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NECESSARY CONDITION OF FUNDAMENTAL PRINCIPLE FOR INVARIANT SUBSPACES ON UNBOUNDED CONVEX DOMAIN

A.S. KRIVOSHEEV, O.A. KRIVOSHEEVA

Abstract. In this paper we study the spaces $H(D)$ of analytic functions in convex domains of the complex plane as well as subspaces $W(\Lambda, D)$ of such spaces. A subspace $W(\Lambda, D)$ is the closure in the space $H(D)$ of the linear span of the system $\mathcal{E}(\Lambda) = \{z^n \exp(\lambda_k z)\}_{k=1,n=0}^{\infty,n_k-1}$, where Λ is the sequence of different complex numbers λ_k and their multiplicities n_k. This subspace is invariant with respect to the differentiation operator. The main problem in the theory of invariant subspaces is to represent all its functions by using the eigenfunctions and associated functions of the differentiation operator, $z^n e^{\lambda_k z}$. In this paper we study the problem of the fundamental principle for an invariant subspace $W(\Lambda, D)$, that is, the problem of representing all its elements by using a series constructed over the system $\mathcal{E}(\Lambda)$. We obtain simple geometric conditions, which are necessary for the existence of a fundamental principle. These conditions are formulated in terms of the length of the arc of the convex domain and the maximum density of the exponent sequence.

Keywords: exponential monomial, convex domain, fundamental principle, length of arc.

Mathematics Subject Classification: 30D10

1. Introduction

Let $\Lambda = \{\lambda_k, n_k\}_{k=1}^{\infty}$ be a sequence of different complex numbers λ_k and their multiplicities n_k. We suppose that $|\lambda_k|$ does not decrease and $|\lambda_k| \to \infty$, $k \to \infty$. Let $D \subset \mathbb{C}$ be a convex domain and $H(D)$ be the space of functions analytic in the domain D with the topology of uniform convergence on compact sets in D. By the symbol $W(\Lambda, D)$ we denote the closure in the space $H(D)$ of the linear span of the system

$$\mathcal{E}(\Lambda) = \{z^n \exp(\lambda_k z)\}_{k=1,n=0}^{\infty,n_k-1}.$$

If the system $\mathcal{E}(\Lambda)$ is incomplete in the space $H(D)$, then $W(\Lambda, D)$ is a nontrivial ($\neq H(D)$, $\{0\}$) closed subspace in $H(D)$. It follows from the definition that it is invariant with respect to the differentiation operator. At the same time, the system $\mathcal{E}(\Lambda)$ is the set of eigenfunctions and adjoint functions of the differentiation operator in $W(\Lambda, D)$, while Λ is its multiple spectrum.

Let $W \subset H(D)$ be a nontrivial closed subspace invariant with respect to the differentiation operator and $\Lambda = \{\lambda_k, n_k\}$ be its multiple spectrum. This is at most countable set with the only accumulation point ∞ [1, Ch. II, Sec. 7]. In the case, when the spectrum W is finite, it coincides with the space of solutions of homogeneous linear differential equation of a finite order with constant coefficients. As a more general example of an invariant subspace, the set of solutions of the convolution equation $\mu(g(z + w)) \equiv 0$ (or of systems of such equations)

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serve, where μ is a linear continuous functional on the space $H(D)$. Particular cases of the convolution equations are linear differential, difference, differential-difference equations with constant coefficients of finite and infinite orders, as well as some kinds of integral equations.

A main problem in the theory of invariant subspaces is to represent of all its functions by means of eigenfunctions and adjoint functions of the differentiation operator, $z^n e^{\lambda_k z}$. If W is the space of solutions of linear differential equation of a finite order with constant coefficients, then it coincides with the linear span of the system $\mathcal{E}(\Lambda)$. This result is known as Leonhard Euler fundamental principle. Because of this, the problem on representing the functions $g \in W$ by means of the series over the elements of the system $\mathcal{E}(\Lambda)$, that is, by the series

$$\sum_{k=1,n=0}^{\infty,n_k-1} a_{k,n} z^n e^{\lambda_k z} \tag{1.1}$$

is known as a fundamental principle problem for an invariant subspace. A first step to representation (1.1) is to resolve the problem of the spectral synthesis, that is, to clarify the conditions, under which the system $\mathcal{E}(\Lambda)$ is complete in the subspace W; in other words, when $W = W(\Lambda, D)$. It is natural to consider the fundamental principle problem only for invariant subspaces admitting the spectral synthesis, that is, for subspaces of form $W(\Lambda, D)$.

The study of the fundamental principle problem has a rich history. Partially it was reflected in work [2]. A complete solution of the fundamental principle problem in the case of a bounded convex domain D was obtained in works [3]–[5]. It was proved that each function $g \in W(\Lambda, D)$ is represented by series (1.1) in the domain D if and only if $S_\Lambda = 0$ and

$$\overline{n}_0(\Lambda(\varphi_1, \varphi_2)) \leqslant \frac{\Upsilon_D(-\varphi_2, -\varphi_1)}{2\pi}, \qquad \varphi_1, \varphi_2 \notin \Phi(\Lambda), \quad 0 < \varphi_2 - \varphi_1 < \pi. \tag{1.2}$$

Here S_Λ is the condensation index of the sequence Λ introduced in work [2], $\overline{n}_0(\Lambda)$ is the maximal density of Λ, $\Phi(\Lambda)$ is some at most countable set, (φ_1, φ_2) is a sequence consisting of all pairs λ_k, n_k such that λ_k is located in the angle

$$\Gamma(\varphi_1, \varphi_2) = \{z = te^{i\varphi} : \varphi \in (\varphi_1, \varphi_2),\ t > 0\},$$

$\Upsilon_D(\varphi_1, \varphi_2)$ is the length of a part the boundary of the domain D, which connects the points, at which the support straight lines

$$L(-\varphi_2, D) = \{z : \mathrm{Re}(ze^{i\varphi_2}) = H(-\varphi_2, D)\}, \quad L(-\varphi_1, D) = \{z : \mathrm{Re}(ze^{i\varphi_1}) = H(-\varphi_1, D)\}$$

touch the boundary ∂D, and $H(\varphi, D)$ is the support function of the domain D.

In work [6] there was obtained a criterion of representation (1.1) in the case $D = \mathbb{C}$. Such representation holds if and only if the inequality $S_\Lambda < \infty$ is satisfied. The case, when the domain D is a half-plane was studied in works [7] and [8]. The criterion of the representation was formulated only in terms of the index S_Λ. In work [9] there was obtained a complete solution of the fundamental principle in the case when $\Theta(\Lambda)$ contains no internal points of the set, at which the support function of the domain D. Here $\Theta(\Lambda)$ is the set of limits of all converging sequences of form $\{\overline{\lambda}_{k_j}/|\lambda_{k_j}|\}_{j=1}^{\infty}$. This solution is also formulated in terms of the index S_Λ.

In the present work we consider arbitrary convex domains D. We prove that inequality (1.2) is necessary for representation (1.1) for all $\varphi_1, \varphi_2 \notin \Phi(\Lambda)$ such that the arc $\{e^{i\varphi} : \varphi \in [-\varphi_2, -\varphi_1]\}$ is located inside the set, on which the function $H(\varphi, D)$ is bounded.

2. Construction of special entire function

By symbols $B(z, r)$ and $S(z, r)$ we denote respectively an open circle and a circumference centered at a point $z \in \mathbb{C}$ of a radius $r > 0$. Let $\Lambda = \{\lambda_k, n_k\}$ and $n(r, \Lambda)$ denote the number of points λ_k counting their multiplicities n_k in the circle $B(0,r)$. We let

$$m(\Lambda)=\varlimsup_{k\to\infty}\frac{n_k}{|\lambda_k|},\qquad \overline{n}(\Lambda)=\varlimsup_{r\to\infty}\frac{n(r,\Lambda)}{r},$$
$$\overline{n}_0(\Lambda,\delta)=\varlimsup_{r\to\infty}\frac{n(r,\Lambda)-n((1-\delta)r,\Lambda)}{\delta r},\qquad \overline{n}_0(\Lambda)=\varlimsup_{\delta\to 0}\overline{n}_0(\Lambda,\delta).$$

The quantities $\overline{n}(\Lambda)$ and $\overline{n}_0(\Lambda)$ are respectively called upper and maximal density of the sequence Λ. We say that Λ possesses a density $n(\Lambda)$ if there exists the limit

$$n(\Lambda)=\lim_{r\to\infty}\frac{n(r,\Lambda)}{r}.$$

By Lemma 2.1 in work [10] we have:

$$\overline{n}(\Lambda)\leqslant\overline{n}_0(\Lambda,\delta)\leqslant\overline{n}_0(\Lambda),\quad \delta\in(0,1). \tag{2.1}$$

If Λ possesses a density, then

$$n(\Lambda)=\overline{n}(\Lambda)=\overline{n}_0(\Lambda,\delta)=\overline{n}_0(\Lambda),\quad \delta\in(0,1). \tag{2.2}$$

Let f be an entire function of exponential type in the complex plane, that is,

$$\ln|f(z)|\leqslant A+B|z|,\qquad z\in\mathbb{C}.$$

The function

$$h_f(\varphi)=\varlimsup_{r\to+\infty}\frac{\ln|f(re^{i\varphi})|}{r},\qquad \varphi\in[0,2\pi],$$

is called an indicator of f. We observe one property of the indicator [11, Ch. I, Sect. 18, Thm. 28]: for each $\varepsilon>0$ there exists $R(\varepsilon)>0$ such that

$$\ln|f(re^{i\varphi})|\leqslant(h_f(\varphi)+\varepsilon)r,\qquad \varphi\in[0,2\pi],\qquad r\geqslant R(\varepsilon). \tag{2.3}$$

The function h_f coincides with the support function

$$H(\varphi,T)=\max_{z\in T}\mathrm{Re}(ze^{-i\varphi})$$

of some convex compact set $T\subset\mathbb{C}$, which is called an indicator diagram of the function f. An adjoint diagram K of the function f is a compact set complex conjugate to the compact set T [1, Ch. I, Sect. 5, Thm. 5.4]. Thus,

$$h_f(\varphi)=H(-\varphi,K),\qquad \varphi\in[0,2\pi].$$

This implies that the function h_f is continuous, and hence, uniformly continuous, on the segment $[0,2\pi]$. This is why for each $\varepsilon_0>0$ there exists $\delta_0\in(0,1)$ such that

$$\begin{gathered}|th_f(\psi)-h_f(\varphi)|=|tH(-\psi,K)-H(-\varphi,K)|\leqslant\varepsilon_0,\\ \varphi\in[0,2\pi],\quad te^{i\psi}\in B(e^{i\varphi},\delta_0).\end{gathered} \tag{2.4}$$

One says [11, Ch. III] that f has a regular growth if

$$h_f(\varphi)=\lim_{r\notin E,r\to+\infty}\frac{\ln|f(re^{i\varphi})|}{r^{\rho(r)}},\qquad \varphi\in[0,2\pi],$$

where $E\subset(0,+\infty)$ is the set of zero relative measure (E_0-set) if

$$\lim_{r\to+\infty}\frac{\mathrm{mes}(E\cap(0,r))}{r}=0$$

(the symbol mes stands for the Lebesgue measure of a set). A classical result by B.Ya. Levin [11, Ch. II, Thm. 2, Ch. III, Thm. 4] states that f has a regular growth if and only if its multiple zero set $\Lambda_f = \{\lambda_k, n_k\}_{k=1}^{\infty}$ is called regularly distributed. At the same time, the identity

$$2\pi n(\Lambda_f(\varphi_1, \varphi_2)) = h'_f(\varphi_2) - h'_f(\varphi_1) + \int_{\varphi_1}^{\varphi_2} h_f(\varphi)d\varphi, \qquad \varphi_1, \varphi_2 \notin \Phi(\Lambda_f), \tag{2.5}$$

holds, where $\Phi(\Lambda)$ is the set of all φ such that

$$\inf_{\alpha>0} \overline{\lim_{r\to\infty}} \frac{n(r, \Lambda(\varphi - \alpha, \varphi + \alpha))}{r}.$$

We note that the set $\Phi(\Lambda_f)$ coincides with the set of numbers φ, for which the derivatives $h'_f(\varphi)$ does not exist. At the same time, one-sided derivatives of the function h_f exist.

One also says that f has a regular growth on the ray $L_\varphi = \{re^{i\varphi}, r > 0\}$ if

$$h_f(\varphi) = \lim_{r\notin E_\varphi, r\to+\infty} \frac{\ln|f(re^{i\varphi})|}{r},$$

where E_φ is a E_0-set. If f has a regular growth on each ray, then the set E_φ, generally depends on $\varphi \in [0, 2\pi]$. However, it turns out that one can find an exceptional E_0-set, which is appropriate for all $\varphi \in [0, 2\pi]$ [11, Ch. III, Sect. 1, Thm. 1]. In other words, the function f has a regular growth if and only if when it has a regular growth on each ray. Another equivalent definition of a function of a regular growth [12, Lm. 4.1] is also known. The function f has a regular growth on the ray L_φ if and only if there exists a sequence $\{z_m\}_{m=1}^{\infty}$ such that

$$\lim_{m\to\infty} |z_m| = \infty, \qquad \lim_{m\to\infty} \frac{z_m}{|z_m|} = e^{i\varphi}, \qquad \lim_{m\to\infty} \frac{|z_{m+1}|}{|z_m|} = 1, \qquad \lim_{m\to\infty} \frac{\ln|f(z_m)|}{|z_m|} = h_f(\varphi). \tag{2.6}$$

By the symbol $\underline{h}_f$ we denote a lower indicator of the function f [13]:

$$\underline{h}_f(\varphi) = \lim_{\delta\to 0} \underline{\lim_{t\to\infty}} \frac{1}{\pi\delta^2} \int_{B(te^{i\varphi}, t)} \frac{\ln|f(z)|}{t} dxdy, \qquad z = x + iy.$$

Then in view of (2.3) we obtain:

$$\underline{h}_f(\varphi) \leqslant h_f(\varphi), \quad \varphi \in [0, 2\pi]. \tag{2.7}$$

If $\underline{h}_f(\varphi) \geqslant c$, then by Lemma 2.7 in work [14] there exists a sequence $\{z_m\}_{m=1}^{\infty}$ such that

$$\lim_{m\to\infty} |z_m| = \infty, \qquad \lim_{m\to\infty} \frac{z_m}{|z_m|} = e^{i\varphi}, \qquad \lim_{m\to\infty} \frac{|z_{m+1}|}{|z_m|} = 1, \qquad \underline{\lim_{m\to\infty}} \frac{\ln|f(z_m)|}{|z_m|} \geqslant c. \tag{2.8}$$

Let D be a convex domain and $\Lambda = \{\lambda_k, n_k\}$. By the symbol $I(D, \Lambda)$ we denote the set of all entire functions f of exponential type such that

$$h_f(\varphi) < H(\varphi, D), \qquad \varphi \in [0, 2\pi],$$

and for each $k \geqslant 1$ the function f vanishes at the point λ_k with the multiplicities at least n_k. In other words, f/f_Λ is an entire function, where

$$f_\Lambda(z) = \prod_{k=1}^{\infty} \left(1 - \frac{z}{\lambda_k}\right)^{n_k} e^{\frac{n_k z}{\lambda_k}}.$$

We note that the function f_Λ is an entire function of first order and probably of an infinite type, that is, generally speaking, it is not an entire function of exponential type. This is the case if and only if $\overline{n}(\Lambda) < \infty$ and

$$\varlimsup_{r\to\infty}\left|\sum_{|\lambda_k|<r}\frac{n_k}{\lambda_k}\right| < \infty.$$

We let

$$J(D) = \{e^{i\varphi} \in S(0,1) : H(\varphi, D) = +\infty\}.$$

We observe that the support function $H(\varphi, D)$ is always lower semi-bounded and is continuous inside the interval, in which it is bounded. In particular, if D is a bounded domain, then $H(\varphi, D)$ is a continuous function.

If D is bounded, then $J(D) = \emptyset$. In the case of an unbounded domain D the following situations are possible:

1) $J(D) = S(0,1)$, that is, $D = \mathbb{C}$,

2) D is the half-plane $\{z \in \mathbb{C} : \mathrm{Re}(ze^{-i\varphi} < a\}$ and $J(D) = S(0,1) \setminus \{e^{i\varphi}\}$,

3) D is the strip $\{z \in \mathbb{C} : b < \mathrm{Re}(ze^{-i\varphi}) < a\}$ and $J(D) = S(0,1) \setminus \{e^{i\varphi}, e^{i\varphi+\pi}\}$,

4) in other cases $J(D)$ is an arc of the unit circle, which is supported by an angle of opening at least π.

By $\mathcal{K}(D) = \{K_p\}_{p=1}^{\infty}$ we denote a sequence of compact sets in the domain D, which strictly exhaust it, that is, (the symbol int stands of the interior of a set)

$$K_p \subset \operatorname{int} K_{p+1}, \quad p \geqslant 1, \quad D = \bigcup_{p=1}^{\infty} K_p.$$

Let $M \subset \mathbb{C}$ and $\rho(z, M)$ denotes the distance from a point z to the set M. We let

$$M^\delta = \bigcup_{z\in M} B(z, \delta|z|).$$

Now we formulate a result, which is a part of the result proven in Theorem 5.1 in work [2]. It follows directly from this theorem.

Lemma 2.1. *Let D be a convex domain and $\Lambda = \{\lambda_k, n_k\}$. Assume that $m(\Lambda) = 0$, the system $\mathcal{E}(\Lambda)$ is incomplete in $H(D)$ and each function $g \in W(\Lambda, D)$ is represented by series (1.1) for all $z \in D$. Then for each $p \geqslant 1$ and each compact set $\mathcal{F} \subset S(0,1) \setminus J(D)$ there exists $f \in I(D, \Lambda)$ such that for each $\delta > 0$ there exist numbers $\beta, T > 0$ obeying the condition: $\lambda_k \in (M_p)^\delta$ if $\rho(\overline{\lambda}_k/|\lambda_k|, \mathcal{F}) < \beta$ and $|\lambda_k| > T$, where*

$$M_p = \{z = re^{i\varphi} : \ln|f(z)| \geqslant rH(-\varphi, K_p)\}, \quad K_p \subset K(D),$$

and $\overline{\lambda}$ is the conjugate of λ.

We employ this result to construct an entire function with needed properties.

Lemma 2.2. *Let D be a convex domain and $\Lambda = \{\lambda_k, n_k\}$. Assume that $m(\Lambda) = 0$, the system $\mathcal{E}(\Lambda)$ is incomplete in $H(D)$, and each function $g \in W(\Lambda, D)$ is represented by series (1.1) for all $z \in D$. Then for all φ_1 and φ_2 such that $0 < \varphi_2 - \varphi_1 < \pi$ and*

$$\{e^{i\varphi} : \varphi \in [-\varphi_2, -\varphi_1]\} \subset S(0,1) \setminus \overline{J(D)}, \tag{2.9}$$

there exists a function $u \in I(\mathbb{C}, \Lambda(\varphi_1, \varphi_2))$ such that

$$\underline{h}_u(\varphi) = h_u(\varphi), \qquad \varphi \in [0, 2\pi], \qquad h_u(\varphi) = H(-\varphi, D), \qquad \varphi \in [\varphi_1, \varphi_2]. \tag{2.10}$$

Proof. By (2.10) there exists $\alpha > 0$ such that

$$\{e^{-i\varphi} : \varphi \in [\varphi_1 - 2\alpha, \varphi_2 + 2\alpha]\} \subset S(0,1) \setminus \overline{J(D)}. \tag{2.11}$$

We let

$$D_1 = \{z : \mathrm{Re}(ze^{-i\varphi}) < H(\varphi, D), \varphi \in [-\varphi_2 - 2\alpha, -\varphi_1 + 2\alpha]\}.$$

Lessening $\alpha > 0$ if it is needed, we can suppose that $\varphi_2-\varphi_1+4\alpha < \pi$. Then D_1 is an unbounded convex domain lying in an angle with the sides on the support straight lines $L(-\varphi_2 - 2\alpha, D)$ and $L(-\varphi_1 + 2\alpha, D)$. At the same time,

$$H(\varphi, D_1) = H(\varphi, D), \qquad \varphi \in [-\varphi_2 - 2\alpha, -\varphi_1 + 2\alpha].$$

Let $z_1 \in \partial D_1 \cap L(-\varphi_1 + 2\alpha, D)$ and $z_2 \in \partial D_1 \cap L(-\varphi_2 - 2\alpha, D)$. We have:

$$\mathrm{Re}(z_1e^{-i\varphi}) < H(\varphi, D_1), \qquad \mathrm{Re}(z_2e^{-i\varphi}) < H(\varphi, D_1), \qquad \varphi \in (-\varphi_2 - 2\alpha, -\varphi_1 + 2\alpha).$$

$$\mathrm{Re}(z_1e^{i(\varphi_1-2\alpha)}) = H(2\alpha - \varphi_1, D_1), \quad \mathrm{Re}(z_2e^{i(\varphi_2+2\alpha)}) = H(-\varphi_2 - 2\alpha, D_1).$$

Let $D_2 = D_1 \cap \Pi$, where Π is a half-plane, the boundary of which contains the segment $T = [z_1, z_2]$ such that D_2 is a bounded domain. By the above relations we have:

$$H(\varphi, D_2) = H(\phi, D), \qquad \varphi \in [-\varphi_2 - 2\alpha, -\varphi_1 + 2\alpha], \tag{2.12}$$

$$H(\varphi, D_2) > H(\varphi, T), \qquad \varphi \in (-\varphi_2 - 2\alpha, -\varphi_1 + 2\alpha), \tag{2.13}$$

$$H(\varphi, D_2) = H(\varphi, T), \qquad e^{i\varphi} \in S(0, 1) \setminus \{e^{-i\vartheta} : \vartheta \in (-\varphi_2 - 2\alpha, -\varphi_1 + 2\alpha)\}. \tag{2.14}$$

We let

$$\psi_0(z) = \varlimsup_{w\to z} \sup\{\psi(w) : \psi \in SH(\mathbb{C}), \psi(w) + \ln|f_\Lambda(w)| \leqslant rH(-\varphi, D_2), w \in \mathbb{C}\},$$

where $w = re^{i\varphi}$ and $SH(\mathbb{C})$ is the space of subharmonic in the plane functions. The function ψ_0 also belongs to this space and satisfies the estimate of form

$$\psi_0(z) \leqslant C_0 + a_0|z|^2, \qquad z \in \mathbb{C}.$$

Then by Theorem 5 from work [15] there exists an entire function u_0 such that

$$|\ln|u_0(z)| - \psi_0(z)| \leqslant B_0 \ln|z|, \quad z \in \mathbb{C} \setminus E, \tag{2.15}$$

where $B_0 > 0$, and the exceptional set E can be covered by the circles $B(\xi_j, r_j)$, $j \geqslant 1$, such that $\sum r_j = A < \infty$.

Let $u = u_0f_\Lambda$. Then u is an entire function. We are going to show that u vanishes at the points $\lambda_k \in \Lambda(\varphi_1, \varphi_2)$ with the multiplicities at least n_k. By (2.15),

$$|\ln|u(z)| - \psi_0(z) - \ln|f_\Lambda(z)|| \leqslant B_0 \ln|z|, \quad z \in \mathbb{C} \setminus E. \tag{2.16}$$

Since

$$\ln|f_\Lambda(z)| = \varlimsup_{w\to z} \ln|f_\Lambda(w)|,$$

it follows from the definition of ψ_0 that

$$\psi(z) + \ln|f_\Lambda(z)| \leqslant rH(-\varphi, D_2), \qquad z = re^{i\varphi} \in \mathbb{C}.$$

Then in view of (2.16) we obtain

$$\ln|u(z)| \leqslant rH(-\varphi, D_2) + B_0 \ln r, \qquad z = re^{i\varphi} \in \mathbb{C} \setminus E. \tag{2.17}$$

Let $|w| > 3A$. Since the sum of the diameters of the circles $B(\xi_j, r_j)$, $j \geqslant 1$, is equal to $2A$, then in the circle $B(w, 3A)$ there exists a circumference, on which (2.17) holds true. Then by the maximum modulus principle we obtain

$$\ln|u(w)| \leqslant \sup_{z\in B(w,3A)} (rH(-\varphi, D_2) + B_0 \ln r).$$

This and (2.4) imply that for each $\varepsilon > 0$ there exists $t(\varepsilon) \geqslant 3A$ such that

$$\ln|u(z)| \leqslant r(H(-\varphi, D_2) + \varepsilon), \qquad |w| \geqslant t(\varepsilon).$$

This is why the inequality holds: $h_u(\varphi) \leqslant H(-\varphi, D_2) + \varepsilon$, $\varphi \in [0, 2\pi]$. Since $\varepsilon > 0$ is arbitrary, then

$$h_u(\varphi) \leqslant H(-\varphi, D_2), \quad \varphi \in [0, 2\pi]. \tag{2.18}$$

Thus, $u \in I(\mathbb{C}, \Lambda(\varphi_1, \varphi_2))$.

We are going to prove identities (2.10). Suppose that for each number $\varphi_0 \in [0, 2\pi]$ and $\varepsilon \in (0, \varepsilon_0)$ the inequality holds:

$$\underline{h}_u(\varphi_0) < H(-\varphi_0, D_2) - 4\varepsilon.$$

Then by Proposition 9.3 from work [16] there exists $\delta_0 \in (0, 1/3)$ and a sequence $\{t_m\}$ such that $t_m \to +\infty$, $m \to \infty$, and

$$\frac{\ln |u(t_m e^{i\varphi})|}{t_m} \leqslant H(-\varphi_0, D_2) - 3\varepsilon, \qquad e^{i\varphi} \in B(e^{i\varphi_0}, 2\delta_0), \qquad m \geqslant 1.$$

Lessening if needed $\delta_0 \in (0, 1/3)$, by (2.4) we obtain:

$$\ln |u(re^{i\varphi})| \leqslant r(H(-\varphi, D_2) - 2\varepsilon), \qquad re^{i\varphi} \in B(t_m e^{i\varphi_0}, 2\delta_0 t_m), \qquad m \geqslant 1.$$

Then in accordance with (2.16) we have:

$$\psi_0(z) + \ln |f_\Lambda(z)| \leqslant r(H(-\varphi, D_2) - 2\varepsilon), \qquad re^{i\varphi} \in B(t_m e^{i\varphi_0}, 2\delta_0 t_m) \setminus E, \qquad m \geqslant 1. \tag{2.19}$$

By (2.11) there exists a compact set $K_l \in \mathcal{K}(D)$ such that

$$H(-\varphi, K_l) \geqslant H(-\varphi, D) - \varepsilon, \qquad \varphi \in [-\varphi_2 - 2\alpha, -\varphi_1 + 2\alpha]. \tag{2.20}$$

Let $\varepsilon_1 \in (0, \varepsilon/20)$ satisfies the condition

$$H(-\varphi, K_l) \leqslant H(-\varphi, D) - 20\varepsilon_1, \qquad \varphi \in [0, 2\pi]. \tag{2.21}$$

We choose a compact set $K_p \in \mathcal{K}(D)$ such that

$$H(-\varphi, K_p) \geqslant H(-\varphi, D) - \varepsilon_1, \qquad \varphi \in [-\varphi_2 - 2\alpha, -\varphi_1 + 2\alpha]. \tag{2.22}$$

We let $\mathcal{F} = \{e^{i\varphi} : \varphi \in [-\varphi_2, -\varphi_1]\}$, and let $f \in I(D, \Lambda)$ be a function from Lemma 2.1. According to (2.11), there exists $\delta \in (0, \delta_0/36)$ such that

$$\begin{gathered} |tH(-\psi, D) - H(-\varphi, D)| + 19\varepsilon_1 |1 - t| \leqslant \varepsilon_1, \\ \varphi \in [-\varphi_2 - 2\alpha, -\varphi_1 + 2\alpha], \qquad te^{i\psi} \in B(e^{i\varphi}, 36\delta). \end{gathered} \tag{2.23}$$

By (2.3) we can suppose that

$$\ln |f(re^{i\varphi})| \leqslant (h_f(\varphi) + \varepsilon_1) r, \qquad \varphi \in [0, 2\pi], \qquad r \geqslant R(\varepsilon_1) \geqslant 1.$$

Since $f \in I(D, \Lambda)$, then $h_f(\varphi) < H(\varphi, D)$, $\varphi \in [0, 2\pi]$. This is why

$$\ln |f(re^{i\varphi})| \leqslant (H(\varphi, D) + \varepsilon_1) r, \qquad \varphi \in [0, 2\pi], \qquad r \geqslant R(\varepsilon_1) \geqslant 1. \tag{2.24}$$

We can also suppose that $e^{i\varphi} \in \{e^{-i\vartheta} : \vartheta \in [\varphi_1 - \alpha, \varphi_2 + \alpha]\}$ for each point $re^{i\varphi} \in B(z, 36\delta|z|)$ and each circle $B(z, \delta|z|)$, which contains at least one $\lambda_k \in \Gamma(\varphi_1, \varphi_2)$.

Let $\lambda_k \in \Gamma(\varphi_1, \varphi_2)$ and $|\lambda_k| > \max\{T, 2R(\varepsilon_1)\}$. According to (2.22) and Lemma 2.1 there exists $w_k = te^{i\vartheta}$ such that

$$\ln |f(w_k)| \geqslant tH(-\vartheta, K_p) \geqslant t(H(-\vartheta, D) - \varepsilon_1) \tag{2.25}$$

and $\lambda_k \in B(w_k, \delta t)$. We can suppose that $\delta t \geqslant A$. By (2.23) and (2.24) we have:

$$\ln |f(z)| \leqslant t(H(-\vartheta, D) + 2\varepsilon_1), \qquad z \in B(w_k, 36\delta t).$$

Then by (2.25) we obtain

$$\ln |f_k(z)| \leqslant 3\varepsilon_1 t, \qquad z \in B(w_k, 36\delta t), \qquad f(z) = \frac{f(z)}{f(w_k)}.$$

Then by the theorem on the lower bound for the absolute value of an analytic function [1, Ch. I, Thm. 4.2]

$$\ln|f_k(z)| \geqslant -18\varepsilon_1 t, \qquad z \in B(w_k, 6\delta t) \setminus E_w,$$

where E_w is the union of the circles, the sum of radii of which is equal to δt. We choose a circumference $S(w_k, \delta t(w_k))$, which does not intersect $E_w \cup E$ such that $t(w_k) \in (t, 6t)$. By (2.25)

$$\ln|f(z)| \geqslant t(H(-\vartheta, D) - 19\varepsilon_1), \qquad z \in S(w_k, \delta t(w_k)).$$

Taking into consideration also (2.23) and (2.21), we obtain:

$$\ln|rf(re^{i\varphi})| \geqslant r(H(-\varphi, D) - 20\varepsilon_1) \geqslant rH(-\varphi, K_l), \qquad re^{i\varphi} \in S(w_k, \delta t(w_k)). \tag{2.26}$$

Let $\lambda_k \in \Gamma(\varphi_1, \varphi_2)$ and $|\lambda_k| \leqslant \max\{T, 2R(\varepsilon_1)\}$. We choose a circle $B(\lambda_k, \tau_k) \subset \Gamma(\varphi_1, \varphi_2)$, which contains no other points λ_j. We let

$$b = \min_{\lambda_k} \min_{re^{i\varphi} \in S(\lambda_k, \tau_k)} \left(\ln|rf(re^{i\varphi})| - rH(-\varphi, K_l)\right),$$

where the first minimum is taken over all mentioned λ_k. We consider the set

$$\{z = re^{i\varphi} : r\ln|f(z)| < rH(-\varphi, K_l) - |b|\}.$$

Let Ω be the union of all its connected components, each of which contains at least one point $\lambda_k \in \Gamma(\varphi_1, \varphi_2)$. Then the set Ω contains all points $\lambda_k \in \Gamma(\varphi_1, \varphi_2)$, and it is contained in the union of the circles $B(w_k, \delta t(w_k))$ and $B(\lambda_j, \tau_j)$. It follows from (2.26) and the definition of the number b that $\overline{\Omega} \subset \Gamma(\varphi_1 - \alpha, \varphi_2 + \alpha)$.

We let

$$\psi_1(z) = \ln|zf(z)|, \qquad \psi_2(z) = \psi_1(re^{i\varphi}) = rH(-\varphi, K_l) - |b|, \qquad \psi_3(z) = rH(-\varphi, T),$$

$$\psi_4(z) = \begin{cases} \psi_1(z) - \ln|f_\Lambda(z)|, & z \in \overline{\Omega}, \\ \max\limits_{j=1,2}(\psi_j(z) - \ln|f_\Lambda(z)|), & z \in \mathbb{C} \setminus \overline{\Omega}. \end{cases}$$

The functions $\psi_2(z)$ and $\psi_3(z)$ are convex on the entire plane. This is why $\psi_2, \psi_3 \in SH(\mathbb{C})$. Since the function $zf(z)/(f_\Lambda(z))$ is entire, then the function f_Λ has no zeroes on the set $\mathbb{C}\setminus\Omega$ (in particular, $\ln|f_\Lambda(z)|$ is a harmonic function), then it follows directly from the definition of the function ψ_4 that it is subharmonic in $\mathbb{C} \setminus \partial\Omega$. Let $z \in \partial\Omega$. Then by the upper semi-continuity of the functions $\psi_j(z) - \ln|f_\Lambda(z)|$, $j = 1, 2$, and the definition of the set Ω the relations

$$\psi_4(z) = \psi_1(z) - \ln|f_\Lambda(z)| \geqslant \varlimsup_{w \to z}(\psi_1(w) - \ln|f_\Lambda(w)|) = \varlimsup_{w \to z} \psi_4(w)$$

hold, that is, $\psi(w)$ is upper semi-continuous at the point z. Moreover, for sufficiently small $\tau > 0$,

$$\psi_4(z) = \ln\left|\frac{zf(z)}{f_\Lambda(z)}\right| \leqslant \frac{1}{\pi\delta^2} \int\limits_{B(z,\tau)} (\ln|wf(w)| - \ln|f_\Lambda(w)|)\,dxdy \leqslant \frac{1}{\pi\delta^2} \int\limits_{B(z,\tau)} \psi_4(w)\,dxdy.$$

Thus, $\psi_4 \in SH(\mathbb{C})$. Since $f \in I(D, \Lambda)$, in view of (2.3) there exists a compact set $K_s \subset \mathcal{K}(D)$, $s \geqslant l$, and a number $b_1 > 0$ such that

$$\ln|f(re^{i\varphi})| - b_1 \leqslant rH(-\varphi, K_s), \qquad re^{i\varphi} \in \mathbb{C}.$$

Then in view of the definition of ψ_4 we obtain:

$$\psi_4(z) - b_1 + \ln|f_\Lambda(z)| \leqslant rH(-\varphi, K_s), \quad re^{i\varphi} \in \mathbb{C}. \tag{2.27}$$

We let

$$\psi(z) = \begin{cases} \max\{\psi_4(z) - b_1, \psi_3(z) - \ln|f_\Lambda(z)|\}, & z \in \Gamma(\varphi_1 - 2\alpha, \varphi_2 + 2\alpha), \\ \psi_3(z) - \ln|f_\Lambda(z)|, & z \in \mathbb{C} \setminus \Gamma(\varphi_1 - 2\alpha, \varphi_2 + 2\alpha). \end{cases}$$

Since $K_s \subset \mathcal{K}(D)$, then by (2.12), (2.14) and (2.27)

$$\psi_3(z) - \ln |f_\Lambda(z)| \geqslant \psi_4(z) - b_1, \qquad z \in \partial\Gamma(\varphi_1 - 2\alpha, \varphi_2 + 2\alpha).$$

Then, as above, we have: $\psi \in SH(\mathbb{C})$. Moreover, relations (2.27), (2.12)–(2.14) and the definition of the function ψ imply the inequality

$$\psi(z) + \ln |f_\Lambda(z)| \leqslant rH(-\varphi, D_2), \qquad z = re^{i\varphi} \in \mathbb{C}.$$

Then in view of (2.19) and the definition of the function ψ_0 we obtain:

$$\psi(z)+\ln |f_\Lambda(z)| \leqslant r(H(-\varphi, D_2) - 2\varepsilon), \qquad z = re^{i\varphi} \in B(t_m e^{i\varphi_0}, 2\delta_0 t_m) \setminus E, \qquad m \geqslant 1. \quad (2.28)$$

Let the circle $B(t_{m(j)}e^{i\varphi_0}, 3\delta_0 t_{m(j)}/2)$, $j \geqslant 1$, contains none of the points w_k. This means that

$$B(t_{m(j)}e^{i\varphi_0}, \delta_0 t_{m(j)}) \cap \Omega = \emptyset, \qquad j \geqslant j_0. \quad (2.29)$$

We can suppose that $\delta_0 t_{m(j)} > 2A$. Then there exists a point $\nu_j \in B(t_{m(j)}e^{i\varphi_0}, \delta_0 t_{m(j)})$ such that

$$\psi(\nu_j) + \ln |f_\Lambda(\nu_j)| \leqslant \rho_j(H(-\theta_j, D_2) - 2\varepsilon), \quad \nu_j = \rho_j e^{i\theta_j}, \qquad j \geqslant j_0. \quad (2.30)$$

By (2.29) and the definition of the function ψ we get the inequalities

$$\psi(\nu_j) + \ln |f_\Lambda(\nu_j)| \geqslant \rho_j H(-\theta_j, T), \qquad \nu_j \in \mathbb{C} \setminus \Gamma(\varphi_1 - 2\alpha, \varphi_2 + 2\alpha),$$
$$\psi(\nu_j) + \ln |f_\Lambda(\nu_j)| \geqslant \rho_j H(-\theta_j, K_l) - |b| - b_1, \qquad \nu_j \in \mathbb{C} \setminus \Gamma(\varphi_1 - 2\alpha, \varphi_2 + 2\alpha).$$

By (2.14), (2.12) and (2.20) two latter inequalities contradict (2.30).

Suppose now that for all $m \geqslant m_0$ the circle $B(t_m e^{i\varphi_0}, 3\delta_0 t_m/2)$ contains a point $w_{k(m)}$. Then

$$S(w_{k(m)}, \delta t(w_{k(m)})) \subset B(t_m e^{i\varphi_0}, 2\delta_0 t_m), \qquad m \geqslant m_0.$$

Since the circumference $S(w_{k(m)}, \delta t(w_{k(m)}))$ does not intersect the set E, at each its point both inequalities (2.26) and (2.28) are satisfied. In view of (2.12) and (2.20) we get a contradiction.

Thus,

$$\underline{h}_u(\varphi) \geqslant H(-\varphi, D_2), \qquad \varphi \in [0, 2\pi].$$

Together with (2.7), (2.12) and (2.18) this gives (2.10). The proof is complete. □

3. Fundamental principle

Let D be a convex domain and z_1, z_2 the points on its boundary ∂D. By $s(z_1, z_2, D)$ we denote the length of the arc $\gamma \subset \partial D$ connecting z_1 and z_2 and the motion from z_1 to z_2 is made in the positive direction (counterclockwise). For each $\varphi \in \mathbb{R}$ such that $e^{i\varphi} \in S(0,1) \setminus \overline{J(D)}$ the intersection

$$L(\varphi) = \{z : \mathrm{Re}(ze^{-i\varphi}) = H_K(e^{i\varphi})\} \cap \partial D$$

(of the support straight line and the boundary of the domain) is either a point $z(\varphi)$ or a segment. The set $\Phi(D)$ of directions φ, for which $L(\varphi)$ is a segment, is at most countable set. We let

$$S_D(\varphi_1, \varphi_2) = \sup_{z_1 \in L(\varphi_1), z_2 \in L(\varphi_2)} s(z_1, z_2, D).$$

The function $S_D(\varphi_1, \varphi_2)$ is non-decreasing in φ_2 and is non-increasing in φ_1, while the set of its discontinuity points in both variables coincide with $\Phi(D)$. If $\varphi_1, \varphi_2 \notin \Phi(D)$, then

$$S_D(\varphi_1, \varphi_2) = s(z(\varphi_1), z(\varphi_2), D).$$

Using formula (1.114) from book [11], we obtain:

$$S_D(\varphi_1, \varphi_2) = H'(\varphi_2, D) - H'(\varphi_1, D) + \int_{\varphi_1}^{\varphi_2} H(\varphi, D) d\varphi, \qquad \varphi_1, \varphi_2 \notin \Phi(D). \quad (3.1)$$

We note that the set $\Phi(D)$ coincides with the set of numbers φ, for which the derivative $H'(\varphi, D)$ does not exist. At the same time, one-sided derivatives of the function $H(\varphi, D)$ exist.

Let $\Lambda = \{\lambda_k, n_k\}$. By the symbol $\Theta(\Lambda)$ we denote the set of limits of all converging sequences of form $\left\{\overline{\lambda_{k(j)}}/|\lambda_{k(j)}|\right\}_{j=1}^{\infty}$. It is obvious that $\Theta(\Lambda)$ is a closed subset of the circumference $S(0,1)$. We let

$$m(\Lambda, \mu) = \sup \varlimsup_{k\to\infty} \frac{n_{k(j)}}{|\lambda_{k(j)}|},$$

where the supremum is taken over all subsequences $\{\lambda_{k(j)}\}$ such that $\overline{\lambda_{k(j)}}/|\lambda_{k(j)}| \to \mu$. If $\mu \notin \Theta(\Lambda)$, we let $m(\Lambda, \mu) = 0$. It is easy to see that $m(\Lambda) = 0$ if and only if $m(\Lambda, \mu) = 0$, $\mu \in \Theta(\Lambda)$.

Theorem 3.1. *Let D be a convex domain and $\Lambda = \{\lambda_k, n_k\}$. Suppose that the system $\mathcal{E}(\Lambda)$ is incomplete in $H(D)$ and each function $g \in W(\Lambda, D)$ is represented by series (1.1), which converges uniformly on compact sets in the domain D. Then for all $\varphi_1, \varphi_2 \notin \Phi(\Lambda_f)$ such that $0 < \varphi_2 - \varphi_1 < \pi$ and*

$$\{e^{i\varphi} : \varphi \in [-\varphi_2, -\varphi_1]\} \subset S(0,1) \setminus \overline{J(D)},$$

the inequality

$$\overline{n}_0(\Lambda(\varphi_1, \varphi_2)) \leqslant \frac{1}{2\pi} S_D(-\varphi_2, -\varphi_1) \tag{3.2}$$

holds.

Proof. If the assumptions of this theorem are satisfied, by Theorem 4.2 from work [7], $m(\Lambda, \mu) = 0$, $\mu \in \{e^{i\varphi} : \varphi \in [-\varphi_2, -\varphi_1]\}$. This implies that $m(\Lambda(\varphi_1, \varphi_2)) = 0$. By assumption, each function $g \in W(\Lambda, D)$ is represented by series (1.1) for all $z \in D$. In particular, this concerns all functions $g \in W(\Lambda(\varphi_1, \varphi_2), D)$. The system $\mathcal{E}(\Lambda(\varphi_1, \varphi_2))$ is incomplete in $H(D)$, that is, the system $\mathcal{E}(\Lambda)$. Thus, sequence $\Lambda = \Lambda(\varphi_1, \varphi_2)$ satisfies all assumptions of Lemma 2.2. According to this lemma, there exists a function $u \in I(\mathbb{C}, \Lambda(\varphi_1, \varphi_2))$ such that (2.10) holds.

By the first identity in (2.10), in view of (2.6)–(2.8) we find that u has a regular growth. Then relation (2.5) holds. By this identity, (3.1) and the second identity in (2.10) we have:

$$n(\Lambda_u(\varphi_1, \varphi_2)) = \frac{1}{2\pi} S_D(-\varphi_2, -\varphi_1).$$

Together with (2.2) this gives:

$$\overline{n}_0(\Lambda_u(\varphi_1, \varphi_2)) = \frac{1}{2\pi} S_D(-\varphi_2, -\varphi_1).$$

Since $u \in I(\mathbb{C}, \Lambda(\varphi_1, \varphi_2))$, the latter identity leads us to (3.2). The proof is complete. □

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Alexander Sergeevich Krivosheev,
Institute of Mathematics,
Ufa Federal Research Center, RAS,
Chernyshevsky str. 112,
450008, Ufa, Russia
E-mail: `kriolesya2006@yandex.ru`

Olesya Alexandrovna Krivosheeva,
Ufa University of Science and Technologies,
Zaki Validi str. 32,
450076, Ufa, Russia
E-mail: `kriolesya2006@yandex.ru`