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ON CONTINUOUS LINEAR FUNCTIONALS IN SOME SPACES OF FUNCTIONS ANALYTIC IN DISK

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Abstract. The issue on description of linear continuous functionals on the spaces of analytic functions has been studied since the middle of 20th century. Historically, the structure of linear continuous functionals on the Hardy spaces H^p for $p \geqslant 1$ was first established by A. Taylor in 1951. In the spaces H^p, $0 < p < 1$, this problem was solved by P. Duren, B. Romberg, A. Shields in 1969. We note that an estimate for the coefficient multipliers in these spaces was employed in the proof. In the present paper, by developing the method proposed in the work by P. Duren et al, we describe linear continuous functionals on area Privalov classes and Nevanlinna-Dzhrbashjan type spaces. The considered classes generalize the area Nevanlinna classes well-known in scientific literature. The idea of the proof of the main result is as follows: the issue on finding the general form of a continuous linear functional is reduced to finding a form of an arbitrary coefficient multiplier acting from a studied space into the space of bounded analytic functions. The latter problem in a simplified form can be formulated as follows: by what factors we should multiply the Taylor coefficients of the functions in a studied class in order to make them Taylor coefficients of some bounded analytic function.

Keywords: area Privalov spaces, Nevanlinna-Dzhrbashjan type spaces, linear continuous functionals, coefficient multipliers.

Mathematics Subject Classification: 30H99, 32C15, 46E10.

1. Introduction

Let $\mathbb{C}$ be the complex plane, D be unit circle on $\mathbb{C}$ and $H(D)$ be the set of all functions analytic in D. For an arbitrary function $f \in H(D)$ we denote $M(r,f) = \max_{|z|=r} |f(z)|$, $0 < r < 1$, and by $T(r,f)$ we denote the Nevanlinna characteristics of the function f [2]:

$$T(r,f) = \frac{1}{2\pi} \int_{-\pi}^{\pi} \ln^+ |f(re^{i\theta})| d\theta, \quad 0 < r < 1.$$

For all values of the parameter $0 < p < +\infty$ we introduce Hardy classes on the circle:

$$H^p := \left\{ f \in H(D) : \sup_{0<r<1} \int_{-\pi}^{\pi} |f(re^{i\varphi})|^p d\varphi < +\infty \right\},$$

H^∞ is the class of bounded analytic in D functions.

For all $0 < q < +\infty$ we define the Privalov class Π_q:

$$\Pi_q = \left\{ f \in H(D) : \sup_{0<r<1} \frac{1}{2\pi} \int_{-\pi}^{\pi} \left(\ln^+ |f(re^{i\theta})| \right)^q d\theta < +\infty, \right\}$$

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where $\ln^+ a = \max(\ln a, 0)$, $a > 0$.

First the classes Π_q were introduced by I.I. Privalov in [3]. As $q = 1$, the Privalov class coincides with the well-known in the scientific literature class of functions of bounded type or Nevanlinna class N [2]. The following chain of inclusions holds:

$$H^\infty \subset H^p\,(p > 0) \subset \Pi_q\,(q > 1) \subset N \subset \Pi_q\,(0 < q < 1).$$

For $0 < q < +\infty$ we introduce an area Privalov class

$$\tilde{\Pi}_q = \left\{ f \in H(D) : \int_0^1 \int_{-\pi}^{\pi} \left(\ln^+ |f(re^{i\theta})|\right)^q d\theta dr < +\infty \right\}.$$

The class $\tilde{\Pi}_q$ is a generalization of the well-known area Nevanlinna class and as $q = 1$ it coincides with this class. We note that the spaces $\tilde{\Pi}_q$ naturally arise in studying the issues of differentiation in Privalov classes, see [16].

For all $\alpha > -1$, $0 < q < +\infty$ we consider the classes S^q_α:

$$S^q_\alpha = \left\{ f \in H(D) : \int_0^1 (1-r)^\alpha T^q(r, f) dr < +\infty \right\}.$$

The classes S^q_α were introduced and studied in [11] by F.A. Shamoyan, they generalize the well-known Nevanlinna-Djrbashian classes, see [2].

Using the Hölder inequality, it is easy to show that

$$\tilde{\Pi}_q \subset S^q_0 \quad \text{as} \quad q > 1,$$

and

$$\tilde{\Pi}_q \supset S^q_0 \quad \text{as} \quad 0 < q < 1.$$

In the present work we study linear continuous functionals on the spaces $\tilde{\Pi}_q$ and S^q_α. The notion of a linear continuous functional plays an important role in the functional analysis. The question on description of linear continuous functional has been studied till the middle of the 20th century. Historically, first in work by the structure of linear continuous functionals on the Hardy spaces H^p as $p \geqslant 1$ was found by A. Taylor in 1951 [17]. In the space H^p, $0 < p < 1$, which in contrast to the case $p \geqslant 1$ are not Banach but just F-spaces, the linear continuous functionals were first described by P. Duren, B. Romberg, A. Shields in 1969 [12]. We note that an estimate for the coefficient multipliers in the spaces H^p was employed in the proof. In 1973, basing on work [12], N. Yanagihara found a general form of linear continuous functionals in the Smirnov spaces [18]. In 1999, developing the method proposed by N. Yanagihara, R. Meštrović and A.V. Subbotin described the linear continuous functionals on the Privalov spaces for all $q > 1$, see [1].

We extend the latter mentioned results for the area Privalov classes and the classes S^q_α. The idea of the proof is as follows: the issue on finding the general form of linear continuous functionals on the Privalov spaces is reduced to finding an arbitrary coefficient multiplier acting from the studied space into the space of bounded analytic functions.

To formulate the results of the work, we introduce additional definitions and notation.

Let X and Y be some classes of functions analytic in the unit circle D.

Definition 1.1. *The sequence of complex numbers $\Lambda = \{\lambda_k\}_{k=1}^{+\infty}$ is called a coefficient multiplier from the class X into the class Y if for an arbitrary function $f \in X$, $f(z) = \sum_{k=0}^{+\infty} a_k z^k$, we have $\Lambda(f)(z) = \sum_{k=0}^{+\infty} \lambda_k a_k z^k \in Y$. The notation is $CM(X, Y)$.*

The paper is organized as follows. In the next section we formulate and prove auxiliary statement, while in the third section we prove the main result.

2. Auxiliary statements

In the proof of the results of work we use an analogue of the Mergelyan theorem in the studied spaces.

Theorem 2.1 ([4]). *If $f \in S^q_\alpha$, then*

$$\ln^+ M(r,f) = o\left(\frac{1}{(1-r)^{\frac{\alpha+1}{q}+1}}\right), \quad r \to 1-0, \tag{2.1}$$

and this estimate is sharp, that is, for each positive function $\omega(r)$, $0<r<1$, such that $\omega(r)=o(1)$, $r \to 1-0$, there exists a function $f \in S^q_\alpha$ such that

$$\ln^+ M(r,f) \neq O\left(\frac{\omega(r)}{(1-r)^{\frac{\alpha+1}{q}+1}}\right), \quad r \to 1-0.$$

Theorem 2.2 ([4]). *If $f(z)=\sum\limits_{k=0}^{+\infty} a_k z^k$ is the Taylor series of a function $f(z)$, $f \in S^q_\alpha$, then*

$$\ln^+ |a_k| = o\left(k^{\frac{\alpha+q+1}{\alpha+2q+1}}\right), \quad k \to +\infty. \tag{2.2}$$

This estimate is sharp, that is, for each positive sequence $\{\delta_k\}$, $\delta_k = o(1)$, $k \to +\infty$, there exists a function $f \in S^q_\alpha$ such that

$$\ln^+ |a_k| \neq O\left(\delta_k k^{\frac{\alpha+q+1}{\alpha+2q+1}}\right), \quad k \to +\infty.$$

Theorem 2.3 ([9]). *If $f \in \tilde{\Pi}_q$, then*

$$\ln^+ M(r,f) = o((1-r)^{-2/q}), \quad r \to 1-0. \tag{2.3}$$

This estimate is sharp, that is, for each positive function $\omega(r)$, $0<r<1$, such that $\omega(r)=o(1)$, $r \to 1-0$, there exists a function $f \in \tilde{\Pi}_q$ such that

$$\ln^+ M(r,f) \neq O(\omega(r)(1-r)^{-2/q}), \quad r \to 1-0.$$

Theorem 2.4 ([9]). *If $f(z)=\sum\limits_{k=0}^{+\infty} a_k z^k$ is the Taylor series of a function $f(z)$, $f \in \tilde{\Pi}_q$, then*

$$\ln^+ |a_k| = o\left(k^{\frac{2}{2+q}}\right), \quad k \to +\infty. \tag{2.4}$$

This estimate is sharp, that is, for each positive sequence $\{\delta_k\}$, $\delta_k = o(1)$, $k \to +\infty$ there exists a function $f \in \tilde{\Pi}_q$ such that

$$\ln^+ |a_k| \neq O\left(\delta_k k^{\frac{2}{2+q}}\right), \quad k \to +\infty.$$

For all $q>0$ in the spaces $\tilde{\Pi}_q$ and S^q_α we introduce the metrics

$$\rho_{\tilde{\Pi}_q}(f,g) = \left(\int_0^1 \int_{-\pi}^{\pi} \ln^q\left(1+|f(re^{i\theta})-g(re^{i\theta})|\right) d\theta dr\right)^{\alpha_q/q}, \qquad f,g \in \tilde{\Pi}_q; \tag{2.5}$$

$$\rho_{S^q_\alpha}(f,g) = \left(\int_0^1 (1-r)^\alpha \left(\int_{-\pi}^{\pi} \ln\left(1+|f(re^{i\theta})-g(re^{i\theta})|\right) d\theta\right)^q dr\right)^{\alpha_q/q}, \qquad f,g \in S^q_\alpha, \tag{2.6}$$

where $\alpha_q = \min(q,1)$.

The classes $\tilde{\Pi}_q$ and S^q_α are linear spaces. Let us show that they form F-spaces with respect to the introduced metrics, see [6], [9].

We recall that a metric space (X,ρ) is an F-space if [10]

a) $\rho(f,g)=\rho(f-g,0)$ (*invariancy with respect to the shifts*);

b) (X,ρ) is a complete metric space;

c) If $f,\, f_n \in X$ and $\rho(f_n, f) \to 0$, $n \to +\infty$, then $\rho(\beta f_n, \beta f) \to 0$, $n \to +\infty$ for each $\beta \in \mathbb{C}$ (*continuity of the multiplication by a scalar in the vector variable*);

d) If $\beta_n,\, \beta \in \mathbb{C}$ and $\beta_n \to \beta$, $n \to +\infty$, then $\rho(\beta_n f, \beta f) \to 0$, $n \to +\infty$ for each function $f \in X$ (*continuity of the multiplication in the scalar variable*).

In the proofs of the auxiliary statements we shall make use of the following useful estimate, which can be established easily.

Lemma 2.1. *For all $a \geqslant 0$, $b \geqslant 0$ the inequality $(a+b)^q \leqslant (a^q + b^q)$ holds as $0 < q \leqslant 1$ and $(a+b)^q \leqslant 2^q(a^q + b^q)$ as $q > 1$.*

Lemma 2.2. *With respect to the introduced metrics, S^q_α forms an F-space and the convergence in metrics* (2.6) *of this space is not weaker than the uniform convergence on the compact subsets D.*

Proof. We provide the proof for the case $0 < q \leqslant 1$. The case $q > 1$ can be considered similarly.

a) The identity $\rho(f, g) = \rho(f - g, 0)$ is obvious.

b) Let us show that S^q_α is a complete metric spaces.

Let $\{f_n(z)\}$ be an arbitrary fundamental sequence in the class S^q_α, that is, for each $\varepsilon > 0$ there exists an index $N(\varepsilon) > 0$ such that for all $n,\, m > N$ we have $\rho(f_n, f_m) < \varepsilon$. We are going to show that this sequence converges to some function $f \in S^q_\alpha$. Our first step is to confirm that the fundamental property of the sequence $\{f_n\}$ in S^q_α implies its uniform convergence inside the circle D. Due to the subharmonicity of the function $u(z) = \ln(1 + |f_n(z) - f_m(z)|)$ in D we have

$$\ln(1 + |f_n(re^{i\varphi}) - f_m(re^{i\varphi})|)$$
$$\leqslant \frac{1}{2\pi}\int\limits_{-\pi}^{\pi} \frac{R^2 - r^2}{R^2 - 2rRcos(\theta - \varphi) + r^2} \ln\left(1 + |f_n(Re^{i\theta}) - f_m(Re^{i\theta})|\right) d\theta$$
$$\leqslant \frac{1}{2\pi}\frac{R+r}{R-r}\int\limits_{-\pi}^{\pi} \ln\left(1 + |f_n(Re^{i\theta}) - f_m(Re^{i\theta})|\right) d\theta, \quad 0 < r < R < 1, \quad \varphi \in [-\pi, \pi].$$

This yields:

$$(R - r)^q \left(\ln(1 + |f_n(re^{i\varphi}) - f_m(re^{i\varphi})|)\right)^q \leqslant \frac{1}{\pi^q}\left(\int\limits_{-\pi}^{\pi} \ln\left(1 + |f_n(Re^{i\theta}) - f_m(Re^{i\theta})|\right) d\theta\right)^q.$$

We multiply both sides of the inequality by $(1-R)^\alpha$ and fixing $r \in [0, 1)$, we integrate in $R \in \left[\frac{1+r}{2}, 1\right)$:

$$\int\limits_{\frac{1+r}{2}}^{1} (1 - R)^\alpha (R - r)^q \left(\ln(1 + |f_n(re^{i\varphi}) - f_m(re^{i\varphi})|)\right)^q dR$$
$$\leqslant \frac{1}{\pi^q}\int\limits_{\frac{1+r}{2}}^{1} (1 - R)^\alpha \left(\int\limits_{-\pi}^{\pi} \ln\left(1 + |f_n(Re^{i\theta}) - f_m(Re^{i\theta})|\right) d\theta\right)^q dR.$$

Since the integrand is a non-negative function, we obtain that the right hand side of the inequality is majorized by metrics $\rho(f_n, f_m)$ and this is why

$$\left(\ln(1 + |f_n(re^{i\varphi}) - f_m(re^{i\varphi})|)\right)^q \int\limits_{\frac{1+r}{2}}^{1} (1 - R)^\alpha (R - r)^q dR \leqslant \frac{1}{\pi^q}\rho(f_n, f_m),$$

which implies

$$\ln(1 + |f_n(re^{i\varphi}) - f_m(re^{i\varphi})|) \leqslant \frac{c_{\alpha,q}}{(1 - r)^{(\alpha+1+q)/q}}(\rho(f_n, f_m))^{1/q},$$

for all $0 < r < 1$, $\varphi \in [-\pi, \pi]$. And finally,

$$|f_n(re^{i\varphi}) - f_m(re^{i\varphi})| \to 0, \quad n,\, m \to +\infty,$$

for all $0 < r < 1$, $\varphi \in [-\pi, \pi]$. Thus, the sequence $\{f_n\}$ converges uniformly inside the circle D to some function $f \in H(D)$. It is obvious that $\{f_n\}$ converges to f in the metrics of the space S_α^q. That is, for each $\varepsilon > 0$ there exists an index $N > 0$ such that for all $n > N$, $\rho(f_n, f) < \varepsilon$.

Let us prove that $f \in S_\alpha^q$.

$$\int_0^1 (1-r)^\alpha T^q(r,f)dr \leqslant \int_0^1 (1-r)^\alpha \left(\int_{-\pi}^{\pi} \ln\left(1 + |f(re^{i\theta})|\right) d\theta \right)^q dr = \rho(f, 0).$$

But $\rho(f,0) \leqslant \rho(f, f_n) + \rho(f_n, 0) < \varepsilon + c$ for all $n > N$ and this is why

$$\int_0^1 (1-r)^\alpha T^q(r,f)dr \leqslant const.$$

Thus, $f \in S_\alpha^q$ and the space S_α^q is complete.

c) Let f, $f_n \in S_\alpha^q$ and $\rho(f_n, f) \to 0$, $n \to +\infty$. Let us show that $\rho(\beta f_n, \beta f) \to 0$, $n \to +\infty$ for each $\beta \in \mathbb{C}$;

Let $|\beta| < 1$, then $\ln(1 + |\beta| x) \leqslant \ln(1+x)$ for all $x \geqslant 0$ and the property is immediately implied by the inequality $0 \leqslant \rho(\beta f_n, \beta f) \leqslant \rho(f_n, f)$.

For all $|\beta| \geqslant 1$ and $x \geqslant 0$ the estimate $(1 + |\beta| x) \leqslant (1+x)^{|\beta|}$ holds, which implies property c):

$$\begin{aligned}\rho(\beta f_n, \beta f) &= \int_0^1 (1-r)^\alpha \left(\int_{-\pi}^{\pi} \ln(1 + |\beta| \cdot |f_n(re^{i\theta}) - f(re^{i\theta})|) d\theta \right)^q dr \\ &\leqslant \int_0^1 (1-r)^\alpha \left(\int_{-\pi}^{\pi} \ln(1 + |f_n(re^{i\theta}) - f(re^{i\theta})|)^{|\beta|} d\theta \right)^q dr \\ &\leqslant |\beta|^q \int_0^1 (1-r)^\alpha \left(\int_{-\pi}^{\pi} \ln(1 + |f_n(re^{i\theta}) - f(re^{i\theta})|) d\theta \right)^q dr = |\beta|^q \rho(f_n, f).\end{aligned}$$

d) Let $f \in S_\alpha^q$ and $\beta_n \to \beta$, $n \to +\infty$. We are going to show that $\rho(\beta_n f, \beta f) \to 0$, $n \to +\infty$ for each function $f \in S_\alpha^q$;

We estimate:

$$\rho(\beta_n f, \beta f) = \int_0^1 (1-r)^\alpha \left(\int_{-\pi}^{\pi} \ln(1 + |f(re^{i\theta})||\beta_n - \beta|) d\theta \right)^q dr = J.$$

We partition the integral J into two parts:

$$J = \int_0^{r_0} \ldots + \int_{r_0}^1 \ldots = J_1 + J_2.$$

We choose $0 < r_0 < 1$ so that $J_2 < \frac{\varepsilon}{2}$, where $\varepsilon > 0$ is an arbitrary sufficiently small number. We estimate J_1 by using inequality (2.1) from Theorem 2.1:

$$J_1 \leqslant (2\pi)^q \ln^q \left(1 + |\beta_n - \beta| \exp \frac{\delta}{(1-r_0)^{\frac{\alpha+1}{q}+1}} \right) \cdot \frac{1 - (1-r_0)^{\alpha+1}}{\alpha+1},$$

where $\delta > 0$ is an arbitrary small number.

Since $|\beta_n - \beta| \to 0$, $n \to +\infty$, then $J_1 \leqslant \frac{\varepsilon}{2}$ as $n > N(\varepsilon)$. Thus, we have established d). The proof is complete. □

Lemma 2.3. *With respect to the introduced metrics, $\tilde{\Pi}_q$ forms an F-space and the convergence in the metrics (2.5) of this space is not weaker than the uniform convergence on the compact subsets D.*

Proof. We consider only the case $0 < q \leqslant 1$, the case $q > 1$ can be treated in the same way.

a) The identity $\rho(f,g) = \rho(f-g,0)$ is obvious.

b) $\tilde{\Pi}_q$ is a complete metric space.

Let $\{f_n\}$ be an arbitrary fundamental sequence in the class $\tilde{\Pi}_q$, that is, for each $\varepsilon > 0$ there exists an index $N(\varepsilon) > 0$ such that for all n, $m > N$ the inequality $\rho(f_n, f_m) < \varepsilon$ holds. Let us show that it converges to some function $f \in \tilde{\Pi}_q$. We note that the functions $\ln(1+|f_n|)$ are subharmonic in D and this is the following estimate holds [13]:

$$\ln^q(1+|f_n(Re^{i\theta}) - f_m(Re^{i\theta})|) \leqslant \frac{c(q)}{(1-R)^2}\cdot \rho(f_n, f_m),$$

which implies

$$|f_n(Re^{i\theta}) - f_m(Re^{i\theta})| \to 0, \quad n,\, m \to +\infty,$$

for all $0 < R < 1$, $\theta \in [-\pi, \pi]$. Thus, the fundamental sequence $\{f_n\} \in \tilde{\Pi}_q$ converges uniformly inside the disk D to some function $f \in H(D)$. It is obvious that $\{f_n\}$ converges to f also in the metrics of the space $\tilde{\Pi}_q$.

Let us prove that $f \in \tilde{\Pi}_q$:

$$\int_0^1\int_{-\pi}^{\pi} (\ln^+|f(re^{i\theta})|)^q d\theta\, dr \leqslant \int_0^1\int_{-\pi}^{\pi} (\ln(1+|f(re^{i\theta})|)^q d\theta\, dr$$
$$\leqslant \int_0^1\int_{-\pi}^{\pi} \ln^q\left(1+|f(re^{i\theta}) - f_n(re^{i\theta})| + |f_n(re^{i\theta})|\right) d\theta\, dr.$$

In view of Lemma 2.1, by the latter estimate we have:

$$\int_0^1\int_{-\pi}^{\pi} (\ln^+|f(re^{i\theta})|)^q d\theta \leqslant \int_0^1\int_{-\pi}^{\pi} \left[\ln^q(1+|f(re^{i\theta}) - f_n(re^{i\theta})|) + \ln^q(1+|f_n(re^{i\theta})|)\right] d\theta dr \leqslant const.$$

Thus, $\tilde{\Pi}_q$ is complete.

The proof of properties c), d) is similar to Lemma 2.2. The proof is complete. □

We also observe that F-spaces can be treated as complete quasinormed spaces.

Lemma 2.4 ([19])**.** *The continuity of a linear operator on quasinormed spaces is equivalent to its boundedness, that is, to the property that it maps bounded sets into bounded ones.*

We denote $f_\zeta(z) = f(\zeta z)$, $\zeta \in D$.

Lemma 2.5. *Let $f \in X$, where $X = S^q_\alpha$ or $X = \tilde{\Pi}_q$. Then the family of the functions $\{f_\zeta(z)\}$ is bounded in X.*

Proof. We consider an η-neighbourhood of 0, that is, $V = \{g \in X : \rho(g,0) < \eta\}$. We choose α' such that $\rho(\alpha' f, 0) < \frac{\eta}{2}$.

We denote $f_r(z) = f(rz)$, $0 < r < 1$. It is obvious that $\rho(f, f_r) \to 0$, $r \to 1-0$. We choose $r_0 \leqslant r < 1$ close enough to 1 so that $\rho(f, f_r) < \frac{\eta}{2}$.

We denote $f_{(\theta)}(z) = f(e^{i\theta}z)$, $f_{r(\theta)}(z) = f_r(e^{i\theta}z) = f(re^{i\theta}z)$. Then

$$\rho(\alpha' f_{(\theta)}, 0) = \rho(\alpha' f, 0) < \frac{\eta}{2},$$

and

$$\rho(\alpha' f_{r(\theta)}, \alpha' f_{(\theta)}) = \rho(\alpha' f_r, \alpha' f) \leqslant \rho(f_r, f) < \frac{\eta}{2}.$$

If $\zeta = re^{i\theta}$, then $f_\zeta = f_{r(\theta)}$. For all $r \geqslant r_0$ we obtain:

$$\rho(\alpha' f_\zeta, 0) = \rho(\alpha' f_{r(\theta)}, 0) \leqslant \rho(\alpha' f_{r(\theta)}, \alpha' f_{(\theta)}) + \rho(\alpha' f_{(\theta)}, 0) = \rho(\alpha' f_r, \alpha' f) + \rho(\alpha' f, 0) < \eta.$$

For all $0 \leqslant r \leqslant r_0$ we can choose α'' small enough so that

$$\rho(\alpha'' f_\zeta, 0) \leqslant \rho(\alpha'' f_r, 0) < \eta.$$

Letting then $\alpha=\min(\alpha',\alpha'')$, we obtain $\{\alpha f_\zeta\}\subset V$. The proof is complete. □

In the proof of the main result we employ the description of coefficient multipliers acting from the studied spaces into the Hardy spaces.

Theorem 2.5 ([5]). *Let $\Lambda=\{\lambda_k\}_{k=1}^{+\infty}\subset\mathbb{C}$. Then $\Lambda=CM(S_\alpha^q,X)$, where $X=H^p$, $(0<p\leqslant+\infty)$, if and only if*

$$|\lambda_k|=O\left(\exp\left(-c\cdot k^{\frac{\alpha+q+1}{\alpha+2q+1}}\right)\right),\quad k\to+\infty.$$

for some $c>0$.

Theorem 2.6 ([9]). *Let $\Lambda=\{\lambda_k\}_{k=1}^{+\infty}\subset\mathbb{C}$. Then $\Lambda=CM(\tilde{\Pi}_q,X)$, where $X=H^p$, $(0<p\leqslant+\infty)$, if and only if*

$$|\lambda_k|=O\left(\exp\left(-c\cdot k^{\frac{2}{q+2}}\right)\right),\quad k\to+\infty.$$

for some $c>0$.

3. Main results

We are in position to formulate main results of the work, namely, a discrete description of linear continuous functionals in the spaces S_α^q and in the Privalov classes by area.

Theorem 3.1. *Each continuous linear functional Φ on the area Privalov class $\tilde{\Pi}_q$, $q>0$, is determined by the formula*

$$\Phi(f)=\sum_{k=0}^{+\infty}a_kb_k, \tag{3.1}$$

where $\{a_k\}$ are the Taylor coefficients of the function $f\in\tilde{\Pi}_q$, while the numbers $\{b_k\}$ obeying the condition

$$|b_k|=O\left(\exp\left(-c\cdot k^{\frac{2}{2+q}}\right)\right),\quad k\to+\infty,\quad c>0. \tag{3.2}$$

are the Taylor coefficients of some analytic function in D and the series in the right hand side of (3.1) converges absolutely.

And vice versa, each sequence $\{b_k\}$ obeying condition (3.2) determines a linear continuous functional Φ on $\tilde{\Pi}_q$ by formula (3.1).

Theorem 3.2. *Each linear continuous functional Φ on the space S_α^q is determined by the formula*

$$\Phi(f)=\sum_{k=0}^{+\infty}a_kb_k, \tag{3.3}$$

where the numbers $\{b_k\}$ obey the condition

$$|b_k|=O\left(\exp\left(-c\cdot k^{\frac{\alpha+q+1}{\alpha+2q+1}}\right)\right),\quad c>0,\quad k\to+\infty, \tag{3.4}$$

and are the Taylor coefficient of some analytic function in D, while $\{a_k\}$ are the Taylor coefficients of the function $f\in S_\alpha^q$. The series in the right hand side in (3.3) converges absolutely.

And vice versa, each sequence $\{b_k\}$ obeying condition (3.4) determines a linear continuous functional Φ on S_α^q by formula (3.3).

Proof of Theorem 3.1. Let Φ be an arbitrary linear continuous functional on the space $\tilde{\Pi}_q$. With each function $f\in\tilde{\Pi}_q$ we associate a function $F_\zeta=\Phi(f_\zeta)$, $\zeta\in D$. The Taylor series of the function $f_\zeta(z)=f(\zeta z)=\sum\limits_{k=0}^{+\infty}a_kz^k\zeta^k$ converges absolutely and uniformly on the closed unit circle $\bar{D}$ and therefore it converges in the metrics of the space $\tilde{\Pi}_q$ and by the continuity and linearity of the functional Φ we have:

$$F(\zeta)=\Phi(f_\zeta)=\lim_{N\to+\infty}\Phi\left(\sum_{k=0}^{N}a_kz^k\zeta^k\right)=\sum_{k=0}^{+\infty}a_kb_k\zeta^k,\quad\zeta\in D, \tag{3.5}$$

where $b_k = \Phi(z^k)$ and the series in the right hand side of (3.5) converges. Thus, $F \in H(D)$. By Lemma 2.5, the family of the functions $\{f_\zeta\}$ is bounded in $\tilde{\Pi}_q$, and this is why the function F is bounded in D by Lemma 2.4, that is, $F \in H^\infty$. Thus, by the definition the sequence $\{b_k\}$ is a coefficient multiplier from $\tilde{\Pi}_q$ into H^∞ and by Theorem 2.6 estimate (3.2) holds. Then, taking into consideration estimate (2.4), we see that the series $\sum\limits_{k=0}^{+\infty} a_k b_k \zeta^k$ converges absolutely and uniformly by the Weierstrass theorem.

Using the Abel theorem on power series, we conclude that

$$\sum_{k=0}^{+\infty} a_k b_k = \lim_{r\to 1-0} \sum_{k=0}^{+\infty} a_k b_k r^k.$$

On the other hand, since $\rho_{\tilde{\Pi}_q}(f, f_r) \to 0$, $r \to 1-0$, and by continuity of the functional Φ, we get

$$\lim_{r\to 1-0} \sum_{k=0}^{+\infty} a_k b_k r^k = \lim_{r\to 1-0} \Phi(f_r) = \Phi(f).$$

Thus, we have proved (3.1) and the necessity has been established.

We proceed to proving the opposite statement. Let a sequence of complex numbers $\{b_k\}$ obeys condition (3.2). Taking into consideration estimate (2.4), we obtain that the series $\sum\limits_{k=0}^{+\infty} a_k b_k$ converges absolutely for each function $f = \sum_{k=1}^{+\infty} a_k z^k \in \tilde{\Pi}_q$. This is why the functional Φ is well-defined by formula (3.1). It is linear by the linearity of each Taylor coefficient as a functional over the space of holomorphic in the circle D functions. To prove the continuity, we represent the functional Φ as

$$\Phi_N = \sum_{k=1}^{N} a_k b_k.$$

The linearity and continuity of this functional follows from the same properties of each Taylor coefficient as a functional over the space of holomorphic in the circle D functions with the topology of uniform convergence on the compact sets and by the fact that topology of the convergence in the metrics of the space $\tilde{\Pi}_q$ is not weaker. The limit $\lim\limits_{N\to+\infty} \Phi_N$ is well-defined and finite and this is why the sequence $\{\Phi_N\}$ is pointwise bounded. Hence, it is equicontinuous by the general principle of the uniform boundedness for the F-space and hence, their pointwise limit, the functional Φ, is also continuous. The sufficiency is proved and the proof of the theorem is complete. □

Theorem 3.2 can be proved in the same way.

It is clear that we can pass from the discrete form of writing the functional to the usual integral form using the general theory of Fourier series.

The results of this work were announced in [8], [7], [15].

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