ISSN 2074-1863 Ufa mathematical journal. Volume 3. № 1 (2011). Pp. 3-15.

UDC 517.5

UNCONDITIONAL EXPONENTIAL BASES IN HILBERT SPACES

K.P. ISAEV, R.S. YULMUKHAMETOV

Annotation. In the present paper, we consider the existence of unconditional exponential bases in general Hilbert spaces $H = H(E)$ consisting of functions defined on some set $E \subset \mathbb{C}$ and satisfying the following conditions.

1. The norm in the space H is weaker than the uniform norm on E, i.e. the following estimate holds for some constant A and for any function f from H :

$$||f||_H \leq A \sup_{z\in E} |f(z)|.$$

2. The system of exponential functions $\{\exp(\lambda z),\ \lambda \in \mathbb{C}\}$ belongs to the subset H and it is complete in H.

It is proved that unconditional exponential bases cannot be constructed in H unless a certain condition is carried out.

Sufficiency of the weakened condition is proved for spaces defined more particularly.

Keywords: series of exponents, unconditional bases, Hilbert space.

1. Introduction

The notion of unconditional exponential bases is one of generalizations of the classical Fourier systems in the space $L_2([-\pi;\pi])$. The undivided attention of numerous mathematicians was first of all attracted by unconditional exponential bases in weighted spaces $L_2(I, w)$. The current status of research in this field is described in the monograph [13]. The work [14] started investigation of unconditional exponential bases in Hilbert subspaces of the space $H(D)$, that are analytical in the convex domain $D \subset \mathbb{C}$ of functions. Unconditional exponential bases were constructed for the Smirnov space $E_2(D)$ over a convex polygon. A futile attempt to construct exponential bases in $E_2(D)$ over a convex domain with a smooth boundary was made in [15]. The dissertation [16] proves that unconditional exponential bases do not exist in Smirnov spaces over convex domains with a smooth arc on the boundary. Finally, it is demonstrated in [7] that unconditional exponential bases do not exist in Bergman spaces over convex domains with a point of a non-zero curvature on the boundary. This result was extended to weighted spaces on intervals in the dissertation [12].

The present paper generalizes methods of the above woks to general Hilbert spaces and proves sufficient conditions for nonexistence of unconditional exponential bases.

The second section considers more specific weighted spaces on intervals.

Let $H(E)$ be a Hilbert space of functions given on a bounded set $E \subset \mathbb{C}$. Assume that the following conditions are met.

1. A norm in the space H is weaker than a uniform norm in E, i.e. the estimate

$$||f||_H \leq A \sup_{z\in E} |f(z)|$$

holds for an arbitrary constant A and for any bounded function f of H.

Submitted on 18 December 2010.

This work is supported by the Russian Foundation for Basic Research (grant 10-01-00233-a).

2. Exponents $\exp(\lambda z)$, $\lambda \in \mathbb{C}$, belong to the space E, and the system is complete in the space H.

According to Banach's theorem, the second condition entails that the Laplace transform

$$L : S \longrightarrow \widehat{S}(\lambda) := S(e^{\lambda z}),\ \lambda \in \mathbb{C},\ S \in H^*$$

is an injective mapping from a conjugate space, and it follows from the fist condition that the Laplace transform injects the conjugate space H^* into a space of entire functions $H(\mathbb{C})$. The image with this mapping $L(H^*)$ is designated by $\widehat{H}$. We consider an induced structure of the Hilbert space in the space $\widehat{H}$, i.e. if $F_1, F_2 \in \widehat{H}$, $F_j = L(S_j)$, then

$$(F_1, F_2)_{\widehat{H}} = (S_1, S_2)_{H^*}.$$

A system of elements e_k, $k = 1, 2, ...$ in a Hilbert space is termed as an unconditional basis (see [1]), if it is complete and there are numbers $c, C > 0$, such that the relation

$$c\sum_{j=1}^{n} |c_k|^2||e_k||^2 \le ||\sum_{j=1}^{n} c_k e_k||^2 \le C\sum_{j=1}^{n} |c_k|^2||e_k||^2$$

holds for any set of numbers $c_1, c_2, ..., c_n$. It is known (see [2],[3]), that if a system e_k, $k = 1, 2, ...$ is un unconditional basis, then any element of the space H is uniquely represented in the form of a series

$$x = \sum_{k=1}^{\infty} x_k e_k,$$

and

$$c\sum_{k=1}^{\infty} |x_k|^2||e_k||^2 \le ||x||^2 \le C\sum_{k=1}^{\infty} |x_k|^2||e_k||^2.$$

2. Unconditional exponential bases in Hilbert spaces of functions

This section is devoted to unconditional exponential bases in Hilbert spaces satisfying the conditions 1, 2 of Introduction.

Theorem 1. *Let*

$$K(\lambda) = ||e^{\lambda z}||^2.$$

If the system $\{e^{\lambda_k t}\}$ is an unconditional basis in the space H, then there exists an entire function L with simple zeroes at points λ_k, $k = 1, 2, ...$ for which the following correlation holds:

$$\frac{1}{P}K(\lambda) \le \sum_{k=1}^{\infty} \frac{|L(\lambda)|^2 K(\lambda_k)}{|L'(\lambda_k)|^2|\lambda - \lambda_k|^2} \le PK(\lambda),\ \lambda \in \mathbb{C}. \quad (1)$$

Here P is a positive constant.

Proof. Let

$$e^{\lambda z} = \sum_{k=1}^{\infty} c_k(\lambda)e^{\lambda_k z},\ z \in E,$$

and S_k be a biorthogonal system of functionals on H. Then,

$$c_k(\lambda) = \widehat{S}_k(\lambda).$$

Hence, if $L(\lambda) = c_1(\lambda)(\lambda - \lambda_1)$, then due to completeness of the system $(e^{\lambda_k z})$

$$c_k(\lambda) = \frac{L(\lambda)}{L'(\lambda_k)(\lambda - \lambda_k)},\ k = 1, 2, ...$$

The statement of the theorem follows from unconditional basis character of the system of exponents $(e^{\lambda_k z})$. Theorem 1 is proved.

Theorem 2. *Let us assume that*

$$K(\lambda) = ||e^{\lambda z}||^2$$

and that $\delta_\lambda : F \longrightarrow F(\lambda)$ *is a point functional in the space* $\widehat{H}$. *Then,* δ_λ *is a linear continuous functional on* $\widehat{H}$ *and*

$$K(\lambda) = ||\delta_\lambda||^2_{\widehat{H}^*},$$

i.e. $\sqrt{K(\lambda)}$ *is the Bergman function (see [4]) of the space* $\widehat{H}$. *Moreover,* $\ln K(\lambda)$ *is a continuous subharmonic function on a plane.*

Proof. On the one hand, if $F \in \widehat{H}$, then $F = \widehat{S}$ for a functional $S \in H^*$. Hence,

$$\delta_\lambda(F) = F(\lambda) = S(e^{\lambda z}).$$

Therefore,

$$|\delta_\lambda(F)| = |S(e^{\lambda z})| \le ||S|| \cdot ||e^{\lambda z}|| = ||F||\sqrt{K(\lambda)}.$$

Thus,

$$||\delta_\lambda|| \le \sqrt{K(\lambda)},\ \lambda \in \mathbb{C}.$$

On the other hand, when $\lambda \in \mathbb{C}$ is fixed, it is the function $e^{\lambda z} \in H$ that generates some linear continuous functional $E \in H^*$. One has

$$\delta_\lambda(\widehat{E}) = \widehat{E}(\lambda) = E(e^{\lambda z}) = (e^{\lambda z}, e^{\lambda z})_H =$$

$$= ||e^{\lambda z}||^2 = \sqrt{K(\lambda)}||e^{\lambda z}||_H = \sqrt{K(\lambda)}||E||_{H^*} = \sqrt{K(\lambda)}||\widehat{E}||_{\widehat{H}}.$$

Since the function $\ln K(\lambda)$ is an upper family envelope of subharmonic functions $\{\ln|F(\lambda)|,\ F \in \widehat{H},\ ||F|| \le 1\}$, it follows that it is subharmonic. Theorem 2 is proved.

Let us introduce a characteristics for functions u continuous on a plane. Let z be a fixed point on a plane. Let us designate a circle $\{w:\ |w - z| < r\}$ for any positive number $r > 0$ by $B(z, r)$ and assume that

$$\|f\|_r = \max_{w \in \overline{B}(z,r)} |f(w)|$$

for the function f continuous in $\overline{B}(z, r)$. Let $d(f, z, r)$ be a distance from the function f to the space of functions harmonic in $B(z, r)$:

$$d(f, z, r) = \inf\{\|f - H\|_r,\ H \text{ is harmonic in } B(z, r)\}.$$

Suppose that

$$\tau(u, z, p) = \sup\{r:\ d(u, z, r) \le p\}$$

for a positive number p. It follows directly from the definition that if $\exists z_0\ :\ \tau(u, z_0, p) = \infty$, then $\tau(u, z, p) = \infty$ for all z. If $\exists z_0\ :\ \tau(u, z_0, p) < \infty$, then $\tau(u, z, p) < \infty$ for all z. The following statement holds.

Lemma 1. *Let* $\exists z\ :\ \tau(u, z, p) < \infty$. *Then, the function* $\tau(z) = \tau(u, z, p)$ *satisfies the Lipschitz condition: for all* z_1 *and* z_2

$$|\tau(z_1) - \tau(z_2)| \le |z_1 - z_2|.$$

Proof. By definition, there is a harmonic function $h_1(z)$ in the circle $B(z_1, \tau(z_1))$ satisfying the condition

$$|u(z) - h_1(z)| \le p.$$

If $|z_1 - z_2| < \tau(z_1)$, then this inequality holds in the circle $B(z_2, \tau(z_1) - |z_1 - z_2|)$ as well. Thus,

$$\tau(z_2) \ge \tau(z_1) - |z_1 - z_2|.$$

Or $\tau(z_1) - \tau(z_2) \le |z_1 - z_2|$. If $|z_1 - z_2| \ge \tau(z_1)$, then all the more so

$$\tau(z_1) - \tau(z_2) \le |z_1 - z_2|.$$

Let us interchange z_1 and z_2:

$$\tau(z_2) - \tau(z_1) \leq |z_1 - z_2|.$$

Thus,

$$|\tau(z_1) - \tau(z_2)| \leq |z_1 - z_2|.$$

It is demonstrated in [6] (Lemma 1.1), that if u is a continuous subharmonic function, the value $\tau = \tau(u, \lambda, p)$ is completely defined by the following condition. If $H(z)$ is a harmonic majorant of u in the circle $B(\lambda, \tau)$, then

$$\max_{z \in \overline{B}(\lambda,\tau)} (H(z) - u(z)) = 2p. \tag{2}$$

Let us determine this value for the function $u(\lambda) = \ln K(\lambda)$ and the number $\ln(5P)$, where P is a constant from (1). In what follows, it is designated just by $\tau(\lambda)$. Thus,

$$\inf_{v \in A(B(\lambda,\tau(\lambda)))} \max_{z \in \overline{B}(\lambda,\tau(\lambda))} |\ln K(z) - v(z)| = \ln(5P),$$

where $A(B(\lambda, \tau))$ indicates a set of functions harmonic in the circle $B(\lambda, \tau(\lambda))$ and continuous in the closure $\overline{B}(\lambda, \tau(\lambda))$.

Theorem 3. *Let $L(\lambda)$ be an entire function with simple zeroes λ_k, $k = 1, 2, ...$ that satisfies the two-sided estimate*

$$\frac{1}{P} K(\lambda) \leq \sum_{k=1}^{\infty} \frac{|L(\lambda)|^2 K(\lambda_k)}{|L'(\lambda_k)|^2 |\lambda - \lambda_k|^2} \leq PK(\lambda)$$

when P is arbitrary. Then,

1) there is at least one zero λ_k of the function LB in any circle $B(\lambda, 2\tau(\lambda))$;

2) the inequality

$$|\lambda_k - \lambda_n| \geq \frac{\max(\tau(\lambda_k), \tau(\lambda_n))}{10P^{\frac{3}{2}}}$$

holds for any n, k, $n \neq k$;

3) the relation

$$\frac{1}{5^6 P^8} K(\lambda) \leq \frac{K(\lambda_k)|L(\lambda)|^2}{|L'(\lambda_k)|^2 |\lambda - \lambda_k|^2} \leq PK(\lambda)$$

holds for any k in the circle $B(\lambda_k, \frac{\tau(\lambda_k)}{20P^{\frac{3}{2}}})$.

(see [7], Theorem 1).

Theorem 4. *Let us assume that λ_k, $k = 1, 2, ...$, are zeroes of the function $L(\lambda)$, satisfying the conditions of the above theorem. Then, in any finite set of zeroes B, containing at least two zeroes, there is an index n such that*

$$\sum_{\lambda_k \in B, k \neq n} \frac{\tau^2(\lambda_k)}{|\lambda_k - \lambda_n|^2} \leq (4P)^{12}. \tag{3}$$

Proof. By condition of the theorem the estimate

$$\sum_{\lambda_k \in B} \frac{K(\lambda_k)|L(\lambda)|^2}{|L'(\lambda_k)|^2 |\lambda - \lambda_k|^2} \leq PK(\lambda) \tag{4}$$

holds for any λ. Since the set B is finite, then there is such a number n, that

$$\frac{K(\lambda_n)\tau^2(\lambda_n)}{|L'(\lambda_n)|^2} = \min_{\lambda_k \in B} \left(\frac{K(\lambda_k)\tau^2(\lambda_k)}{|L'(\lambda_k)|^2} \right).$$

According to the statement 3 of Theorem 3, the estimate

$$\frac{1}{5^6P^8}K(\lambda) \le 20^2P^3\frac{K(\lambda_n)|L(\lambda)|^2}{|L'(\lambda_n)|^2\tau^2(\lambda_n)}$$

holds for the points λ, lying on the boundary of the circle $B\left(\lambda_n, \frac{1}{20P^{\frac{3}{2}}}\tau(\lambda_n)\right)$, or

$$\frac{K(\lambda)}{|L(\lambda)|^2} \le 4^2 5^8 P^{11}\frac{K(\lambda_n)}{|L'(\lambda_n)|^2\tau^2(\lambda_n)}.$$

This together with estimate (4) yields

$$4^2 5^8 P^{11}\frac{K(\lambda_n)}{|L'(\lambda_n)|^2\tau^2(\lambda_n)} \ge \frac{1}{P}\sum_{\lambda_k\in B}\frac{K(\lambda_k)}{|L'(\lambda_k)|^2|\lambda-\lambda_k|^2}.$$

Hence,

$$4^2 5^8 P^{12}\frac{K(\lambda_n)}{|L'(\lambda_n)|^2\tau^2(\lambda_n)} \ge \sum_{\lambda_k\in B}\frac{K(\lambda_k)}{|L'(\lambda_k)|^2\tau^2(\lambda_k)}\cdot\frac{\tau^2(\lambda_k)}{|\lambda-\lambda_k|^2}$$

for the points λ, lying on the boundary of the circle $B\left(\lambda_n, \frac{1}{20P^{\frac{3}{2}}}\tau(\lambda_n)\right)$.

Invoking selection of the number n for the points λ on the boundary $B\left(\lambda_n, \frac{1}{20P^{\frac{3}{2}}}\tau(\lambda_n)\right)$, one has

$$4^2 5^8 P^{12}\frac{K(\lambda_n)}{|L'(\lambda_n)|^2\tau^2(\lambda_n)} \ge \frac{K(\lambda_n)}{|L'(\lambda_n)|^2\tau^2(\lambda_n)}\sum_{\lambda_k\in B}\frac{\tau^2(\lambda_k)}{|\lambda-\lambda_k|^2}$$

or

$$\sum_{\lambda_k\in B}\frac{\tau^2(\lambda_k)}{|\lambda-\lambda_k|^2} \le 4^2 5^8 P^{12}. \tag{5}$$

According to the statement 2 of Theorem 3, the estimate

$$|\lambda-\lambda_k| \le |\lambda-\lambda_n| + |\lambda_n-\lambda_k| = \frac{\tau(\lambda_n)}{20P^{\frac{3}{2}}} + |\lambda_n-\lambda_k| \le \frac{3}{2}|\lambda_n-\lambda_k|$$

holds for the indicated points λ when $k \ne n$. Therefore, (5) entails the estimate

$$\sum_{\lambda_k\in B, k\ne n}\frac{\tau^2(\lambda_k)}{|\lambda_n-\lambda_k|^2} \le (4P)^{12}.$$

Theorem 4 is proved.

Corollary.

Let us assume that $\lambda_k,\ k=1,2,...,$ *are zeroes of the function* $L(\lambda)$, *satisfying conditions of the above theorem and* $b=\frac{1}{20P^{\frac{3}{2}}}$. *Then, for any finite set of zeroes* B, *containing at least two zeroes, there is an index* n *such that*

$$\sum_{\lambda_k\in B, k\ne n}\int_{B(\lambda_k, b\tau(\lambda_k))}\frac{dm(\lambda)}{|\lambda-\lambda_n|^2} \le 4^{10}P^9. \tag{6}$$

Proof. Since one has

$$|\lambda-\lambda_n| \ge |\lambda_k-\lambda_n| - |\lambda-\lambda_k| \ge \frac{1}{2}|\lambda_k-\lambda_n|$$

for points $\lambda\in B(\lambda_k, b\tau(\lambda_k))$, then

$$\int_{B(\lambda_k, b\tau(\lambda_k))}\frac{dm(\lambda)}{|\lambda-\lambda_n|^2} \le \frac{4\pi b^2\tau^2(\lambda_k)}{|\lambda_k-\lambda_n|^2}.$$

Hence,

$$\sum_{\lambda_k\in B, k\neq n}\int_{B(\lambda_k, b\tau(\lambda_k))}\frac{dm(\lambda)}{|\lambda-\lambda_n|^2}\leq 4\pi b^2(4P)^{12}=\frac{4\pi}{400P^3}(4P)^{12}\leq 4^{10}P^9.$$

Theorem 5. *Let us assume that $H(E)$ is a Hilbert space, satisfying the conditions 1, 2 of Introduction and $\sqrt{K(\lambda)}$ is a Bergman function of the space $\widehat{H}$. Suppose that for any positive number p, there is a number $\delta=\delta(p)>0$, such that the function $\tau(\lambda)=\tau(\ln K(z),\lambda,p)$ satisfies the condition*

$$\min_{z\in B(\lambda, 2\tau(\lambda))}\tau(z)\geq\delta\tau(\lambda) \tag{7}$$

for all $\lambda\in\mathbb{C}$ and $\tau(\lambda)=o(|\lambda|)$ when $|\lambda|\longrightarrow\infty$. Then, there are no unconditional exponential bases in the space H.

Proof. Let us make use of the following statement (see [8], p.216).

Lemma (Lemma on covering with spheres)

Let us assume that the set $A\subset\mathbb{R}^p$ is covered with spheres so that every point $x\in A$ is a centre of a sphere $S(x)$ with the radius $r(x)$. If $\sup_{x\in A}r(x)<\infty$, then one can single out from the system $\{S(x)\}$ no more than a countable system $\{S(x_k)\}$, covering all the set A and having the order not exceeding a number $N(p)$, depending only on the dimension of the space.

One can readily verify that $N(2)=6$.

Let us prove by contradiction. Let us assume that conditions of the theorem are met, but an unconditional exponential basis $\{e^{\lambda_k z}\}$ exists in the space $\widehat{H}$. Then, Theorems 1,3 and 4 hold. Suppose that $p=\ln(5P)$ in a condition of the theorem under consideration and let $\tau(\lambda)=\tau(\ln K(z),\lambda,\ln(5P))$.

Let us select an arbitrary $\varepsilon>0$, and consider the number R large enough to meet the condition

$$\max_{|\lambda|\leq R}\tau(\lambda)\leq\varepsilon R. \tag{8}$$

Such R can be found by condition on $\tau(\lambda)$. Indeed, there is such R' that $\tau(\lambda)<\varepsilon|\lambda|$ holds when $|\lambda|\geq R'$. If we assume that $R=\frac{2R'}{\varepsilon}$, then one has $\tau(\lambda)<\varepsilon|\lambda|\leq\varepsilon R$ when $|\lambda|\in[\frac{\varepsilon}{2}R;R]$. The correlation $\tau(\lambda)\leq\tau(0)+|\lambda|$ holds due to Lemma 1. Therefore, if $|\lambda|\in[\tau(0);\frac{\varepsilon}{2}R]$, then $\tau(\lambda)\leq 2|\lambda|\leq\varepsilon R$. Finally, selecting R larger than $\frac{1}{\varepsilon}\max_{|z|\leq\tau(0)}\tau(z)$, one obtains (8).

Consider a system of circles $B(\lambda,2\tau(\lambda))$, $\lambda\in B(0,R)$. According to the statement 1 in Theorem 3, every circle contains at least one coefficient λ_k, and these circles cover all the circle $B(0,R)$. Due to Lemma on covering with circles, one can single out no more than a countable set of circles $B_n=B(z_n,2\tau(z_n))$, covering the circle $B(0,R)$, and every point of this circle gets in no more than $N(2)=6$ covering circles. Let us select one index $\lambda_{k(n)}$ in every circle B_n. Meanwhile, some indices $\lambda_{k(n)}$ can appear to be selected more than once, but due to properties of the singled out covering, the multiplicity of the choice of one index is not larger than six. Let us renumber the system of selected indices and assign them the number of the circle where this index is selected. We obtain a set of indices (w_n), where every index occurs no more than six times. Let us apply Theorem 4 to the resulting selection. There is a number m such that the estimate

$$\sum_{w_n\neq w_m}\frac{\tau^2(w_n)}{|w_n-w_m|^2}\leq 6(4P)^{12} \tag{9}$$

holds if multiplicity is taken into account. In our notation $w_n\in B_n=B_n(z_n,2\tau(z_n))$. Further, consider such n, that $w_m\notin B_n'=B_n(z_n,3\tau(z_n))$. Then, one has $|w-w_m|\geq\tau(z_n)$ for any $w\in B_n$. Hence, one has

$$|w_n-w_m|=|w_n-w|+|w-w_m|\leq 4\tau(z_n)+|w-w_m|\leq 5|w-w_m|,$$

or

$$\frac{1}{|w-w_m|^2} \le \frac{25}{|w_n-w_m|^2},\ w \in B_n,\ w_m \notin B'_n$$

for the point w, lying on the intersection of the length $[w_n; w_m]$ with the boundary of the circle B_n. Integrating this inequality with respect to the circle B_n, one obtains

$$\int_{B_n} \frac{dm(w)}{|w-w_m|^2} \le \frac{100\pi\tau^2(z_n)}{|w_n-w_m|^2},\ w_m \notin B'_n.$$

Since $w_n \in B(z_n, 2\tau(z_n))$, then $\tau^2(w_n) \ge \delta^2\tau^2(z_n)$ according to the condition (7). Thus, the latter estimate and (9) provide

$$\sum_{w_n \ne w_m \notin B'_n} \int_{B_n} \frac{dm(w)}{|w-w_m|^2} \le \frac{100\pi}{\delta^2} \sum_{w_n \ne w_m \notin B'_n} \frac{\tau^2(w_n)}{|w_n-w_m|^2} \le \frac{600(4P)^{12}}{\delta^2} := C. \qquad (10)$$

If the number n is such that $w_m \in B'_n$, then one has

$$|w-z_m| \le |w-w_m| + |w_m-z_m| \le |w-z_n| + |z_n-w_m| + 2\tau(z_m) \le$$

$$\le 2\tau(z_n) + 3\tau(z_n) + 2\tau(z_m) \le 5\tau(z_n) + 2\tau(z_m)$$

for any $w \in B_n$. According to the selection of the number R, one has $|w - z_m| \le 7\varepsilon R$, i.e. the circles B_n lie completely in the circle $B(z_m, 7\varepsilon R)$. It means that the covering circles whose numbers take part in summation in (10) cover the set $C(R) = B(0,R) \setminus B(z_m, 7\varepsilon R)$. Hence,

$$\int_{C(R)} \frac{dm(w)}{|w-w_m|^2} \le C.$$

Let us substitute the variables $w = R\zeta$, $w_m = R\zeta_m$, $\zeta_m \in B(0,1)$, and obtain

$$\int_{B(0,1)\setminus B(z_m,7\varepsilon)} \frac{dm(\zeta)}{|\zeta-\zeta_m|^2} \le C.$$

The number $\varepsilon > 0$ was chosen arbitrarily. If ε tends to zero one, arrives to a contradiction.

Theorem 5 is proved.

3. Unconditional exponential bases in weighted Hilbert spaces on a bounded interval

In what follows, Theorem 4 and its Corollary are applied to more specific weighted Hilbert spaces of functions on a bounded interval of the real axis.

Let us assume that I is a bounded interval of the real axis, $h(t)$ is a convex function on this interval, and $L^2(I,h)$ is a space of locally integrable functions on I, satisfying the condition

$$||f|| := \sqrt{\int_I |f(t)|^2 e^{-2h(t)}\, dt} < \infty.$$

It is a Hilbert space with a scalar product

$$(f,g) = \int_I f(t)\overline{g}(t) e^{-2h(t)}\, dt.$$

One can readily verify that the space $L^2(I,h)$ satisfies the conditions 1 and 2 from Introduction.

The space $\widehat{L}^2(I,h)$ is described in the works [9],[10],[11]. It is proved that the space $\widehat{L}^2(I,h)$ is isomorphic (as a Banach space) to the space of entire functions F, satisfying the conditions

$$|F(z)| \le C_F\sqrt{K(z)},\ z \in \mathbb{C}.$$

$$||F||^2 = \int_{\mathbb{R}} \int_{\mathbb{R}} \frac{|F(x+iy)|^2}{K(x)} d\widetilde{h}'(x) dy < \infty,$$

where

$$K(z) = \int_I |e^{2zt}| e^{-2h(t)} dt, \ \widehat{h}(x) = \sup_{t \in I}(xt - h(t)).$$

Theorem 6. *Let us assume, that for any $p > 0$ there is a number $\delta = \delta(p) > 0$ with the following property. There exists a sequence $x_k \in \mathbb{R}$, $k \in \mathbb{Z}$, such that the intervals*

$$I_k = \{x : \ |x - x_k| \le 2\tau(\ln K(z), x_k, p)\}$$

are pairwise disjoint and

$$\min_{x \in I_k} \tau(\ln K(z), x, p) \ge \delta(p)\tau(\ln K(z), x_k, p).$$

Further, suppose that for any $\varepsilon > 0$ there is an interval $[m; s]$, $s > m$ of an integer-valued series with the following properties.

1) If $I_{m,s} = \bigcup_{m \le k \le s} I_k$, $I^0_{m,s}$ is the smallest segment of the real axis containing $I_{m,s}$, then $d_{m,s}$ is the sum of lengths of intervals composing $I_{m,s}$, and $d^0_{m,s}$ is the length of the interval $I^0_{m,s}$, then $d_{m,s} \ge (1 - \varepsilon)d^0_{m,s}$.

2) The estimate $\max_{k \in [m,s]} \tau(\ln K(z), x_k, p) \le \varepsilon d^0_{m,s}$ holds.

Then, there are no Riesz exponential bases in the space $L_2(I, h)$.

Proof. Let us use proof by contradiction. Suppose that there exists an unconditional exponential basis $\{e^{\lambda_k t}\}_{k=1}^{\infty}$ in the space $L_2(I, h)$. Then, Theorems 1,3,4 hold. Let us follow the notation introduced in these theorems. In particular, $\tau(\lambda)$ designates a function $\tau(\ln K(z), \lambda, \ln(5P))$, while

$$K(z) = ||e^{zt}||^2 = \int_I |e^{zt}|^2 e^{-2h(t)} dt = \int_I e^{2Re\, zt - 2h(t)} dt = K(Re\, z),$$

and the constant P comes from properties of the basis (correlation (1)). The value $\delta(\ln(5P))$ is designated just by δ. Assume that $\tau(\ln K(z), x_k, \ln(5P)) = \tau_k$,

$$Q_{k,n} = \{x + iy : \ x \in I_k, \ 4n\tau_k \le y < 4(n+1)\tau_k\}, \ k = 0, 1, 2, ..., \ n \in \mathbb{Z}.$$

Since the square $Q_{k,n}$ contains the circle $B(x_k + i(4n+2)\tau_k, 2\tau_k)$ then, according to the statement 1 of Theorem 3, every square contains at least one zero of the function L. Let us select one zero $\lambda_{k,n}$ in every square $Q_{k,n}$.

Take a positive ε and the interval $[m, s]$, mentioned in the conditions of the theorem. Consider a set of zeroes $B = \{\lambda_{k,n} : \ k \in [m, s], \ |n| \le M\}$ for a large positive integer M. Let us apply Theorem to the set of zeroes and find the corresponding index. A zero value with this index is denoted by $\lambda^* = x^* + iy^*$. Without loss of generality, we suppose that $y^* \le 0$. Thus, the point λ^* is one of zeroes and depends on parameters m, s, M.

Let us assume that

$$n(k) = \left[\frac{y^*}{4\tau_k} + \frac{4}{3}\right]$$

for every $k \in [m, s]$. Here $[t]$ indicates the integer part t. Let $\tau_{k,n} = \tau(\lambda_{k,n})$. If $n \ge n(k)$, then the square $Q_{k,n}$ and the circle $B_{k,n} = B(\lambda_{k,n}, p\tau_{k,n})$ (recall that $p = \frac{1}{20P^{\frac{3}{2}}}$) lie in a half-plane $Im\, z \ge y^* + \tau_k$. Indeed, by definition of the number $n(k)$, if $\lambda \in Q_{k,n}$, then

$$Im\, \lambda \ge 4n\tau_k \ge 4n(k)\tau_k \ge y^* + \frac{4}{3}\tau_k.$$

If $x \in I_k$ then, due to Lemma 1,

$$\tau(x) \le \tau(x_k) + |x_k - x| \le 3\tau_k.$$

Therefore, one has

$$Im\,\lambda \geq Im\,\lambda_{k,n} - p\tau_{k,n} \geq Im\,\lambda_{k,n} - 3p\tau_k \geq y^* + \tau_k$$

for the points $\lambda \in B_{k,n}$. Note that $p < 1/20$. If λ belongs to a half-plane $Im\,z \geq y^* + \tau_k$ then,

$$|\lambda - \lambda^*| \geq Im\,(\lambda - \lambda^*) \geq \tau_k$$

or

$$\tau_k \leq |\lambda - \lambda^*|.$$

Thus, if points λ, w lie in $A = Q_{k,n} \bigcup B_{k,n}$, $n \geq n(k)$ then,

$$|\lambda - \lambda^*| \leq |w - \lambda^*| + |\lambda - w| \leq |w - \lambda^*| + (4\sqrt{2} + 3p)\tau_k \leq 7|w - \lambda^*|.$$

Here, we use the fact that $p < 1/20$ again. Hence,

$$\alpha := \max_{z \in A} \frac{1}{|z - \lambda^*|^2} \leq 49 \min_{z \in A} \frac{1}{|z - \lambda^*|^2} := 49\beta.$$

Since $\tau_{k,n}^2 \geq \delta^2 \tau_k^2$ due to condition of the theorem, one has

$$\int_{Q_{k,n}} \frac{dm(z)}{|z - \lambda^*|^2} \leq 16\alpha\tau_k^2 \leq \frac{16\alpha\tau_k^2}{\beta\pi(p\tau_{k,n})^2}\beta\pi(p\tau_{k,n})^2 \leq \frac{784}{\pi p^2 \delta^2} \int_{B_{k,n}} \frac{dm(z)}{|z - \lambda^*|^2}.$$

Summation of the resulting inequalities first over every $n(k) \leq n \leq M$ when k is fixed, and then over every $k \in [m, s]$ provides

$$\sum_{k=m}^{s} \sum_{n=n(k)}^{M} \int_{Q_{k,n}} \frac{dm(z)}{|z - \lambda^*|^2} \leq \frac{784}{\pi p^2 \delta^2} \sum_{k=m}^{s} \sum_{n=n(k)}^{M} \int_{B_{k,n}} \frac{dm(z)}{|z - \lambda^*|^2}.$$

The point λ^* is selected by Theorem 4. Then, by virtue of (6), one has

$$\sum_{k=m}^{s} \sum_{n=n(k)}^{M} \int_{Q_{k,n}} \frac{dm(z)}{|z - \lambda^*|^2} \leq \frac{784}{\pi p^2 \delta^2} 4^{10} P^9 := C. \tag{11}$$

By definition of squares $Q_{k,n}$

$$\bigcup_{n=n(k)}^{M} Q_{k,n} = \{x + iy : \ x \in I_k, \ 4n(k)\tau_k \leq y \leq 4(M+1)\tau_k\}.$$

Therefore,

$$\sum_{n=n(k)}^{M} \int_{Q_{k,n}} \frac{dm(z)}{|z - \lambda^*|^2} = \int_{I_k} \int_{4n(k)\tau_k}^{4(M+1)\tau_k} \frac{dydx}{|z - \lambda^*|^2}.$$

By definition of the number $n(k)$, we have $4n(k)\tau_k < y^* + 6\tau_k$. Let us substitute the variables $w = z - y^*$ in the latter integral. We assume that $y^* \leq 0$. By virtue of the choice of the number $n(k)$, we have

$$\sum_{n=n(k)}^{M} \int_{Q_{k,n}} \frac{dm(z)}{|z - \lambda^*|^2} \geq \int_{I_k} \int_{6\tau_k}^{4(M+1)\tau_k} \frac{dm(w)}{|w - x^*|^2}.$$

If we had $y^* > 0$, then likewise, we would obtain the estimate

$$\sum_{n=-M}^{-n(k)} \int_{Q_{k,n}} \frac{dm(z)}{|z - \lambda^*|^2} \geq \int_{I_k} \int_{-4(M+1)\tau_k}^{-6\tau_k} \frac{dm(w)}{|w - x^*|^2},$$

which is equivalent to the previous estimate because the integrand function with respect to y is even. Summation of these estimates over all $k\in[m,s]$ provides

$$\sum_{k=m}^{s}\sum_{n=n(k)}^{M}\int_{Q_{k,n}}\frac{dm(z)}{|z-\lambda^*|^2}\geq\sum_{k=m}^{s}\int_{I_k}\int_{6\tau_k}^{4(M+1)\tau_k}\frac{dm(w)}{|w-x^*|^2}.$$

Let us use the estimate (11)

$$\sum_{k=m}^{s}\int_{I_k}\int_{6\tau_k}^{4(M+1)\tau_k}\frac{dm(w)}{|w-x^*|^2}\leq C.$$

By definition $x^*=Re\,\lambda^*$, and λ^* is one of the points $\lambda_{k,n}$, $k\in[m,s]$, $|n|\leq M$. Thus, when the interval $[m,s]$ is fixed, the number x^* can vary within the interval $I^0_{m,s}$ with the change of the number M. Thus, one can select the sequence M_n, extending to $+\infty$ or $-\infty$ so that the corresponding values of x^*_n converge to a limiting value x^*.

Invoking that the integrals are limited, one can turn to the limit

$$\sum_{k=m}^{s}\int_{I_k}\int_{6\tau_k}^{+\infty}\frac{dm(w)}{|w-x^*|^2}\leq C. \tag{12}$$

Let s make use of the evident estimates. If $p\in[0;1)$, then

$$p\int_p^\infty\frac{dt}{1+t^2}\geq p\int_1^\infty\frac{dt}{1+t^2}=\frac{\pi}{4}p\geq\frac{1}{2}p,$$

and if $p\geq 1$, then

$$p\int_p^\infty\frac{dt}{1+t^2}\geq p\int_p^\infty\frac{dt}{2t^2}=\frac{1}{2}.$$

Hence, one has

$$p\int_p^\infty\frac{dt}{1+t^2}\geq\frac{1}{2}\min(p,1)$$

with any $p\geq 0$. Whence,

$$\int_a^\infty\frac{dy}{x^2+y^2}=\frac{1}{a}\cdot\frac{a}{|x|}\int_{\frac{a}{|x|}}^\infty\frac{dt}{1+t^2}\geq\frac{1}{2a}\min\left(\frac{a}{|x|},1\right)=\frac{1}{2}\min\left(\frac{1}{|x|},\frac{1}{a}\right)$$

for any $a\geq 0$ and $x\in\mathbb{R}$.

Thus, the estimate

$$\int_{6\tau_k}^{+\infty}\frac{dy}{(x-x^*)^2+y^2}\geq\frac{1}{2}\min\left(\frac{1}{|x-x^*|},\frac{1}{6\tau_k}\right)$$

holds. The latter formula together with (12) provides

$$\sum_{k=m}^{s}\int_{I_k}\min\left(\frac{1}{|x-x^*|},\frac{1}{6\tau_k}\right)dx\leq 2C. \tag{13}$$

Let us denote the interval $(x^*-6\tau_k;x^*+6\tau_k)$ by I^*_k, $k\in[m,s]$. Separate the set of all indices $k\in[m,s]$ in two parts $A_1=\{k\in[m,s]:\ I_k\bigcap I^*_k=\emptyset\}$ and $A_2=\{k\in[m,s]:\ I_k\bigcap I^*_k\neq\emptyset\}$.

The summarized length of all intervals I_k with respect to $k\in A_j$, $j=1,2$ is indicated by d_j.

If $k\in A_2$ then $|x_k-x^*|\leq 8\tau_k$. Hence, all the interval I_k belongs to $\{x:\ |x-x^*|\leq 10\tau_k\}$. In other words, all the intervals I_k, $k\in A_2$ lie in the interval $\{x:\ |x-x^*|\leq 10\tau^*\}$, where $\tau^*=10\max_{k\in[m,s]}\tau_k$. It means that $d_2\leq 20\tau^*$ and by condition 2) of the theorem, one has

$$d_2\leq 20\varepsilon d^0_{m,s}, \tag{14}$$

$$d_1=d_{m,s}-d_2\geq(1-20\varepsilon)d^0_{m,s}.$$

The set of indices A_1 is separated in two parts: A_1^+ are those indices from A_1 for which $x_k \geq x^*$, and A_1^- are the remaining indices from A_1. The symbol $d_1^\pm$ indicates the summarized length of intervals I_k with indices from $A_1^\pm$. One of the values is not less than the half d_1. Let $d_1^+ \geq \frac{d_1}{2}$. The previous estimate demonstrates that in this case

$$d_1^+ \geq \frac{1-20\varepsilon}{2} d_{m,s}^0.$$

In particular, if $I_{m,s}^0 = (A;B)$ then

$$B - x^* \geq d_1^+ \geq \frac{1-20\varepsilon}{2} d_{m,s}^0. \tag{15}$$

Let us shift the intervals I_k, $k \in A_1^+$ to the right end of the interval $(A;B)$ so that the resulting intervals I'_k fill the interval $(B - d_1^+; B)$. The length of the interval $(x^*; B - d_1^+)$ does not exceed the summarized length of intervals I_k, $k \in A_2$ and a linear measure of the Lebesgue set $I_{m,s}^0 \backslash I_{m,s}$. Condition 1) of the theorem and the estimate (14) provide

$$|B - d_1^+ - x^*| \leq 21\varepsilon d_{m,s}^0. \tag{16}$$

Let us extend the estimate (13). Since

$$\int_{I_k} \frac{dx}{|x-x^*|} \geq \int_{I'_k} \frac{dx}{|x-x^*|}$$

for $k \in A_1^+$, then

$$\sum_{k\in A_1^+} \int_{I_k} \min\left(\frac{1}{|x-x^*|}, \frac{1}{6\tau_k}\right) dx = \sum_{k\in A_1^+} \int_{I_k} \frac{dx}{|x-x^*|} \geq \sum_{k\in A_1^+} \int_{I'_k} \frac{dx}{|x-x^*|}.$$

The relation (13) yields

$$\sum_{k\in A_1^+} \int_{I'_k} \frac{dx}{|x-x^*|} \leq 2C$$

or

$$\int_{B-d_1^+}^{B} \frac{dx}{|x-x^*|} \leq 2C.$$

The estimates (14) and (16) provide

$$\ln \frac{1-20\varepsilon}{42\varepsilon} \leq 2C.$$

This is impossible due to arbitrary smallness of ε. The resulting contradiction proves Theorem 5.

The following theorem is proved in [12] (Theorem 2.4).

Theorem 7. *Let us assume that I is an arbitrary interval on $\mathbb{R}$, $h(t)$ is a convex function on this interval*

$$K(\lambda) = \int_I e^{2Re\,\lambda t - 2h(t)}\, dt, \quad J = \{x: \ K(x) < \infty\}.$$

Suppose that for a certain $p > 0$, there exists a sequence of intervals $[a_m; b_m]$ and positive numbers τ_m, $m = 1, 2, ...,$ so that

1) the inequality

$$\delta\tau_m \leq \tau(\ln K(z), x, p) \leq \tau_m, \ m = 1, 2, ...,$$

holds for a certain positive number δ and for all $x \in [a_m; b_m]$

2) the relation

$$\lim_{m\longrightarrow\infty} \frac{b_m - a_m}{\tau_m} = \infty$$

holds, then the Riesz exponential basis does not exist in the space $L^2(I,h)$.

This theorem follows from Theorem 6.

It is known that two-sided estimates

$$\tau(\ln K(z), x, q) \geq \tau(\ln K(z), x, p) \geq \frac{p}{16q}\tau(\ln K(z), x, q)$$

hold when $q \geq p > 0$ (see [5], Lemma 5). Therefore, if the sequence of intervals required in Theorem 7 exists only for a certain $p > 0$ then, such sequence exists for any number $p > 0$. Every interval of $[a_m; b_m]$ in Theorem 7 should be represented in the form of a union of disjoint intervals of the form $\{x : |x - y| \leq 2\tau(y)\}$. It is possible, that we will not be able to cover the interval $[a_m; b_m]$ completely, but we can cover it so that more than a half of its length is covered. Then, we will have a union of intervals from $[a_m; b_m]$ as a set $I_{m,s}$ in Theorem 6. They, or to be more exact, their closures will be the intervals $I^0_{m,s}$ as well. Therefore, the condition 1) of Theorem 6 is met trivially. The condition 2) follows from the condition 2) of Theorem 7.

REFERENCES

1. Nikol'skii N.K., Pavlov B.S., Khrushchev S.V. *Unconditional bases of exponential functions and reproducing kernels. I.* // Preprint No. R-8-80 (LOMI, Leningrad, 1980).(Russian)
2. Gohberg I.C., Krejn M.G. *Introduction into the theory of linear non-self-adjoint operators in a Hilbert space.* Nauka, Moscow, 1965 [Dunod, Paris, 1971].
3. Nikol'skii N.K. *Lectures on the shift operator.* Nauka, Moscow, 1980. (Russian)
4. N. Aronszajn *Theory of reproducing kernels.* // Transactions of the American Mathematical Society, 68:3 (1950), 337–404.
5. Bashmakov R.A., Isaev K.P., Yulmukhametov R.S. *On geometric characteristics of convex functions and Laplace integrals.* // Ufimskii mat. zhurn, 2:1 (2010), 3–16.
6. Yulmukhametov R.S. *Asymptotic approximation of subharmonic functions.* // Sibirsk. Math. Zh. 26:4 (1985), 159–175 [Siberian Math. J. 26 (1985), 603-618].
7. Isaev K.P., Yulmukhametov R.S. *The absence of unconditional bases of exponentials in Bergman spaces on non-polygonal domains.* // Izvestiya Ross. Acad. Nauk, Ser. Mat., 71:6 (2007), 69–90 [Izvestiya: Mathematics, 2007, 71:6, 1145—1166].
8. Landkof N.S. *Foundations of modern potential theory.* Nauka, Moscow, 1966 [Springer, Berlin, 1972].
9. Lutsenko V.I., Yulmukhametov R.S. *A generalization of the Paley-Wiener theorem in weighted spaces.* // Mat. Zametki 48:5 (1990), 80–87 [Math. Notes 48 (1990), 1131-1136].
10. Lutsenko V.I. *The Paley-Wiener theorem on an unbounded interval.* // Investigations in Theorey of Appoximations. Ufa. 1989. Pp. 79–85. (Russian)
11. Napalkov V.V., Bashmakov R.A., Yulmukhametov R.S. *Asymptotic behavior of Laplace integrals and geometric characteristics of convex functions.* // Dokl. Akad. Nauk, Ross. Akad. Nauk 413, No. 1 (2007), 20–22 [Dokl. Math. 75, No. 2 (2007), 190-192].
12. Bashmakov R.A. *Systems of exponents in weighted Hilbert spaces on* $\mathbb{R}$. // PhD thesis. Institute of Mathematics with Computer Center, USC, RAS. Ufa. 2006. (Russian)
13. Sedletskii A.M. *Classes of analytic Fourier transforms and exponential approximations.* Moscow, 2005. 504 p. (Russian)
14. Levin B.Ya., Lyubarskii Yu.I. *Interpolation by means of special classes of entire functions and related expansions in series of exponentials.* // Izv. Akad. Nauk SSSR, Ser. Mat., 39:3 (1975), 657–702 [MATH. USSR IZV., 1975, 9 (3), 621–662].
15. Lyubarskii Yu.I. *Exponential series in Smirnov spaces and interpolation by entire functions of special classes.* // Izv. Akad. Nauk SSSR, Ser. Mat., 52:3 (1988), 559–580 [MATH. USSR. IZV., 1989, 32 (3), 563–586].
16. Lutsenko V.I. *Unconditional exponential bases in Smirnov spaces* // PhD thesis. Institute of Mathematics with Computer Center, USC, RAS. Ufa. 1992. (Russian)

Konstantin Petrovich Isaev,
Institute of Mathematics with Computer Center, Ufa Science Center,

Russian Academy of Sciences,
Chernyshevskii Str., 112,
450008, Ufa, Russia
E-mail: orbit81@list.ru

Rinad Salavatovich Yulmukhametov,
Institute of Mathematics with Computer Center, Ufa Science Center,
Russian Academy of Sciences,
Chernyshevskii Str., 112,
450008, Ufa, Russia
E-mail: Yulmukhametov@mail.ru

Translated from Russian by E.D. Avdonina.