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ON INTEGRABILITY OF $O(3)$–MODEL

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Abstract. A three-dimensional $O(3)$ model for a unit vector $\mathbf{n}(\mathbf{r})$ has numerous application in the field theory and in the physics of condensed matter. We prove that this model is integrable under some differential constraint, that is, under certain restrictions for the gradients of fields $\Theta(\mathbf{r})$, $\Phi(\mathbf{r})$ parametrizing the vector $\mathbf{n}(\mathbf{r})$). Under the presence of the differential constraint, the equations of the models are reduced to a one-dimensional sine-Gordon equation determining the dependence of the field $\Theta(\mathbf{r})$ on an auxiliary field $a(\mathbf{r})$ and to a system of two equations $(\nabla S)(\nabla S) = 0$, $\Delta S = 0$ for a complex-valued function $S(\mathbf{r}) = a(\mathbf{r})+\mathrm{i}\Phi(\mathbf{r})$. We show that the solution of this system provide all known before exact solutions of models, namely, two-dimensional magnetic instantons and three-dimensional structures of hedgehog type. We find an exact solution for the field $S(\mathbf{r})$ as an arbitrary implicity function of two variables, which immediately represents the solution for the fields $\Theta(\mathbf{r})$, $\Phi(\mathbf{r})$ in an implicit form. We show that the found in this way exact solution of the system for the field $S(\mathbf{r})$ leads one to exact solution of equations of $O(3)$–model in the form of an arbitrary implicit function of two variables.

Keywords: integrable system, $O(3)$–model, differential substitution, quasilinear equation, general solution

Mathematics Subject Classification: 35C05, 35J60, 35A08

1. Introduction

$O(3)$–model in a three-dimensional space with an energy density E

$$E = \frac{1}{2}\sum_{j=1}^{3} \nabla n_j \nabla n_j \tag{1.1}$$

for a unit vector $\mathbf{n}$

$$\mathbf{n}^2 = 1 \tag{1.2}$$

possesses an explicit $O(3)$–symmetry corresponding to the rotation of a sphere. It belongs to a wide class of models, for which the space of order parameter belongs to manifolds different from $\mathbb{R}^N$. This model has numerous applications in the field theory. In the physics of condensed matter it is known as a Heisenberg model for describing magnetic structures in the exchange approximation [1] or as a model for a director field for describing the elastic properties of liquid crystalls in one-constant approximation [2].

One-dimensional and two-dimensional exact solutions of $O(3)$–models were studied by many authors. It was shown that in the space $(1, 1)$ the model is integrable by the inverse scattering problem method [3]. A very interesting and popular class of solutions in $D = 2$, instantons, was obtained in work [4]. Finally, in the three-dimensional space exact solutions of hedgehog type

$$\mathbf{n} = \frac{\mathbf{r}}{r} \tag{1.3}$$

A.B. Borisov, On integrability of $O(3)$–model.

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were experimentally found and theoretically studied in nematic liquid crystal, helium [5] and ferromagnetics [6].

The paper is organized as follows. In the second section, to solve the equations in $O(3)$–model, we employ a differential substitution, which transforms these equations to a system of two equations for a complex function $S(\mathbf{r})$ and to the pendulum equation. We show that straightforward solving of this system gives all known before solutions: two-dimensional (instantons) and three-dimensional (hedgehog type structures) models. An exact solution of the system for the field $S(\mathbf{r})$ is provided in the third section. It is determined by an arbitrary function of two variables and gives a general solution to the $O(3)$–model.

2. Differential substitution

For further presentation it is convenient to parametrize the unit vector $\mathbf{n}$ by the fields Θ, Φ:

$$\mathbf{n} = (\cos\Phi\sin\Theta, \sin\Phi\sin\Theta, \cos\Theta).$$

Then in terms of the variables Θ, Φ the equations of model (1.1) cast into a system of nonlinear differential equations:

$$\Delta\Theta = \frac{1}{2}(\nabla\Phi)^2\sin 2\Theta, \qquad \nabla\left[(\nabla\Phi)^2\sin^2\Theta\right] = 0. \tag{2.1}$$

Equation (1.1) is invariant with respect to the group of spin and spatial rotations $SO(3)\times SO(3)$. Such symmetry allows us to find a wide class of exact solutions. Analytic resolving of equation (2.1) is possible only in certain classes of solutions. In order to select one of them, we need to generalize a procedure proposed in [7] and to assume that the field Θ is locally dependent on an auxiliary field $\Theta(a[x_1, x_2, x_3])$. Then by straightforward calculations we confirm that the equations

$$\Theta''(a) = \frac{\sin 2\Theta(a)}{2}, \tag{2.2}$$

$$\Delta a = \Delta\Phi = 0, \quad (\nabla a)^2 = (\nabla\Phi)^2, \quad \nabla a\nabla\Phi = 0 \tag{2.3}$$

imply equations (1.2). Hereinafter $\Delta = \sum_{i=1}^{3}\partial_{x_i}^2$ is the three-dimensional Laplace operator.

To analyze the textures of Θ, we employ a solution to equation (2.2) of a 2π–soliton

$$\Theta = 2\arctan\exp(-a) \tag{2.4}$$

or lattices of solitons

$$\sin\frac{\Theta}{2} = \mathrm{cn}\left(\frac{a}{k}, k\right), \qquad 0 < k < 1. \tag{2.5}$$

As we shall see later, differential substitution (2.3) leads us to an exact solution of non-integrable model (1.1).

We proceed to solving equations (2.3). We introduce a complex-valued field

$$S = a + \mathrm{i}\Phi \tag{2.6}$$

and write system (2.3) as a system of two equations for a complex field S:

$$(\nabla S)(\nabla S) = 0, \tag{2.7}$$

$$\Delta S = 0. \tag{2.8}$$

This system possesses a splendid property of invariance with respect to the changes of the field $S \to S' = F(S)$ with an arbitrary function $F(S)$, which we shall use in what follows.

First we are going to show that the simplest solutions of system (2.7), (2.8) include all aforementioned known two-dimensional and three-dimensional solutions of $O(3)$–model. A wide class of solutions found in [4] is described by a simple formula:

$$w = \cot\frac{\Theta}{2}\exp[\mathrm{i}\Phi] = U[z], \qquad z = x + \mathrm{i}y, \tag{2.9}$$

where U is an analytic function.

In the two-dimensional case, equations (2.7), (2.8) are identically satisfied as $S = S(z)$ and, according (2.4), the relation with the field is determined by a simple identity

$$w = \exp[S]. \tag{2.10}$$

An easiest way to find solutions is to integrate straightforwardly (2.7), (2.8) by separating the variables. Then in the spherical coordinate system (R, θ, φ) we find a family of solutions

$$S(\theta, \varphi) = F\left(s\left[\mathrm{i}\varphi + \ln\tan\frac{\theta}{2}\right]\right), \qquad s = \pm 1, \tag{2.11}$$

where F is an arbitrary function. Expression (2.11) for $F = 1$, $s = 1$ and (2.6) lead us to hedgehog structures (1.3), as well as to other types of hedgehogs under a certain choice of the function F.

3. Exact solutions to model

Now we discuss exact solutions to equations (2.7), (2.8). Equation (2.7) implies immediately the expression for the field $S_{,x_3} = \partial_{x_3} S$:

$$S_{,x_3} = \sqrt{-S_{,x_1}^2 - S_{,x_2}^2}. \tag{3.1}$$

Then after the substitution (3.1) equation (2.8) casts into a simple form

$$S_{,x_1}^2 S_{,x_2,x_2} - 2S_{,x_1,x_2} S_{,x_1} S_{,x_2} + S_{,x_2}^2 S_{,x_1,x_1} = 0. \tag{3.2}$$

This equation belongs to wide class of Monge-Ampère equations [8], [9]. In order to solve it, we employ the following procedure. We introduce new variables

$$\alpha(x_1, x_2, x_3) = \frac{S_{,x_1}}{\sqrt{S_{,x_1}^2 + S_{,x_2}^2}}, \qquad \beta(x_1, x_2, x_3) = \frac{S_{,x_2}}{\sqrt{S_{,x_1}^2 + S_{,x_2}^2}}. \tag{3.3}$$

It is easy to confirm that the equation

$$\alpha_{,x_1} + \beta_{,x_2} = 0 \tag{3.4}$$

is equivalent to (3.2). The fields α, β are related by the identity

$$\alpha^2(x_1, x_2, x_3) + \beta^2(x_1, x_2, x_3) = 1.$$

This is why after the parametrization of the fields

$$\alpha(x_1, x_2, x_3) = \sin[B(x_1, x_2, x_3)], \qquad \beta(x_1, x_2, x_3) = \cos[B(x_1, x_2, x_3)] \tag{3.5}$$

and substituting them into (3.1), (3.3) we obtain equations

$$S_{,x_1} = S_{,x_2}\tan[B], \qquad S_{,x_3} = -\mathrm{i}S_{,x_2}\sec[B]. \tag{3.6}$$

The compatibility condition for system (3.6) gives rise to a closed equation for the field B

$$B_{,x_3} + \mathrm{i}(\cos[B]B_{,x_2} + \sin[B]B_{,x_1}) = 0. \tag{3.7}$$

It follows from the theory of first order nonlinear partial differential equations for (3.7) that the field B is determined by an implicit equation

$$G[H_1, H_2, H_3] = 0 \tag{3.8}$$

with an arbitrary function G, where the quantities H_1, H_2, H_3 are the integrals of the characteristic system of equations for the coordinates $x_1(t)$, $x_2(t)$, $x_3(t)$:

$$\begin{aligned} &\frac{\mathrm{d}}{\mathrm{d}t}x_3(t) = 1, &&\frac{\mathrm{d}}{\mathrm{d}t}x_1(t) = \mathrm{i}\sin B(x_1, x_2, x_3), \\ &\frac{\mathrm{d}}{\mathrm{d}t}x_2(t) = \mathrm{i}\cos B(x_1, x_2, x_3), &&\frac{\mathrm{d}}{\mathrm{d}t}B(x_1, x_2, x_3) = 0. \end{aligned} \tag{3.9}$$

The integrals read as

$$
\begin{aligned}
H_1 &= -\mathrm{i}x_3 \sin B(x_1, x_2, x_3) + x_1,\\
H_2 &= -\mathrm{i}x_3 \cos B(x_1, x_2, x_3) + x_2,\\
H_3 &= B(x_1, x_2, x_3).
\end{aligned}
\tag{3.10}
$$

Apart of (3.7), the fields $B(x_1, x_2, x_3)$ should also satisfy equations (3.4), (3.5):

$$B_{,x_1} - \tan[B]B_{,x_2} = 0. \tag{3.11}$$

Substituting the derivatives of the field B

$$
\begin{aligned}
&B_{,x_1} = -\mathrm{i}\frac{G_{,H_1}}{U}, \qquad B_{,x_2} = -\mathrm{i}\frac{G_{,H_2}}{U},\\
&B_{,x_3} = -\frac{G_{,H_1} \sin H_3 + G_{,H_2} \cos H_3}{U},\\
&U = x_3(G_{,H_1} \cos H_3 - G_{,H_2} \sin H_3) + \mathrm{i}G_{,H_3},
\end{aligned}
\tag{3.12}
$$

found (3.8), (3.10) into this equation, we obtain an additional restriction for equation (3.8):

$$G_{,H_1} - G_{,H_2} \tan H_3 = 0. \tag{3.13}$$

This implies that the solutions for the field B are determined by an implicit equation

$$G[H_1 \sin H_3 + H_2 \cos H_3, H_3] = 0. \tag{3.14}$$

Non-uniqueness and singularity of solutions to (3.14) in a general cases are most interesting in studying singular defects in condensed matter.

We mention an important fact. By (3.7), (3.9) and (3.1), (3.6) we immediately obtain the relations

$$B_{,x_1}S_{,x_2} - B_{,x_2}S_{,x_1} = 0, \qquad B_{,x_3}S_{,x_2} - B_{,x_2}S_{,x_3} = 0. \tag{3.15}$$

This is why a general solution to equations (2.7), (2.8) is determined by formula (3.14) and an arbitrary function F:

$$S(x_1, x_2, x_3) = F(B(x_1, x_2, x_3)). \tag{3.16}$$

Indeed, substituting (3.16) into equations (2.7), (2.8), we obtain a system of equations

$$(\nabla B)(\nabla B) = 0, \qquad \Delta B = 0. \tag{3.17}$$

By straightforward calculating of the derivatives of implicit functions (3.14) we easily confirm that this system holds true.

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