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A ONE-RADIUS THEOREM ON A SPHERE WITH PRICKED POINT

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Abstract. We considers local properties of mean periodicity on the two-dimensional sphere $\mathbb{S}^2$. According to the classical properties of periodic functions, each function continuous on the unit circle $\mathbb{S}^1$ and possessing zero integrals over any interval of a fixed length $2r$ on $\mathbb{S}^1$ is identically zero if and only if the number r/π is irrational. In addition, there is no non-zero continuous function on $\mathbb{R}$ possessing zero integrals over all segments of fixed length and their boundaries. The aim of this paper is to study similar phenomena on a sphere in $\mathbb{R}^3$ with a pricked point. We study smooth functions on $\mathbb{S}^2 \setminus (0, 0, -1)$ with zero integrals over all admissible spherical caps and circles of a fixed radius. For such functions, we establish a one-radius theorem, which implies the injectivity of the corresponding integral transform. We also improve the well-known Ungar theorem on spherical means, which gives necessary and sufficient conditions for the spherical cap belong to the class of Pompeiu sets on $\mathbb{S}^2$. The proof of the main results is based on the description of solutions $f \in C^\infty(\mathbb{S}^2 \setminus (0, 0, -1))$ of the convolution equation $(f * \sigma_r)(\xi) = 0$, $\xi \in B_{\pi-r}$, where $B_{\pi-r}$ is the open geodesic ball of radius $\pi - r$ centered at the point $(0, 0, 1)$ on $\mathbb{S}^2$ and σ_r is the delta-function supported on ∂B_r. The key tool used for describing f is the Fourier series in spherical harmonics on $\mathbb{S}^1$. We show that the Fourier coefficients $f_k(\theta)$ of the function f are representable by series in Legendre functions related with the zeroes of the function $P_\nu(\cos r)$. Our main results are consequence of the above representation of the function f and the corresponding properties of the Legendre functions. The results obtained in the work can be used in solving problems associated with ball and spherical means.

Keywords: spherical means, Pompeiu transform, Legendre functions, convolution

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1. Introduction

Let r be a fixed positive number. An obvious property of non-zero $2r$-periodic functions on the real axis is the absence of the anti-period equal to $2r$. In other words, if a function f defined on $\mathbb{R}$ satisfies the relations

$$f(x + r) - f(x - r) = 0, \qquad f(x + r) + f(x - r) = 0, \quad x \in \mathbb{R},$$

then $f \equiv 0$. In terms of integral means, this implies that each continuous function on $\mathbb{R}$ having zero integrals over all segments $K_r = [x - r, x + r]$ and over its boundaries $\partial K_r = \{x \pm r\}$ vanishes identically; as usually, the integral over ∂K_r is introduced as the sum of the values of the functions at points in the set ∂K_r.

This fact admits non-trivial generalizations for various multi-dimensional spaces, see [1]–[5]. In particular, if the function $f \in C(\mathbb{R}^n)$, $n \geqslant 2$, has zero integrals over all balls and spheres of a fixed radius, then $f \equiv 0$, see [2]. The statements of such kind are called one radius theorems.

In the present work we study functions on a pricked two-dimensional sphere having zero integrals over all admissible spherical caps and circumferences of a fixed radius. For such

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functions we establish a new one radius theorem specifying one of the results in work [6]. We also observe that an intermediate result of the work is an improving of the known theorem by P. Ungar on spherical means [7], see Theorem 4.1 in Section 4.

2. Main result

Let $\mathbb{S}^2 = \{\xi \in \mathbb{R}^3 : |\xi| = 1\}$, ξ_1, ξ_2, ξ_3 be Cartesian coordinates of a point $\xi \in \mathbb{S}^2$,

$$\mathbb{S}' = \{\xi \in \mathbb{S}^2 : \xi_3 \neq -1\}.$$

The distance $d(\xi, \eta)$ between points $\xi, \eta \in \mathbb{S}^2$ is calculated by the formula

$$d(\xi, \eta) = \arccos(\xi_1\eta_1 + \xi_2\eta_2 + \xi_3\eta_3).$$

In particular,

$$d(\xi, 0) = \arccos \xi_3, \quad \text{where} \quad 0 = (0, 0, 1).$$

The set

$$B_r(\eta) = \{\xi \in \mathbb{S}^2 : d(\xi, \eta) < r\}, \quad 0 < r < \pi,$$

is called an open geodesic ball (spherical cap) on $\mathbb{S}^2$ of the radius r centered at the point η. Its boundary

$$\partial B_r(\eta) = \{\xi \in \mathbb{S}^2 : d(\xi, \eta) = r\}$$

is a geodesic circumference of the radius r on $\mathbb{S}^2$ centered at the point η. In the same way, the set

$$\overline{B_r(\eta)} = B_r(\eta) \cup \partial B_r(\eta)$$

is called a closed geodesic ball on $\mathbb{S}^2$ of the radius r centered at the point η.

We denote by $d\xi$ and $dl(\xi)$ the differential of the area and the length on $\mathbb{S}^2$, respectively.

The main result of the present work is the following spherical analogue of a one radius theorem.

Theorem 2.1. *Let r be a fixed number in the interval $(0; \pi)$, $f \in C^\infty(\mathbb{S}')$ and the following conditions hold:*

1) *the function f has zero integrals with respect to the measure $d\xi$ over each closed geodesic ball of radius r on $\mathbb{S}^2$ lying in $\mathbb{S}'$;*
2) *the function f has non-zero integrals with respect to the measure $dl(\xi)$ over each geodesic circumference of the radius r on $\mathbb{S}^2$ located in $\mathbb{S}'$.*

Then $f \equiv 0$.

It is interesting to compare this result with a one radius theorem obtained in work [6]. Theorem 2 in [6] shows that if $0 < r \leqslant \pi/2$, $f \in C(\mathbb{S}')$ and

$$\int_B f(\xi) d\xi = \int_{\partial B} f(\xi) dl(\xi) = 0 \tag{2.1}$$

for each closed geodesic ball B of radius r located in $\mathbb{S}'$, then $f \equiv 0$. Moreover, as $\pi/2 < r < \pi$, there exist non-zero functions f satisfying conditions (2.1). If f is smooth on $\mathbb{S}'$ and satisfies conditions 1), 2) in Theorem 2.1 for some $r \in (0; \pi)$, then $f \equiv 0$.

For other one radius theorem, we refer to [2]–[5] and the references therein.

3. Main notations

Let $\mathbb{N}$, $\mathbb{Z}$, $\mathbb{Z}_+$, $\mathbb{C}$ be the sets of natural, integer, non-negative integer and complex numbers, respectively. We denote by P_ν^μ $(\mu,\nu\in\mathbb{C})$ the Legendre functions of first kind on $(-1,1)$, that is,

$$P_\nu^\mu(x)=\frac{1}{\Gamma(1-\mu)}\left(\frac{1+x}{1-x}\right)^{\frac{\mu}{2}}F\left(-\nu,\nu+1;1-\mu;\frac{1-x}{2}\right),\qquad \mu\notin\mathbb{N},$$

$$P_\nu^\mu(x)=(-1)^\mu(1-x^2)^{\frac{\mu}{2}}\left(\frac{d}{dx}\right)^\mu P_\nu(x),\qquad \mu\in\mathbb{N},$$

where F is the Gauss hypergeometric function, Γ is the Gamma function and $P_\nu=P_\nu^0$, see [8, Ch. 3, Sect. 3.4, Eq. (6); Sect. 3.6.1, Eq. (6)]. They satisfy Meler-Dirichlet integral representation

$$P_\nu^\mu(\cos\theta)=\sqrt{\frac{2}{\pi}}\frac{(\sin\theta)^\mu}{\Gamma\left(\frac{1}{2}-\mu\right)}\int_0^\theta(\cos t-\cos\theta)^{-\mu-\frac{1}{2}}\cos\left(\left(\nu+\frac{1}{2}\right)t\right)dt \tag{3.1}$$

as $\theta\in(0;\pi)$, $\mathrm{Re}\mu<\frac{1}{2}$. The Legendre functions of second kind on $(-1,1)$ are defined by the identity

$$\frac{(1-x^2)^{\mu/2}Q_\nu^\mu(x)}{2^\mu\pi^{3/2}}=\cot\left(\frac{\pi}{2}(\nu+\mu)\right)\frac{xF\left(\frac{1-\nu-\mu}{2},\frac{\nu-\mu}{2}+1;\frac{3}{2};x^2\right)}{\Gamma\left(\frac{1+\nu-\mu}{2}\right)\Gamma\left(-\frac{\nu+\mu}{2}\right)}-$$

$$-\frac{1}{2}\tan\left(\frac{\pi}{2}(\nu+\mu)\right)\frac{F\left(-\frac{\nu+\mu}{2},\frac{1+\nu-\mu}{2};\frac{1}{2};x^2\right)}{\Gamma\left(\frac{1-\nu-\mu}{2}\right)\Gamma\left(1+\frac{\nu-\mu}{2}\right)},\qquad -\nu-\mu\notin\mathbb{N},$$

$$Q_\nu=Q_\nu^0,\qquad -\nu\notin\mathbb{N}.$$

They are related with P_ν^μ as follows:

$$P_\nu^\mu(-x)=P_\nu^\mu(x)\cos\big(\pi(\nu+\mu)\big)-\frac{2}{\pi}Q_\nu^\mu(x)\sin\big(\pi(\nu+\mu)\big), \tag{3.2}$$

see [8, Ch. 3, Sect. 3.4, Eqs. (14), (15), (20), (21)]. Moreover,

$$(1-x^2)\left(P_\nu^\mu(x)\frac{d}{dx}Q_\nu^\mu(x)-Q_\nu^\mu(x)\frac{d}{dx}P_\nu^\mu(x)\right)=2^{2\mu}\frac{\Gamma\left(1+\frac{\nu+\mu}{2}\right)\Gamma\left(\frac{1+\nu+\mu}{2}\right)}{\Gamma\left(\frac{1+\nu-\mu}{2}\right)\Gamma\left(1+\frac{\nu-\mu}{2}\right)}, \tag{3.3}$$

see [8, Ch. 3, Sect. 3.4, Formula (25)].

Hereafter r is a fixed number in the interval $(0;\pi)$. It follows from (3.1) that the function

$$h(\nu)=P_\nu(\cos r)=P_\nu^0(\cos r)$$

is an entire function in the variable ν of exponential type r. It possesses infinitely many zeroes, all zeroes are real, simple and are located symmetrically with respect to the point $-\frac{1}{2}$ and lie outside the segment $[-1;0]$, see [3, Part 2, Ch. 3]. We denote the set of the zeroes of this function in the interval $(0;+\infty)$ by the symbol $N(r)$, that is,

$$N(r)=\{\nu>0:P_\nu(\cos r)=0\}.$$

We also let

$$\mathcal{Z}(r)=\{l\in\mathbb{N}:P_l(\cos r)=0\}.$$

We note that

$$\mathcal{Z}(\pi/2)=N(\pi/2)=\{2k+1,k\in\mathbb{Z}_+\}.$$

Moreover, the set $\{r\in(0,\pi):\mathcal{Z}(r)\neq\varnothing\}$ is countable and everywhere dense in the interval $(0,\pi)$, see [9].

We introduce spherical coordinates φ, θ on $\mathbb{S}^2$ as follows:

$$\xi_1=\sin\theta\sin\varphi,\qquad \xi_2=\sin\theta\cos\varphi,\qquad \xi_3=\cos\theta,\qquad \varphi\in(0,2\pi),\qquad \theta\in(0,\pi);$$

as above $\xi_1,\ \xi_2,\ \xi_3$ are the Cartesian coordinates of a point $\xi \in \mathbb{S}^2$. We let

$$p_{\nu,k}(\theta) = P_\nu^{-k}(\cos\theta), \tag{3.4}$$

$$S_{\nu,k}(\xi) = p_{\nu,|k|}(\theta)e^{ik\varphi}, \qquad \nu \in \mathbb{C}, \quad k \in \mathbb{Z}.$$

The function $S_{\nu,k}$ is real analytic on $\mathbb{S}'$. At that,

$$L(S_{\nu,k}) = -\nu(\nu+1)S_{\nu,k}, \tag{3.5}$$

where L is the Laplace operator on $\mathbb{S}^2$, that is,

$$L = \frac{\partial^2}{\partial\theta^2} + \cot\theta\frac{\partial}{\partial\theta} + \frac{1}{\sin^2\theta}\frac{\partial^2}{\partial\varphi^2},$$

see the proof of Lemma 4.1 below.

With each function $f \in C(\mathbb{S}')$, we associate the Fourier series

$$f \sim \sum_{k\in\mathbb{Z}} f^k, \tag{3.6}$$

whose terms are defined by the identities

$$f^k(\xi) = f_k(\theta)e^{ik\varphi}, \quad f_k(\theta) = \frac{1}{2\pi}\int_0^{2\pi} f(\sin\theta\sin\alpha, \sin\theta\cos\alpha, \cos\theta)e^{-ik\alpha}d\alpha.$$

If $f \in C^\infty(\mathbb{S}')$, then series (3.6) converges to f in the standard topology of the space $C^\infty(\mathbb{S}')$, see [4, Ch. 11, Sect. 11.1]. Relation (3.6) implies the formula

$$f^k(\xi) = \frac{1}{2\pi}\int_0^{2\pi} f(\tau_\alpha\xi)e^{ik\alpha}d\alpha, \tag{3.7}$$

where τ_α is the rotation of $\mathbb{R}^3$ in the plane (x_1, x_2) by the angle α, that is,

$$\tau_\alpha\xi = (\xi_1\cos\alpha - \xi_2\sin\alpha, \xi_1\sin\alpha + \xi_2\cos\alpha, \xi_3).$$

Let $O(3)$ be an orthogonal group in $\mathbb{R}^3$,

$$B_r = B_r(0) = \{\xi \in \mathbb{S}^2 : \xi_3 > \cos r\} = \{(\varphi,\theta) : 0 \leqslant \theta < r\},$$
$$S_r = S_r(0) = \{\xi \in \mathbb{S}^2 : \xi_3 = \cos r\} = \{(\varphi,\theta) : \theta = r\}.$$

We let

$$U_r(\mathbb{S}') = \Big\{f \in C(\mathbb{S}') : \int_{S_r} f(\tau\xi)dl(\xi) = 0 \quad \forall\tau \in O(3) : \tau\overline{B}_r \subset \mathbb{S}'\Big\}.$$

The class $U_r(\mathbb{S}')$ can be regarded as a set of the functions $f \in C(\mathbb{S}')$ satisfying the convolution equation $f * \sigma_r = 0$ in the ball $B_{\pi-r}$, where σ_r is the delta-function supported on S_r.

4. Auxiliary statements

We denote by D_k the differential operator defined on the space $C^1(0,\pi)$ as follows:

$$(D_k u)(\theta) = (\sin\theta)^k\frac{d}{d\theta}\left(\frac{u(\theta)}{(\sin\theta)^k}\right), \qquad u \in C^1(0,\pi).$$

Let Id be the identity mapping.

Lemma 4.1. *The identities hold:*

$$D_k\, p_{\nu,k} = (k-\nu)(k+\nu+1)p_{\nu,k+1}, \qquad D_{-k}\, p_{\nu,k} = p_{\nu,k-1}, \tag{4.1}$$

$$(L + \nu(\nu+1)Id)(p_{\nu,k}(\theta)e^{ik\varphi}) = 0. \tag{4.2}$$

Proof. Employing the formula

$$(1-x^2)\frac{dP_\nu^\mu(x)}{dx} = -\nu x P_\nu^\mu(x) + (\nu+\mu)P_{\nu-1}^\mu(x),$$

see [8, Ch. 3, Sect. 3.8, Eq. (19)], we find

$$p'_{\nu,k}(\theta) = \nu\, ctg\,\theta\, p_{\nu,k}(\theta) + \frac{(k-\nu)}{\sin\theta} p_{\nu-1,k}(\theta).$$

This implies

$$D_k\, p_{\nu,k}(\theta) = \frac{(k-\nu)}{\sin\theta}(p_{\nu-1,k}(\theta) - \cos\theta\, p_{\nu,k}(\theta)), \tag{4.3}$$

$$D_{-k}\, p_{\nu,k}(\theta) = \frac{1}{\sin\theta}((\nu+k)\cos\theta\, p_{\nu,k}(\theta) - (\nu-k)p_{\nu-1,k}(\theta)). \tag{4.4}$$

Since

$$P_{\nu-1}^\mu(x) - xP_\nu^\mu(x) = (\nu-\mu+1)\sqrt{1-x^2}P_\nu^{\mu-1}(x),$$
$$(\nu-\mu)xP_\nu^\mu(x) - (\nu+\mu)P_{\nu-1}^\mu(x) = \sqrt{1-x^2}P_\nu^{\mu+1}(x),$$

see [8, Ch. 3, Sect. 3.8, Eqs. (15), (17)], by (4.3) and (4.4) we arrive at (4.1).

On a function u of the form $u(\xi) = v(\theta)e^{ik\varphi}$, the operator L acts by the rule

$$(Lu)(\xi) = (\ell_k v)(\theta)e^{ik\varphi}, \tag{4.5}$$

where

$$\ell_k = \frac{d^2}{d\theta^2} + ctg\,\theta\frac{d}{d\theta} - \frac{k^2}{\sin^2\theta}Id.$$

The operator ℓ_k can be represented as

$$\ell_k = D_{-k-1}D_k - k(k+1)Id = D_{k-1}D_{-k} - k(k-1)Id. \tag{4.6}$$

Now relation (4.2) follows (4.6) and (4.1). □

Lemma 4.2. (i) *Let $\varepsilon, \theta \in (0,\pi)$, $k \in \mathbb{Z}_+$. Then as $\nu \to \infty$ and $|\arg\nu| < \pi - \varepsilon$, the asymptotic identity*

$$p_{\nu,k}(\theta) = \sqrt{\frac{2}{\pi\sin\theta}}\frac{\cos((\nu+\frac{1}{2})\theta - \frac{\pi}{4}(2k+1))}{(\nu+\frac{1}{2})^{k+\frac{1}{2}}} + O\left(\frac{e^{\theta|\mathrm{Im}\,\nu|}}{|\nu|^{k+\frac{3}{2}}}\right) \tag{4.7}$$

holds uniformly in θ over each segment $[\alpha,\beta] \subset (0,\pi)$.

(ii) *If $\nu \in \mathbb{C}$, $\theta \in (0,\pi)$, $k \in \mathbb{Z}_+$, then*

$$|p_{\nu,k}(\theta)| \leqslant \frac{1}{k!}\left(\sin\frac{\theta}{2}\right)^k \left(\cos\frac{\theta}{2}\right)^{-k-1} e^{\theta|\mathrm{Im}\,\nu|}. \tag{4.8}$$

(iii) *Let $0 < a < \pi$, $s, k \in \mathbb{Z}_+$. Then*

$$\max_{\theta\in[0,a]}\left|\frac{d^s p_{\nu,k}(\theta)}{d\theta^s}\right| = O(\nu^{s-k}), \qquad \nu \to +\infty. \tag{4.9}$$

Proof. Taking into consideration (3.4), by formula (3.1) we have

$$p_{\nu,k}(\theta) = \frac{(\sin\theta)^{-k}}{\sqrt{2\pi}\Gamma\left(k+\frac{1}{2}\right)}\int_{-\theta}^{\theta}(\cos t - \cos\theta)^{k-\frac{1}{2}}e^{i\left(\nu+\frac{1}{2}\right)t}dt. \tag{4.10}$$

By (4.10) and asymptotic expansion of Fourier integrals, see [10, Ch. 2, Proof of Theorem 10.2], we obtain (4.7).

To prove (4.8), we again employ (4.10). Then

$$|p_{\nu,k}(\theta)| \leqslant \frac{(\sin\theta)^{-k}}{\sqrt{2\pi}\Gamma\left(k+\frac{1}{2}\right)} \int_{-\theta}^{\theta} (\cos t - \cos\theta)^{k-\frac{1}{2}} dt\, e^{\theta|\mathrm{Im}\,\nu|}.$$

The integral in the right hand side is estimated as follows:

$$\begin{aligned}\int_0^{\theta} (\cos t - \cos\theta)^{k-\frac{1}{2}} dt &= \int_{\cos\theta}^{1} (x-\cos\theta)^{k-\frac{1}{2}} \frac{dx}{\sqrt{1-x^2}} \\ &\leqslant \frac{1}{\sqrt{1+\cos\theta}} \int_{\cos\theta}^{1} (x-\cos\theta)^{k-\frac{1}{2}} (1-x)^{-\frac{1}{2}} dx \\ &= \frac{\sqrt{\pi}2^{k-\frac{1}{2}}\Gamma\left(k+\frac{1}{2}\right)}{k!} \left(\sin\frac{\theta}{2}\right)^{2k} \left(\cos\frac{\theta}{2}\right)^{-1},\end{aligned}$$

and this proves estimate (4.8).

Finally, let us prove (4.9). As $a < \pi/2$, estimate (4.9) is implied by the integral representation

$$p_{\nu,-k}(\theta)e^{ik\varphi} = i^k \frac{\Gamma(\nu+k+1)}{2\pi\Gamma(\nu+1)} \int_{-\pi}^{\pi} (\cos\theta + i\sin\theta\cos(\psi-\varphi))^{\nu} e^{ik\psi} d\psi, \quad \theta \in (0,\pi/2)$$

and the identity

$$p_{\nu,-k}(\theta) = (-1)^k \frac{\Gamma(\nu+k+1)}{\Gamma(\nu-k+1)} p_{\nu,k}(\theta),$$

see [8, Ch. 3, Sect. 3.7, Eqs. (25), (26); Sect. 3.3.1, Eq. (7); Sect. 3.4, Eq. (5)]. On the other hand, asymptotic expansion (4.7) and the second relation in (4.1) show that

$$\max_{0<\alpha\leqslant\theta\leqslant\beta<\pi} \left|\frac{d^s p_{\nu,k}(\theta)}{d\theta^s}\right| = O(\nu^{s-k-1/2}), \qquad \nu \to +\infty.$$

Employing these two cases, we obtain statement (iii). □

Lemma 4.3. (i) *The identity holds*

$$\mathcal{Z}(r) = \mathcal{Z}(\pi - r).$$

(ii) *If* $p_{\nu,0}(r) = 0$, *then* $p_{\nu,1}(r) \neq 0$.
(iii) *If* $p_{\nu,0}(r) = 0$, *then* $Q_\nu(\cos r) \neq 0$.

Proof. Statement (i) is implied by the definition of the set $\mathcal{Z}(r)$ and the relation

$$P_n(-x) = (-1)^n P_n(x), \quad n \in \mathbb{Z}_+,$$

see [8, Ch. 3, Sect. 3.4, Formula (19)].

We assume that $p_{\nu,0}(r) = p_{\nu,1}(r) = 0$ for some $\nu \in \mathbb{C}$. Then

$$p_{\nu,0}(r) = p'_{\nu,0}(r) = 0$$

and

$$\frac{d^2}{d\theta^2} p_{\nu,0}(\theta) + ctg\,\theta \frac{d}{d\theta} p_{\nu,0}(\theta) + \nu(\nu+1) p_{\nu,0}(\theta) = 0,$$

see (4.1), (4.2) and (4.5). Then by the uniqueness of the solution to the Cauchy problem for a second order ordinary differential equation we obtain $p_{\nu,0} \equiv 0$ and this contradicts the definition of P_ν.

Finally, the formula

$$(1-x^2)\left(P_\nu(x)\frac{d}{dx}Q_\nu(x) - Q_\nu(x)\frac{d}{dx}P_\nu(x)\right) = 1,$$

see (3.3), shows that the identities $P_\nu(\cos r) = 0$ and $Q_\nu(\cos r) = 0$ can not hold simultaneously. This completes the proof. □

Lemma 4.4. *Let*

$$\delta(\mu,\nu)=\int_0^r p_{\nu,0}(\theta)p_{\mu,0}(\theta)\sin\theta d\theta, \qquad \mu,\nu\in N(r).$$

Then $\delta(\mu,\nu)=0$ *as* $\mu\neq\nu$ *and*

$$\delta(\nu,\nu)>\frac{c}{\nu^2}, \tag{4.11}$$

where constant $c>0$ *is independent of* ν.

Proof. As $\mu\neq\nu$, the statement is implied by the identity

$$(\mu-\nu)(\mu+\nu+1)\int_0^r p_{\nu,0}(\theta)p_{\mu,0}(\theta)\sin\theta d\theta=\sin r(p_{\mu,0}(r)p'_{\nu,0}(r)-p_{\nu,0}(r)p'_{\mu,0}(r)),$$

see [8, Ch. 3, Sect. 3.12, Formula (3)]. It is sufficient to prove inequality (4.11) for sufficiently large $\nu\in N(r)$. Suppose that $\nu>\frac{\pi}{4r}-\frac{1}{2}$. We let

$$g(\theta,t)=(\cos t-\cos\theta)^{-\frac{1}{2}}, \qquad 0\leqslant t\leqslant\theta\leqslant\pi. \tag{4.12}$$

Then by (3.1) we get

$$\begin{aligned}\delta(\nu,\nu)&=\int_0^r (p_{\nu,0}(\theta))^2\sin\theta d\theta=\frac{2}{\pi^2}\int_0^r\sin\theta\left(\int_0^\theta g(\theta,t)\cos\left(\nu+\frac{1}{2}\right)tdt\right)^2 d\theta\\ &\geqslant\frac{2}{\pi^2}\int_0^{\frac{\pi}{4(\nu+1/2)}}\sin\theta\left(\int_0^\theta g(\theta,t)\cos\left(\nu+\frac{1}{2}\right)tdt\right)^2 d\theta\\ &\geqslant\frac{1}{\pi^2}\int_0^{\frac{\pi}{4(\nu+1/2)}}\sin\theta\left(\int_{\frac{\theta}{2}}^\theta g(\theta,t)dt\right)^2 d\theta.\end{aligned}\tag{4.13}$$

An internal integral in (4.13) is estimated as follows:

$$\begin{aligned}\int_{\frac{\theta}{2}}^\theta g(\theta,t)dt&=\int_{\cos\theta}^{\cos\frac{\theta}{2}}(x-\cos\theta)^{-\frac{1}{2}}\frac{dx}{\sqrt{1-x^2}}\\ &\geqslant\frac{1}{\sin\theta}\int_{\cos\theta}^{\cos\frac{\theta}{2}}(x-\cos\theta)^{-\frac{1}{2}}dx=2\frac{(\cos\frac{\theta}{2}-\cos\theta)^{\frac{1}{2}}}{\sin\theta}.\end{aligned}\tag{4.14}$$

Taking into consideration that

$$\frac{\cos\frac{\theta}{2}-\cos\theta}{\sin\theta}=\frac{\sin\frac{3\theta}{4}}{2\cos\frac{\theta}{2}\cos\frac{\theta}{4}}\geqslant\frac{1}{2}\sin\frac{3\theta}{4}\geqslant\frac{3\theta}{4\pi}$$

as $0<\theta<\frac{\pi}{4(\nu+1/2)}$, by (4.13) and (4.14) we obtain

$$\delta(\nu,\nu)\geqslant\frac{4}{\pi^2}\int_0^{\frac{\pi}{4(\nu+1/2)}}\frac{3\theta}{4\pi}d\theta$$

and this implies (4.11). □

Lemma 4.5. *Let* $r\in(0,\pi)$, $\nu\in\mathbb{C}$, $k\in\mathbb{Z}$. *Then for each* $\tau\in O(3)$ *such that* $\tau\overline{B}_r\subset B_\pi$ *the identities hold:*

$$\int_{S_r} S_{\nu,k}(\tau\xi)dl(\xi)=2\pi\sin r\,p_{\nu,0}(r)S_{\nu,k}(\tau 0), \tag{4.15}$$

$$\int_{B_r} S_{\nu,k}(\tau\xi)d\xi=2\pi\sin r\,p_{\nu,1}(r)S_{\nu,k}(\tau 0). \tag{4.16}$$

Proof. By Pizzetti formula, see [11, Formula (20)] and (3.5), we have

$$\begin{aligned}\int_{S_r} S_{\nu,k}(\tau\xi)dl(\xi) =&2\pi\sin r\left(S_{\nu,k}(\tau 0)+\sum_{m=1}^{\infty}\frac{L(L+2)\dots(L+(m-1)m)S_{\nu,k}(\tau 0)}{(m!)^2}\left(\sin\frac{r}{2}\right)^{2m}\right)\\ =&2\pi\sin r\, S_{\nu,k}(\tau 0)\\ &\cdot\left(1+\sum_{m=1}^{\infty}\frac{(-\nu(\nu+1))(2-\nu(\nu+1))...(m(m-1)-\nu(\nu+1))}{(m!)^2}\left(\sin\frac{r}{2}\right)^{2m}\right)\\ =&2\pi\sin r\, S_{\nu,k}(\tau 0)\sum_{m=0}^{\infty}\frac{\Gamma(m-\nu)\Gamma(m+\nu+1)}{\Gamma(-\nu)\Gamma(\nu+1)(m!)^2}\left(\sin\frac{r}{2}\right)^{2m}\\ =&2\pi\sin r\, S_{\nu,k}(\tau 0)F\left(-\nu,\nu+1;1;\left(\sin\frac{r}{2}\right)^{2}\right).\end{aligned}$$

Then identity (4.15) is implied by (3.4) and the definition of the Legendre function. Employing (4.15) and (4.1), we obtain

$$\begin{aligned}\int_{B_r} S_{\nu,k}(\tau\xi)d\xi =& \int_0^r\int_{S_\rho} S_{\nu,k}(\tau\xi)dl(\xi)d\rho = 2\pi\, S_{\nu,k}(\tau 0)\int_0^r \sin\rho\; p_{\nu,0}(\rho)d\rho\\ =&2\pi S_{\nu,k}(\tau 0)\int_0^r \sin\rho\;(D_{-1}p_{\nu,1})(\rho)d\rho = 2\pi S_{\nu,k}(\tau 0)\int_0^r\frac{d}{d\rho}\left(p_{\nu,1}(\rho)\sin\rho\right)d\rho\\ =&2\pi\sin r\, p_{\nu,1}(r)S_{\nu,k}(\tau 0).\end{aligned}$$

This completes the proof. □

Lemma 4.6. *Let $f\in C^\infty(\mathbb{S}')$. Then $f\in U_r(\mathbb{S}')$ if and only if for each $k\in\mathbb{Z}$ the expansion holds:*

$$f^k(\xi)=\sum_{\nu\in N(r)}\alpha_{\nu,k}S_{\nu,k}(\xi),\quad \xi\in\mathbb{S}',$$

where $\alpha_{\nu,k}\in\mathbb{C}$ and

$$\alpha_{\nu,k}=O(\nu^{-a})\quad as\quad \nu\to+\infty\quad for\ each\quad a>0. \tag{4.17}$$

Lemma 4.6 is a particular case of the result established earlier by Vit.V. Volchkov [4, Thm. 16.6(ii)].

According Ungar theorem on spherical means [7], if a function $f\in C(\mathbb{S}^2)$ has zero integrals over all geodesic circumferences of the radius r and $P_l(\cos r)\neq 0$ for each $l\in\mathbb{N}$, then $f\equiv 0$.

The next result specifies this fact.

Theorem 4.1. *Let $f\in C^\infty(\mathbb{S}')$. Then the function f has zero integrals over all geodesic circumferences of the radius r on $\mathbb{S}^2$ lying in $\mathbb{S}'$ if and only if for each $k\in\mathbb{Z}$ the expansion holds true:*

$$f^k(\xi)=\sum_{\nu\in\mathcal{Z}(r)}\alpha_{\nu,k}S_{\nu,k}(\xi),\quad \xi\in\mathbb{S}', \tag{4.18}$$

where the coefficients $\alpha_{\nu,k}$ satisfy condition (4.17).

Proof. First we assume that the integrals of f over all geodesic circumferences of the radius r on $\mathbb{S}^2$ located in $\mathbb{S}'$ vanish. By Lemma 4.6 we have

$$f^k(\xi)=\sum_{\nu\in N(r)}\alpha_{\nu,k}S_{\nu,k}(\xi),\quad \xi\in\mathbb{S}', \tag{4.19}$$

where the coefficients $\alpha_{\nu,k}$ satisfy condition (4.17). By formula (3.7), the integrals of f^k over all geodesic circumferences of the radius r on $\mathbb{S}^2$ lying in $\mathbb{S}'$ are also zero. In particular, since

$S_{\pi-r} = S_r\big((0,0,-1)\big)$, we have

$$\int\limits_{S_{\pi-r}} f^k(a_t\xi)dl(\xi) = 0 \quad \text{as} \quad |t| < r,$$

where

$$a_t\xi = (\xi_1, \xi_2\cos t + \xi_3\sin t, -\xi_2\sin t + \xi_3\cos t).$$

Writing this relation for the right hand side in (4.19) and employing Lemmata 4.2, 4.5, we find

$$\sum_{\nu\in N(r)} \alpha_{\nu,k} P_\nu(-\cos r) p_{\nu,|k|}(t) = 0, \qquad |t| < r. \tag{4.20}$$

We apply the differential operator $D_{-1}\dots D_{-|k|+1}D_{-|k|}$ to both sides of the above identity and taking into consideration (4.9) and (4.1), we obtain

$$\sum_{\nu\in N(r)} \alpha_{\nu,k} P_\nu(-\cos r) p_{\nu,0}(t) = 0, \quad |t| < r.$$

By (4.9) and Lemma 4.4 we then conclude that

$$\alpha_{\nu,k} P_\nu(-\cos r) = 0, \quad \nu \in N(r). \tag{4.21}$$

In view of formula (3.2), identity (4.21) can be rewritten as

$$\alpha_{\nu,k}\sin(\pi\nu) Q_\nu(\cos r) = 0, \qquad \nu \in N(r).$$

Then, in view of Statement (iii) of Lemma 4.3,

$$\alpha_{\nu,k}\sin(\pi\nu) = 0, \quad \nu \in N(r),$$

and hence, $\alpha_{\nu,k} = 0$ as $\nu \in N(r)$, $\nu \notin \mathbb{N}$. In view of (4.19) this proves the necessary condition in Theorem 4.1.

We proceed to the sufficient condition. Assume that for each $k \in \mathbb{Z}$ expansion (4.18) holds true. Then by (4.15) and Statement (i) of Lemma 4.3 we conclude that each Fourier coefficient f^k has zero integrals over all geodesic circumferences of the radius r on $\mathbb{S}^2$ lying in $\mathbb{S}'$. Therefore, the function f possesses the stated property. □

5. Proof of Theorem 2.1

Suppose that a function $f \in C^\infty(\mathbb{S}')$ satisfies the assumptions of Theorem 2.1. Then it follows from the first condition of Theorem 2.1 and Theorem 4.1 that for each $k \in \mathbb{Z}$ representation (4.18) holds true and the coefficients obey estimate (4.17). In view of the second condition of Theorem 2.1 and formula (3.7) we obtain

$$\int_{B_r} f^k(a_t\xi)d\xi = 0, \qquad |t| < \pi - r. \tag{5.1}$$

Employing (5.1), (4.18), (4.16) and Lemma 4.2, we find

$$\sum_{\nu\in\mathcal{Z}(r)} \alpha_{\nu,k}\, p_{\nu,1}(r)\, p_{\nu,|k|}(t) = 0, \qquad |t| < \pi - r.$$

In view of the arguing in the proof of Theorem 4.1) this yields

$$\sum_{\nu\in\mathcal{Z}(r)} \alpha_{\nu,k}\, p_{\nu,1}(r)\, p_{\nu,0}(t) = 0, \qquad |t| < \pi - r,$$

which is equivalent to the identity

$$\sum_{\nu\in\mathcal{Z}(\pi-r)} \alpha_{\nu,k}\, p_{\nu,1}(r)\, p_{\nu,0}(t) = 0, \qquad |t| < \pi - r,$$

see Statement (i) of Lemma 4.3. Now Lemma 4.4 shows that

$$\alpha_{\nu,k}\, p_{\nu,1}(r) = 0, \quad \nu \in \mathcal{Z}(r).$$

But by Statement (ii) in Lemma 4.3, the identities $p_{\nu,0}(r) = 0$ and $p_{\nu,1}(r) = 0$ can not hold simultaneously. This is why $\alpha_{\nu,k} = 0$ as $\nu \in \mathcal{Z}(r)$. This means that $f^k = 0$ and hence, $f = 0$. This completes the proof of Theorem 2.1.

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