu_0), u_0) dt \right)^{-1},$$

where

$$U = \{u_0 \in E : \int_0^1 (T(tu_0), u_0) dt > 0\}.$$

Proof. The formulated theorem is a straightforward corollary of Theorem 2 from [1] and of Theorem 2 from [2]. Indeed, it follows from statements of the theorems $f^{\lambda}(u) < 0$. Whence,

$$\lambda f(u) > (Au, u)/2.$$

By virtue of the condition 2) of the theorem the value $f(u_0) > 0$ for some $u_0 \in E$. Whence,

$$\lambda > \frac{1}{2}(Au_0, u_0)(f(u_0))^{-1}.$$

Assume that

$$U = \{u_0 \in E : f(u_0) > 0\}.$$

Due to the condition 2) of the theorem the given set is not empty. Then, the following upper estimate holds for λ_0 :

$$\lambda_0 \leqslant \inf_{u_0 \in U} \frac{1}{2}(Au_0, u_0)(f(u_0))^{-1}.$$

The definition of a quasipotential provides

$$f(u_0) = \int_0^1 (T(tu_0), u_0) dt.$$

Thus, the following upper estimate for the value λ_0 in spectral problems for equations with discontinuous operators holds:

$$\lambda_0 \leqslant \inf_{u_0 \in U} \frac{1}{2}(Au_0, u_0) \left(\int_0^1 (T(tu_0), u_0) dt \right)^{-1},$$

where

$$U = \{u_0 \in E : \int_0^1 (T(tu_0), u_0) dt > 0\}.$$

The theorem is proved.

Note that the fact that the conditions of the above theorem are met for the corresponding elliptic boundary-value problems with discontinuous nonlinearities is established similarly to the works [1, 2]. Therefore, a similar upper estimate exists for the value of the bifurcation parameter in spectral problems for elliptic-type equations with discontinuous nonlinearities, which confirms results of the works [7, 8].

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