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ON NONLINEAR HYPERBOLIC SYSTEMS RELATED BY BÄCKLUND TRANSFORMS

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Abstract. In this work we describe pairs of nonlinear hyperbolic system of equations $u_{xy} = f(u, u_x, u_y)$, where $u^i_{xy} = f^i$, $i = 1, 2, \dots n$, the linearizations of which are related by the first order Laplace transform. On the base of this Laplace transform we construct Bäcklund transforms relating the solutions of nonlinear systems.

The classical Bäcklund transform is defined for a second-order nonlinear differential equation whose solution is a function of two independent variables. The Bäcklund transform for a pair of nonlinear equations is a system of relations involving functions and their first derivatives and it provides a transform of a solution of one equation into the solution of another and vice versa. The Bäcklund transforms preserve integrability. The Bäcklund problem is to list possible Bäcklund transforms and the equations admitting such transforms.

The Laplace cascade integration method is one of the classical methods for integrating linear partial differential equations. The Laplace transform is a special case of the Bäcklund transform for linear equations. The method used in this paper was previously applied to nonlinear hyperbolic equations. In this paper, this method is employed to describe systems associated with Bäcklund transforms.

Keywords: nonlinear hyperbolic system, Laplace transform, Bäcklund transform, linearization

Mathematics Subject Classification: 35L10, 35L51, 35L70

Dedicated to a blessed memory of my scientific supervisor A.V. Zhiber, who formulated a problem in this paper.

1. Introduction

In the present work we make a classification of nonlinear hyperbolic systems of equations of form

$$u_{xy} = f(u, u_x, u_y), \qquad (u^i_{xy} = f^i, \qquad i = 1, 2, \dots n), \tag{1.1}$$

$$q_{xy} = F(q, q_x, q_y), \qquad (q^i_{xy} = F^i, \qquad i = 1, 2, \dots, n) \tag{1.2}$$

under the condition that their linearizations are related by the first order Laplace transform. On the base of the Laplace transform relating the solutions of linearized system, we construct a Bäcklund transform relating the solutions of nonlinear systems (1.1), (1.2).

The Laplace cascade integration method is a classical one for integrating linear equations of form, see [1]–[4],

$$v_{xy} + a(x, y)v_x + b(x, y)v_y + c(x, y)v = 0.$$

The Laplace transform is a differential substitution (a change involving an unknown function and its derivative) transforming an original equation into an equation of the same form. A pair of differential substitutions gives a transform from one equation to the other and vice versa. A detailed description of the method can be found in [5], [6].

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In works [7], [8], [9], [5], there were considered nonlinear hyperbolic equations. As a definition of exactly integrable equation of Liouville type, there was chosen a property of two-sided break of the Laplace invariants for a linearized equation. In survey [5] there was provided a procedure of finding a general solution to nonlinear hyperbolic equations based on using the Laplace invariants. In works [10], [11] there were described the properties of the Laplace invariants of nonlinear equations possessing differential substitutions.

In work [12] a generalization of the Laplace cascade integration method was proposed for the case of linear hyperbolic systems of equations. On this base, it was proved that the system of equations with a vanishing product of Laplace invariants possesses a complete set of solutions depending on arbitrary functions.

Let us clarify the notion of the Bäcklund transform by a particular example. The equations

$$u_{xy} = \sin u, \qquad v_{xy} = v\sqrt{1 - v_y^2} \tag{1.3}$$

are related by the Bäcklund transform

$$v = u_x, \qquad v_y = \sin u.$$

The latter relations ensure the following passage: if u is a solution to first equation (1.3), then v is a solution to second equation (1.3) and vice versa. In other words, this is a pair of differential substitutions relating solutions to non-linear equations. For the history of the Bäcklund problem, see [13]. A special case of the Bäcklund transform is the Laplace transform for linear equations. The Bäcklund transform preserving the original equation are used to construct exact solutions. For instance, soliton-type solutions were found for the sine-Gordon equation [14]. In papers [15], [16], Bäcklund transforms were employed to solve boundary value problems and construct exact solutions to evolution equations.

A wide class of examples of differential substitutions relating pairs of non-linear second-order hyperbolic equations can be found in [5], [10], [11], [17], [18]. In work [19], there were described non-linear hyperbolic equations, the linearizations of which were related by Laplace transforms, and there were constructed Bäcklund transforms relating solutions of non-linear equations.

This work consists of the following sections. In Section 2 we describe the systems $u_{xy} = f(u)$ and (1.2), the linearizations of which are related by the first-order Laplace transform. We construct a Bäcklund transform, which relates solutions of nonlinear systems. In Section 3 we solve the same problem for a pair of systems (1.1), (1.2). In Section 4 we provide examples.

2. Systems of equations $u_{xy} = f(u)$ and $q_{xy} = F(q, q_x, q_y)$

In this section we describe all nonlinear hyperbolic systems of equations of form

$$u_{xy} = f(u), \quad (u^i_{xy} = f^i, \quad i = 1, 2, \dots n), \tag{2.1}$$

$$q_{xy} = F(q, q_x, q_y), \quad (q^i_{xy} = F^i, \quad i = 1, 2, \dots, n) \tag{2.2}$$

under the condition that their linearizations are related by the first order Laplace transforms.

First we introduce notation and formulate the assumptions that we will use below. In order to do this, we consider a scalar equation of form (1.1). All identities involving the function u must be satisfied on any solution of equation (1.1). In other words, the letter u everywhere denotes an arbitrary solution of equation (1.1). The latter allows to express each mixed derivative of u by means of system (1.1) in terms of the variables u, $u_i = \frac{\partial^i u}{\partial x^i}$, $\bar{u}_i = \frac{\partial^i u}{\partial y^i}$. Therefore, we assume that all functions are infinitely differentiable and depend on a finite number of these variables. It is easy to see that these variables cannot be related to each other using equation (1.1) and this is why we treat them as independent.

By D and $\bar{D}$ we denote the operators of total differentiation with respect to the variables x and y, respectively. The differentiations D and $\bar{D}$ are defined by the relations

$$D(u_i) = u_{i+1}, \qquad \bar{D}(\bar{u}_i) = \bar{u}_{i+1}, \qquad u_0 = \bar{u}_0 = u, \qquad i = 0, 1, 2, \dots,$$

$$D\bar{D}u = f(u, u_1, \bar{u}_1), \qquad [D, \bar{D}] = 0.$$

On functions depending on finitely many variablezs u, u_i, $\bar{u}_i$, the operators D and $\bar{D}$ acts as follows:

$$D = \sum_{i=0}^{\infty} u_{i+1}\frac{\partial}{\partial u_i} + \sum_{i=1}^{\infty} \bar{D}^{i-1}(f)\frac{\partial}{\partial \bar{u}_i},$$
$$\bar{D} = \sum_{i=0}^{\infty} \bar{u}_{i+1}\frac{\partial}{\partial \bar{u}_i} + \sum_{i=1}^{\infty} D^{i-1}(f)\frac{\partial}{\partial u_i}.$$

The action of D and $\bar{D}$ on the vectors and matrices is defined by applying this operator to each their entry.

We consider a linear system of hyperbolic equations

$$v_{xy} + a(x, y)v_x + b(x, y)v_y + c(x, y)v = 0. \tag{2.3}$$

Here v is an n-dimensional vector, a, b and c are matrices of size $n \times n$. System (2.3) can be rewritten as

$$v_{xy} + av_x + bv_y + cv = v_{xy} + av_x + bv_y + (b_y + ab - k)v = \left(\frac{\partial}{\partial y} + a\right)\left(\frac{\partial}{\partial x} + b\right) v - kv = 0.$$

Here

$$k = b_y + ab - c. \tag{2.4}$$

Now it is easy to see that system (2.3) is equivalent to system

$$\left(\frac{\partial}{\partial x} + b\right) v = v_{-1}, \qquad \left(\frac{\partial}{\partial y} + a\right) v_{-1} = kv. \tag{2.5}$$

The first equation defines a so-called Laplace x-transform for system (2.3), which consists in passing from the unknown v to the unknown v_{-1}. If $\det(k) \neq 0$, then by second formula (2.4) we find:

$$v = k^{-1}\left(\frac{\partial}{\partial y} + a\right) v_{-1}.$$

We substitute the latter function into original system (2.3)

$$\left(\frac{\partial}{\partial x} + b\right) k^{-1}\left(\frac{\partial}{\partial y} + a\right) v_{-1} - v_{-1} = 0. \tag{2.6}$$

Differentiating the identity $kk^{-1} = E$ in the variable x, we obtain $k_x k^{-1} + k(k^{-1})_x = 0$, which implies $(k^{-1})_x = -k^{-1}k_x k^{-1}$. By using the latter formula, we transform system (2.6) to

$$k^{-1}\left((v_{-1})_{xy} + a(v_{-1})_x + \big(kbk^{-1} - k_x k^{-1}\big)(v_{-1})_y + \Big(a_x + \big(kbk^{-1} - k_x k^{-1}\big)a - k\Big)v_{-1}\right) = 0.$$

Thus, by applying Laplace x-transform to system (2.3), we obtain system of the same form as the initial one:

$$(v_{-1})_{xy} + a_{-1}(v_{-1})_x + b_{-1}(v_{-1})_y + c_{-1}v_{-1} = 0,$$

where

$$a_{-1} = a, \qquad b_{-1} = \big(kb - k_x\big)k^{-1}, \qquad c = a_x + b_{-1}a_{-1} - k. \tag{2.7}$$

Suppose that the solutions $u(x, y, \tau)$ and $q(x, y, \tau)$ of systems (2.1) and (2.2), respectively, depend on some parameter τ and we define functions $v = u_\tau$ and $p = q_\tau$. Then the functions v and p satisfy linearized systems

$$D\bar{D}v = Cv, \tag{2.8}$$
$$\big(D\bar{D} - A_{-1}D - B_{-1}\bar{D} - C_{-1}\big)p = 0. \tag{2.9}$$

Here we have introduced the following notation:

$$C=\left(\frac{\partial f^i(u)}{\partial u^j}\right),\qquad i,j=1,\dots,n, \tag{2.10}$$

$$A_{-1}=\left(\frac{\partial F^i(q,q_x,q_y)}{\partial q_x^j}\right),\qquad B_{-1}=\left(\frac{\partial F^i(q,q_x,q_y)}{\partial q_y^j}\right),\qquad C_{-1}=\left(\frac{\partial F^i(q,q_x,q_y)}{\partial q^j}\right).$$

Suppose that system (2.9) is obtained from system (2.8) by applying the Laplace x-transform. The problem is to describe corresponding nonlinear systems (2.2) and (2.1). By formulas (2.7) the following relations hold:

$$A_{-1}=0,\qquad B_{-1}=D(k)k^{-1},\qquad C_{-1}=k, \tag{2.11}$$

where, in accordance with formula (2.4),

$$k=C. \tag{2.12}$$

According to formulas (2.5), solutions to system (2.8), (2.9) are related by the identities

$$Dv=p,\qquad \bar{D}p=kv. \tag{2.13}$$

Theorem 2.1. *Let linearized system* (2.9) *be a result of applying the Laplace x-transform to system* (2.8). *Then systems* (2.1) *and* (2.2) *are of the form:*

$$u_{xy}=f(u),\qquad q_{xy}=C\big(f^{-1}(q_y)\big)q, \tag{2.14}$$

where the matrix $C(u)$ is defined by formula (2.10) *and* $\det C\neq 0$.

Proof. We consider relations (2.13). We observe that if

$$u_x=q, \tag{2.15}$$

then first formula (2.13) is true. We apply the differentiation $\bar{D}$ to the both sides of relation (2.15):

$$f(u)=q_y. \tag{2.16}$$

We note that differentiation of identity (2.16) in the parameter τ leads us to second formula (2.13). Applying then the operator D to both sides of relation (2.16), we obtain

$$q^i_{xy}=f^i_{u^1}u^1_x+f^i_{u^2}u^2_x+\dots+f^i_{u^n}u^n_x,\quad i=1,2,\dots,n.$$

Here the vector $u^1,u^2,\dots,u^n$ should be treated as a solution of algebraic systems (2.16). Taking into consideration notation (2.10), we obtain that q satisfies second equation (2.14). The proof is complete. □

Remark 2.1. *In the proof of the theorem we have found the Bäcklund transform*

$$q=u_x,\qquad q_y=f(u),$$

relating the solutions of system (2.14).

3. System of equations of form $u_{xy}=f(u,u_x,u_y)$

We suppose that solutions $u(x,y,\tau)$ and $q(x,y,\tau)$ to systems (1.1) and (1.2), respectively, depend on some parameter τ and we define the functions $v=u_\tau$ and $p=q_\tau$. Then the functions v and p satisfy the linearized systems

$$\big(D\bar{D}-AD-B\bar{D}-C\big)v=0, \tag{3.1}$$

$$\big(D\bar{D}-\tilde{A}D-\tilde{B}\bar{D}-\tilde{C}\big)p=0. \tag{3.2}$$

Here A, B, C, $\tilde{A}$, $\tilde{B}$, $\tilde{C}$ are square matrices of nth order:

$$A=\left(\frac{\partial f^i}{\partial u_1^j}\right),\qquad B=\left(\frac{\partial f^i}{\partial \bar{u}_1^j}\right),\qquad C=\left(\frac{\partial f^i}{\partial u^j}\right),$$

$$\tilde{A} = \left(\frac{\partial F^i}{\partial q_1^j}\right), \qquad \tilde{B} = \left(\frac{\partial F^i}{\partial \bar{q}_1^j}\right), \qquad \tilde{C} = \left(\frac{\partial F^i}{\partial q^j}\right), \qquad i,j = 1,\dots,n.$$

Suppose that system (3.2) is obtained from system (3.1) as a result of applying the Laplace x-transform. The problem is to describe corresponding nonlinear systems (1.2) and (1.1). Then the following relations

$$\big(D - B\big)v = p, \qquad \big(\bar{D} - A\big)p = Hv \tag{3.3}$$

hold. Here $H = -\bar{D}(B) + AB + C$, $\det H \neq 0$. By formulas (2.7) the coefficients of system (3.1), (3.2) should satisfy the following relations:

$$\tilde{A} = A, \qquad \tilde{B} = \big(HB + D(H)\big)H^{-1}, \qquad \tilde{C} = D(A) - \tilde{B}\tilde{A} + H. \tag{3.4}$$

Theorem 3.1. *Let linearized system* (3.2) *be a result of applying the Laplace x-transform to system* (3.1)*. Then systems* (1.1) *and* (1.2) *read as follows:*

$$u_{xy} = \varphi(u, u_1) + \lambda'(u)\bar{u}_1,$$

$$q_{xy} = \frac{\partial\varphi\big(U, q + \lambda(U)\big)}{\partial u_1}\, q_1 + \big(q + \lambda(U)\big)\left(\frac{\partial\varphi\big(U, q + \lambda(U)\big)}{\partial u} + \frac{\partial\varphi\big(U, q + \lambda(U)\big)}{\partial u_1}\,\lambda'(U)\right).$$

Here $\varphi = (\varphi^1, \dots, \varphi^n)^T$, $\lambda = (\lambda^1, \dots, \lambda^n)^T$, $\frac{\partial\varphi}{\partial u} = \left(\frac{\partial\varphi^i}{\partial u^j}\right)$, $\frac{\partial\varphi}{\partial u_1} = \left(\frac{\partial\varphi^i}{\partial u_1^j}\right)$, $\lambda'(U) = \left(\frac{\partial\lambda^i(U)}{\partial u^j}\right)$. *At the same time, the vector-function* $U(q, \bar{q}_1) = \big(U^1(q, \bar{q}_1), \dots, U^n(q, \bar{q}_1)\big)^T$ *is determined by the system*

$$\varphi\big(U, q + \lambda(U)\big) = \bar{q}_1.$$

Remark 3.1. *We have constructed the Bäcklund transform*

$$q = u_1 - \lambda(u), \qquad \bar{q}_1 = \varphi(u, u_1),$$

relating the solutions to the given systems.

Proof. We apply the differentiation in the parameter τ to first relation (3.4):

$$F^i_{q_1^j q^k} p^k + F^i_{q_1^j q_1^k} p_1^k + F^i_{q_1^j \bar{q}_1^k} \bar{p}_1^k = f^i_{u_1^j u^k} v^k + f^i_{u_1^j u_1^k} v_1^k + f^i_{u_1^j \bar{u}_1^k} \bar{v}_1^k.$$

Hereinafter $i, j = 1, \dots, n$, we make a summation over repeating indices from 1 to n. We rewrite the latter relations by using identities (3.3)

$$F^i_{q_1^j q^k}\big(v_1^k - f^k_{\bar{u}_1^s} v^s\big) + F^i_{q_1^j q_1^k}\Big(v_2^k - D(f^k_{\bar{u}_1^s})v^s - f^k_{\bar{u}_1^s} v_1^s\Big)$$
$$+ F^i_{q_1^j}\bar{q}_1^k\Big(h_{ks}v^s + f^k_{u_1^s}\big(v_1^s - f^s_{\bar{u}_1^r} v^r\big)\Big) = f^i_{u_1^j u^k} v^k + f^i_{u_1^j u_1^k} v_1^k + f^i_{u_1^j \bar{u}_1^k} \bar{v}_1^k.$$

Here h_{ks} are the entries of the matrix H. We collect the coefficients at the variables v_2^k and $\bar{v}_1^k$:

$$F^i_{q_1^j q_1^k} = 0, \qquad f^i_{u_1^j \bar{u}_1^k} = 0.$$

By this we specify the functions f^i and F^i:

$$F^i(q, q_1, \bar{q}_1) = \alpha^i_k(q, \bar{q})q_1^k + \beta^i(q, \bar{q}_1), \tag{3.5}$$

$$f^i(u, u_1, \bar{u}_1) = \varphi^i(u, u_1) + \psi^i(u, \bar{u}_1). \tag{3.6}$$

We substitute functions (3.5), (3.6) into third relation (3.4) and we obtain that

$$\beta^i_{q^j} + (\alpha^i_k)_{q^j} q_1^k = D\big(\varphi^i_{u_1^j}\big) - \big((\alpha^i_k)_{\bar{q}_1^s} q_1^k + (\beta^i)_{\bar{q}_1^s}\big)\alpha^s_j + h_{ij}. \tag{3.7}$$

The entries of the matrix H are given by the following formulas:

$$h_{ij} = -\psi^i_{\bar{u}_1^j u^r}\bar{u}_1^r - \psi^i_{\bar{u}_1^j \bar{u}_1^r}\bar{u}_2^r + \varphi^i_{u_1^r}\psi^r_{\bar{u}_1^j} + \varphi^i_{u^j} + \psi^i_{u^j}.$$

Then we substitute h_{ij} into (3.7), differentiate the left and right sides of the obtained relation in the parameter and we collect the coefficients at independent variables $\bar{v}_2^r$:

$$\psi^i_{\bar{u}_1^j \bar{u}_1^r} \equiv 0, \qquad i, j, r = 1, \ldots, n.$$

Then

$$\psi^i(u, \bar{u}_1) = g^i_k(u)\bar{u}_1^k + r^i(u).$$

The functions f^i defined by formulas (3.6) become

$$f^i(u, u_1, \bar{u}_1) = \varphi^i(u, u_1) + g^i_k(u)\bar{u}_1^k + r^i(u).$$

Denoting $\varphi^i + r^i$ by φ^i, we reduce the latter function to the form:

$$f^i(u, u_1, \bar{u}_1) = \varphi^i(u, u_1) + g^i_k(u)\bar{u}_1^k. \tag{3.8}$$

We then observe that if

$$q^i = u_1^i - \lambda^i(u), \tag{3.9}$$

where $\lambda^i_{u^k}(u) = g^i_k(u)$, then the first of formulas (3.3) holds true. Then, by formula (3.8), first sought system (1.1) becomes

$$u^i_{xy} = f^i = \varphi^i(u, u_1) + \lambda^i_{u^k}(u)\bar{u}_1^k \tag{3.10}$$

or, in the matrix form,

$$u_{xy} = \varphi(u, u_1) + \frac{\partial \lambda(u)}{\partial u}\bar{u}_1.$$

We apply the operator $\bar{D}$ to the left and right sides of relation (3.9):

$$\bar{q}_1^i = \varphi^i(u, u_1).$$

In the latter formula we replace u_1 according to (3.9):

$$\bar{q}_1^i = \varphi^i\big(u, q + \lambda(u)\big). \tag{3.11}$$

We observe that the differentiation of identity (3.11) in the parameter τ leads us to the formula coinciding with the second of formulas (3.3). And finally, applying the operator D to the left and right hand sides of relation (3.11) and expressing u_1^k by (3.9), we get:

$$q^i_{xy} = F^i = \varphi^i_{u^k}\big(u, q + \lambda(u)\big)\big(q^k + \lambda^k(u)\big) + \varphi^i_{u_1^k}\big(u, q + \lambda(u)\big)\big(q_1^k + \lambda^k_{u^s}(u)\big(q^s + \lambda^s(u)\big)\big). \tag{3.12}$$

Here the vector function $u(q, \bar{q}_1) = \big(u^1(q, \bar{q}_1), \ldots, u^n(q, \bar{q}_1)\big)^T$ should be treated as a solution to system (3.11). In the vector form system (3.12) can be written as

$$q_{xy} = \frac{\partial \varphi\big(U, q + \lambda(U)\big)}{\partial u_1} q_1 + \big(q + \lambda(U)\big)\left(\frac{\partial \varphi\big(U, q + \lambda(U)\big)}{\partial u} + \frac{\partial \varphi\big(U, q + \lambda(U)\big)}{\partial u_1}\lambda'(U)\right).$$

Here $U(q, \bar{q}_1) = \big(U^1(q, \bar{q}_1), \ldots, U^n(q, \bar{q}_1)\big)^T$ is expressed from system $\bar{q}_1 = \varphi(U, q + \lambda(U))$. The proof is complete. □

4. Examples of nonlinear systems and Bäcklund transform relating their solutions

In the present section we provide examples of nonlinear systems, the linearizations of which are related by the first order Laplace transforms, as well as the Bäcklund transform for each such nonlinear pair.

Example 4.1. A Toda chain of series A_2

$$u_{xy} = -2e^u + e^v, \qquad v_{xy} = e^u - 2e^v$$

and a system

$$q_{xy} = \frac{2}{3}(p_y + 2q_y)q - \frac{1}{3}(q_y + 2p_y)p, \qquad p_{xy} = -\frac{1}{3}(p_y + 2q_y)q + \frac{2}{3}(q_y + 2p_y)p$$

are related by the Bäcklund transform

$$q = u_x, \qquad p = v_x,$$
$$q_y = -2e^u + e^v, \qquad p_y = e^u - 2e^v.$$

Example 4.2. We consider a system

$$u^1_{xy} = e^{u^1-u^2}, \qquad u^2_{xy} = e^{-2u^1+2u^2}, \qquad u^3_{xy} = 3u^3 e^{u^1-u^2} \tag{4.1}$$

and a system

$$q^1_{xy} = q^1_y(q^1 - q^2), \qquad q^2_{xy} = q^2_y(-2q^1 + 2q^2), \qquad q^3_{xy} = 3q^3 q^1_y + q^3_y(q^1 - q^2).$$

System (4.1) possesses a soliton solution [20]. The Bäcklund transform is defined by formulas:

$$q^1 = u^1_x, \qquad q^2 = u^2_x, \qquad q^3 = u^3_x,$$
$$q^1_y = e^{u^1-u^2}, \qquad q^2_y = e^{-2u^1+2u^2}, \qquad q^3_y = 3u^3 e^{u^1-u^2}.$$

Example 4.3. It is known that the following system a soliton solution [20]:

$$u^1_{xy} = e^{u^1-u^2}, \qquad u^2_{xy} = e^{u^2-u^1},$$
$$u^3_{xy} = (3u^3 - u^4)e^{u^1-u^2}, \qquad u^4_{xy} = (3u^4 - u^3)e^{u^2-u^1}.$$

This system is related with a system

$$q^1_{xy} = q^1_y(q^1 - q^2), \qquad q^2_{xy} = q^2_y(q^2 - q^1),$$
$$q^3_{xy} = (3q^3 - q^4)q^1_y + (q^1 - q^2)q^3_y, \qquad q^4_{xy} = (3q^4 - q^3)q^2_y + (q^2 - q^1)q^4_y$$

by the Bäcklund transform

$$q^1 = u^1_x, \qquad q^2 = u^2_x, \qquad q^3 = u^3_x, \qquad q^4 = u^4_x,$$
$$q^1_y = e^{u^1-u^2}, \qquad q^2_y = e^{u^2-u^1},$$
$$q^3_y = (3u^3 - u^4)e^{u^1-u^2}, \qquad q^4_y = (3u^4 - u^3)e^{u^2-u^1}.$$

Example 4.4. As an example we also provide a system

$$u^i_{xy} = u^i_x \sum_{j=1}^{n} a_{ij} u^j.$$

This system is related with

$$q^i_{xy} = \exp\Big(\sum_{j=1}^{n} a_{ij} q^j\Big)$$

by the Bäcklund transform

$$\begin{pmatrix} q^1 \\ \vdots \\ q^n \end{pmatrix} = A^{-1} \begin{pmatrix} \ln u^1_x \\ \vdots \\ \ln u^n_x \end{pmatrix},$$
$$q^i_y = u^i, i = 1, \dots, n.$$

Here $A = (a_{ij})$, $i, j = 1, 2, \dots, n$, is a matrix.

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