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ON CAUCHY PROBLEM FOR LAPLACE EQUATION

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Abstract. The paper is devoted to studying the continuation of a solution and stability estimates for the Cauchy problem for the Laplace equation in a domain G by its known values on the smooth part S of the boundary ∂G. The considered issue is among the problems of mathematical physics, in which there is no continuous dependence of solutions on the initial data. While solving applied problems, one needs to find not only an approximate solution, but also its derivative. In the work, given the Cauchy data on a part of the boundary, by means of Carleman function, we recover not only a harmonic function, but also its derivatives. If the Carleman function is constructed, then by employing the Green function, one can find explicitly the regularized solution. We show that an effective construction of the Carleman function is equivalent to the constructing of the regularized solution to the Cauchy problem. We suppose that the solutions of the problem exists and is continuously differentiable in a closed domain with exact given Cauchy data. In this case we establish an explicit formula for continuation of the solution and its derivative as well as a regularization formula for the case, when instead of Cauchy initial data, their continuous approximations are prescribed with a given error in the uniform metrics. We obtain stability estimates for the solution to the Cauchy problem in the classical sense.

Keywords: Cauchy problem, ill-posed problems, Carleman function, regularized solutions, regularization, continuation formulae.

Mathematics Subject Classification: 47A52; 65N20; 45M10

1. Introduction

Let $x = (x_1, x_2)$ and $y = (y_1, y_2)$ be points in the two-dimensional Euclidean space $\mathbb{R}^2$, G be a bounded simply-connected domain in $\mathbb{R}^2$ with a boundary ∂G consisting of a compact part $T = \{y_1 \in \mathbb{R} : \ a_1 \leqslant y_1 \leqslant b_1\}$ and a smooth arc of the curve $S : \ y_2 = h(y_1)$ lying in the half-plane $y_2 > 0$. We denote $\overline{G} = G \cup \partial G$, $\partial G = S \cup T$ and d/dn is the operator of differentiating along the outward normal to ∂G.

We consider a Cauchy problem in the domain G and we shall construct its solutions as the Cauchy data is give on a part of the boundary S. In the domain G we consider the Laplace equation

$$\frac{\partial^2 U}{\partial y_1^2} + \frac{\partial^2 U}{\partial y_2^2} = 0. \tag{1.1}$$

Formulation of the problem. Find a harmonic function

$$U(y) = U(y_1, y_2) \in C^2(G) \cap C^1(\overline{G})$$

with prescribed values on a part S of the boundary ∂G, that is,

$$U(y)|_S = f(y), \qquad \left.\frac{\partial U(y)}{\partial n}\right|_S = g(y). \tag{1.2}$$

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Here $f(y)$ and $g(y)$ are given functions in the classes $C(S)$ and $C^1(S)$, respectively.

Problem (1.1)–(1.2) is among ill-posed problems of mathematical physics. In work [4], A.N. Tikhonov clarified a real nature of ill-posed problems in mathematical physics. He showed a practical importance of ill-posed problems and showed that if one reduces the class of possible solutions to some compact set, then the existence and uniqueness of the solution imply the stability of the solution, that is, the problem becomes well-posed.

The formulae allowing one to find a solution to an elliptic equation in the case, when the Cauchy data is known only on a part of the domain were named as Carleman type formulae. In [2], Carleman established a formula providing a solution to the Cauchy-Riemann equations in a domain of a special form. Developing his idea, G.M. Goluzin and V.I. Krylov [3] obtained a formula for determining the values of analytic functions by data known only on a part of the boundary for arbitrary domains. They found a formula recovering the solution by its values on a boundary set of a positive Lebesque measure and they also proposed a new version of the continuation formula. Monograph by L.A. Aizenberg [1] was devoted to one-dimensional and multi-dimensional generalizations of Carleman formula. A Carleman type formula involving a fundamental solution of the differential equation with special properties, a Carleman function, was obtained by M.M. Lavrent'ev [6], [7]. In these works, the definition of the Carleman function was given for the case, when the Cauchy data is known approximately and there was provided a regularization scheme for the Cauchy problem for the Laplace equation. Applying this method, Sh.Ya. Yarmukhamedov [8], [9] constructed Carleman functions for a wide class of elliptic operators defined in spatial domains of special form, when the part of the boundary is either a hyperplane or a conical surface.

We note that while solving applied problems, one needs to find approximate values of a solution $U(x)$ and $\frac{\partial U(x)}{\partial x_i}$, $x \in G$, $i = 1, 2$. In the present work we construct a family of the functions

$$U(x, \sigma, f_\delta, g_\delta) = U_{\sigma\delta}(x) \qquad \text{and} \qquad \frac{\partial U(x, \sigma, f_\delta, g_\delta)}{\partial x_i} = \frac{\partial U_{\sigma\delta}(x)}{\partial x_i}, \qquad i = 1, 2,$$

depending on a parameter σ and we prove that under a special choice of the parameter $\sigma = \sigma(\delta)$, as $\delta \to 0$, the family $U_{\sigma\delta}(x)$ and $\frac{\partial U_{\sigma\delta}(x)}{\partial x_i}$ converges to the solution $U(x)$ and its derivative $\frac{\partial U(x)}{\partial x_i}$ at each point $x \in G$. The family of the functions $U(x, \sigma, f_\delta, g_\delta)$ and $\frac{\partial U(x,\sigma,f_\delta,g_\delta)}{\partial x_i}$ with described properties is called a solution regularized in the sense of Lavrent'ev [6].

If under the above conditions, instead of the Cauchy data, their continuous approximations with a known error in the uniform metrics are given, we propose an explicit regularization formula. At that we assume that a solution is bounded on a part T of the boundary.

The method we use to obtain the above results is based on finding explicitly the fundamental solution to the Laplace equation depending on a positive parameter and vanishing together with its derivatives as the parameter tends to infinity on T as the pole of the fundamental solution is located in the half-plane $y_2 > 0$.

Carleman function. Let

$$\sigma > 0, \qquad y' = (y_1, 0), \qquad x' = (x_1, 0), \qquad r = |y - x|,$$
$$\alpha = |y' - x'|, \qquad \alpha^2 = s, \qquad w = i\sqrt{u^2 + \alpha^2} + y_2, \qquad u \geqslant 0.$$

For $\alpha > 0$, we define a function $\Phi_\sigma(x, y)$ by the following identity:

$$-2\pi e^{\sigma x_2^2}\Phi_\sigma(x, y) = \int_0^\infty \operatorname{Im}\left(\frac{e^{\sigma w^2}}{w - x_2}\right)\frac{u\,du}{\sqrt{u^2 + \alpha^2}}, \qquad w = i\sqrt{u^2 + \alpha^2} + y_2. \tag{1.3}$$

We find the imaginary part of the function $\Phi_\sigma(x,y)$:

$$\Phi_\sigma(x,y) = \frac{1}{2\pi} e^{-\sigma(\alpha^2+x_2^2-y_2^2)} \left(\int_0^\infty \frac{e^{-\sigma u^2}\cos 2\sigma y_2\sqrt{u^2+\alpha^2}udu}{u^2+r^2} - \int_0^\infty \frac{e^{-\sigma u^2}(y_2-x_2)\sin 2\sigma y_2\sqrt{u^2+\alpha^2}}{u^2+r^2} \frac{udu}{\sqrt{u^2+\alpha^2}} \right). \tag{1.4}$$

We denote

$$\varphi_\sigma(x,y,u) = \cos\tau\sqrt{u^2+\alpha^2} - \frac{(y_2-x_2)\sin\tau\sqrt{u^2+\alpha^2}}{\sqrt{u^2+\alpha^2}}, \quad \tau = 2\sigma y_2.$$

Then $\Phi_\sigma(x,y)$ becomes:

$$2\pi e^{\sigma(\alpha^2+x_2^2-y_2^2)}\Phi_\sigma(x,y) = \int_0^\infty \frac{\varphi_\sigma(x,y,u)}{u^2+r^2} u e^{-\sigma u^2} du.$$

It was proved in work [9] that as $\sigma > 0$, the function defined by identities (1.3) can be represented as

$$\Phi_\sigma(x,y) = F(r) + G_\sigma(x,y), \tag{1.5}$$

where $F(r) = \frac{1}{2\pi}\ln\frac{1}{r}$, $G_\sigma(x,y)$ is a function harmonic with respect to y in $\mathbb{R}^2$ including $y = x$. This implies that for each $\sigma > 0$ the function $\Phi_\sigma(x,y)$ is a fundamental solution of the Laplace equation for each y. The fundamental solution $\Phi_\sigma(x,y)$ with this property is called the Carleman function for a half-space [6]. This is why for each function $U(y) = U(y_1,y_2) \in C^2(G)\cap C^1(\overline{G})$ and for each $x \in G$, the following integral Green's formula holds:

$$U(x) = \int_{\partial G} \left(\frac{\partial U}{\partial n}\Phi_\sigma(x,y) - U(y)\frac{\partial \Phi_\sigma(x,y)}{\partial n} \right) dS_y. \tag{1.6}$$

2. Continuation formulation and regularization in the sense of Lavrent'ev

We denote

$$U_\sigma(x) = \int_S \left(g(y)\Phi_\sigma(x,y) - f(y)\frac{\partial \Phi_\sigma(x,y)}{\partial n} \right) dS_y. \tag{2.1}$$

Theorem 2.1. *Let a function $U(y) = U(y_1,y_2) \in C^2(G)\cap C^1(\overline{G})$ satisfy condition* (1.2) *on S and on a part T of the boundary ∂G, the inequality holds:*

$$|U(y)| + \left|\frac{\partial U(y)}{\partial n}\right| \leqslant M, \qquad y \in T, \tag{2.2}$$

where $M > 0$. Then for each $x \in G$ and $\sigma > 0$ the estimate hold:

$$|U(x) - U_\sigma(x)| \leqslant \psi_2(\sigma) M e^{-\sigma x_2^2}, \tag{2.3}$$

$$\left| \frac{\partial U(x)}{\partial x_i} - \frac{\partial U_\sigma(x)}{\partial x_i} \right| \leqslant \varphi_i(\sigma,x_2) M e^{-\sigma x_2^2}, \quad i = 1,2, \tag{2.4}$$

where

$$\psi_2(\sigma) = \left(\frac{1}{2} + \frac{\sqrt{\pi}}{4\sqrt{\sigma}} \right), \tag{2.5}$$

$$\varphi_1(\sigma,x_2) = \left(\frac{1}{2} + \frac{\sqrt{\pi}}{4\sqrt{\sigma}x_2} + \frac{\sqrt{\sigma}}{2\sqrt{\pi}} + \frac{1}{2\sqrt{\pi\sigma}x_2^2} \right), \tag{2.6}$$

$$\varphi_2(\sigma, x_2) = \left(\frac{\sqrt{\pi\sigma}x_2}{2} + \frac{\sqrt{\pi}}{2\sqrt{\sigma}x_2} + \frac{\sqrt{\sigma}}{\sqrt{\pi}} + \frac{3}{2\sqrt{\pi\sigma}x_2^2} \right). \tag{2.7}$$

Proof. Estimate (2.3) was proved in work [9]. Let us prove inequality (2.4). We differentiate identities (1.6) and (2.1) with respect to x_1 and we obtain:

$$\begin{aligned} \frac{\partial U(x)}{\partial x_1} &= \int\limits_S \left(\frac{\partial U}{\partial n} \frac{\partial \Phi_\sigma(x,y)}{\partial x_1} - U(y) \frac{\partial}{\partial x_1} \frac{\partial \Phi_\sigma(x,y)}{\partial n} \right) dS_y \\ &+ \int\limits_T \left(\frac{\partial U}{\partial n} \frac{\partial \Phi_\sigma(x,y)}{\partial x_1} - U(y) \frac{\partial}{\partial x_1} \frac{\partial \Phi_\sigma(x,y)}{\partial n} \right) dS_y, \\ \frac{\partial U_\sigma(x)}{\partial x_1} &= \int_S \left(g(y) \frac{\partial \Phi_\sigma(x,y)}{\partial x_1} - f(y) \frac{\partial}{\partial x_1} \frac{\partial \Phi_\sigma(x,y)}{\partial n} \right) dS_y. \end{aligned} \tag{2.8}$$

We denote by $I_{1\sigma}(x)$ a difference of the derivatives:

$$I_{1\sigma}(x) = \frac{\partial U(x)}{\partial x_1} - \frac{\partial U_\sigma(x)}{\partial x_1} = \int\limits_T \left(\frac{\partial U}{\partial n} \frac{\partial \Phi_\sigma(x,y)}{\partial x_1} - U(y) \frac{\partial}{\partial x_1} \frac{\partial \Phi_\sigma(x,y)}{\partial n} \right) dS_y.$$

Then inequality (2.2) yields:

$$|I_{1\sigma}(x)| = \left| \int\limits_T \left(\frac{\partial U}{\partial n} \frac{\partial \Phi_\sigma(x,y)}{\partial x_1} - U(y) \frac{\partial}{\partial x_1} \frac{\partial \Phi_\sigma(x,y)}{\partial n} \right) dS_y \right| \leqslant M N_\sigma(x),$$

where

$$N_\sigma(x) = \int\limits_T \left(\left| \frac{\partial \Phi_\sigma(x,y)}{\partial x_1} \right| + \left| \frac{\partial}{\partial x_1} \frac{\partial \Phi_\sigma(x,y)}{\partial n} \right| \right) dS_y.$$

In order to prove estimate (2.4) as $i = 1$, we are going to prove the following inequality:

$$N_\sigma(x) \leqslant \varphi_1(\sigma, x_2) e^{-\sigma x_2^2}, \quad \sigma > 0. \tag{2.9}$$

In order to do this, we differentiate identity (1.4) with respect to x_1:

$$\begin{aligned} \frac{\partial \Phi_\sigma(x,y)}{\partial x_1} =& 2\sigma(y_1 - x_1)\Phi_\sigma(x,y) + \frac{1}{2\pi} e^{-\sigma(\alpha^2 + x_2^2 - y_2^2)} \\ &\cdot \left(\int\limits_0^\infty \frac{2\sigma y_2 (y_1 - x_1) u e^{-\sigma u^2} \sin 2\sigma y_2 \sqrt{u^2+\alpha^2}}{\sqrt{u^2+\alpha^2}(u^2+r^2)} du \right. \\ &+ \int\limits_0^\infty \frac{2(y_1 - x_1) u e^{-\sigma u^2} \cos 2\sigma y_2 \sqrt{u^2+\alpha^2}}{(u^2+r^2)^2} du \\ &+ \int\limits_0^\infty \frac{2\sigma y_2 (y_1 - x_1)(y_2 - x_2) u e^{-\sigma u^2} \cos 2\sigma y_2 \sqrt{u^2+\alpha^2}}{(u^2+\alpha^2)(u^2+r^2)} du \\ &- \int\limits_0^\infty \frac{2(y_1 - x_1)(y_2 - x_2) u e^{-\sigma u^2} \sin 2\sigma y_2 \sqrt{u^2+\alpha^2}}{\sqrt{u^2+\alpha^2}(u^2+r^2)^2} du \\ &- \left. \int\limits_0^\infty \frac{(y_1 - x_1)(y_2 - x_2) u e^{-\sigma u^2} \sin 2\sigma y_2 \sqrt{u^2+\alpha^2}}{\sqrt{(u^2+\alpha^2)^3}(u^2+r^2)^2} du \right). \end{aligned} \tag{2.10}$$

Letting here $y_2 = 0$, we obtain

$$\frac{\partial \Phi_\sigma(x,y)}{\partial x_1} = \frac{(y_1-x_1)}{\pi} e^{-\sigma(\alpha^2+x_2^2)} \left(\int_0^\infty \frac{\sigma u e^{-\sigma u^2}}{u^2+(y_1-x_1)^2+x_2^2} du + \int_0^\infty \frac{u e^{-\sigma u^2}}{(u^2+(y_1-x_1)^2+x_2^2)^2} du \right). \tag{2.11}$$

Let us estimate the following integral:

$$\int_T \left| \frac{\partial \Phi_\sigma(x,y)}{\partial x_1} \right| dS_y \leqslant \int_{a_1}^{b_1} dy_1 \frac{|y_1-x_1|}{\pi} e^{-\sigma(\alpha^2+x_2^2)} \int_0^\infty \frac{\sigma u e^{-\sigma u^2}}{u^2+(y_1-x_1)^2+x_2^2} du + \int_{a_1}^{b_1} dy_1 \frac{|y_1-x_1|}{\pi} e^{-\sigma(\alpha^2+x_2^2)} \int_0^\infty \frac{u e^{-\sigma u^2}}{(u^2+(y_1-x_1)^2+x_2^2)^2} du.$$

We estimate the first integral:

$$\int_{a_1}^{b_1} dy_1 \frac{|y_1-x_1|}{\pi} e^{-\sigma(\alpha^2+x_2^2)} \int_0^\infty \frac{\sigma u e^{-\sigma u^2}}{u^2+(y_1-x_1)^2+x_2^2} du \leqslant \frac{\sigma}{2\pi} \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} \frac{u\,|y_1-x_1|\, e^{-\sigma(u^2+(y_1-x_1)^2+x_2^2)}}{u^2+(y_1-x_1)^2+x_2^2} du dy_1 \leqslant \frac{\sigma}{2\pi} \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} e^{-\sigma(u^2+(y_1-x_1)^2+x_2^2)} du dy_1 = \frac{e^{-\sigma x_2^2}}{2}.$$

Here we have also employed the inequality

$$\frac{u\,|y_1-x_1|}{u^2+(y_1-x_1)^2+x_2^2} < 1. \tag{2.12}$$

Taking (2.12) into consideration, we estimate the second integral and passing to the polar coordinates, we obtain:

$$\int_{a_1}^{b_1} dy_1 \frac{|y_1-x_1|}{\pi} e^{-\sigma(\alpha^2+x_2^2)} \int_0^\infty \frac{u e^{-\sigma u^2}}{(u^2+(y_1-x_1)^2+x_2^2)^2} du \leqslant \frac{1}{2\pi} \int_{-\infty}^{+\infty} dy_1 \int_{-\infty}^{+\infty} \frac{u\,|y_1-x_1|\, e^{-\sigma(u^2+(y_1-x_1)^2+x_2^2)}}{(u^2+(y_1-x_1)^2+x_2^2)^2} du$$

$$\leqslant \frac{e^{-\sigma x_2^2}}{2\pi} \int_{-\infty}^{+\infty} dy_1 \int_{-\infty}^{+\infty} \frac{e^{-\sigma(u^2+(y_1-x_1)^2)}}{u^2+(y_1-x_1)^2+x_2^2} du = \frac{e^{-\sigma x_2^2}}{2\pi} \int_0^{2\pi} d\varphi \int_0^{+\infty} \frac{t e^{-\sigma t^2}}{t^2+x_2^2} dt$$

$$= e^{-\sigma x_2^2} \int_0^{+\infty} \frac{t e^{-\sigma t^2}}{t^2+x_2^2} dt \leqslant \frac{e^{-\sigma x_2^2}}{2x_2} \int_0^{+\infty} e^{-\sigma t^2} dt = \frac{\sqrt{\pi} e^{-\sigma x_2^2}}{4\sqrt{\sigma} x_2}.$$

This leads us to the inequality

$$\int\limits_T \left|\frac{\partial \Phi_\sigma(x,y)}{\partial x_1}\right| dS_y \leqslant e^{-\sigma x_2^2}\left(\frac{1}{2}+\frac{\sqrt{\pi}}{4\sqrt{\sigma}x_2}\right). \tag{2.13}$$

We calculate the derivative:

$$\begin{aligned}
\frac{\partial}{\partial x_1}\frac{\partial \Phi_\sigma(x,y)}{\partial n} &= \frac{\partial}{\partial x_1}\frac{\partial \Phi_\sigma(x,y)}{\partial y_1}\cos\gamma + \frac{\partial}{\partial x_1}\frac{\partial \Phi_\sigma(x,y)}{\partial y_2}\sin\gamma \\
=&\frac{\sigma(y_1-x_1)}{\pi}e^{-\sigma(x_2^2-y_2^2+(y_1-x_1)^2)} \\
&\cdot\frac{(y_1-x_1)\cos\tau(y_1-x_1)-(y_2-x_2)\sin\tau(y_1-x_1)}{r^2}\cos\gamma \\
&-\frac{e^{-\sigma(-y_2^2+x_2^2+(y_1-x_1)^2)}}{2\pi}\Bigg(\frac{-\cos\tau(y_1-x_1)+\tau(y_1-x_1)\sin\tau(y_1-x_1)}{r^2} \\
&+\frac{\tau(y_2-x_2)\cos\tau(y_1-x_1)}{r^2} \\
&+\frac{2(y_1-x_1)^2\cos\tau(y_1-x_1)-2(y_1-x_1)(y_2-x_2)\sin\tau(y_1-x_1)}{r^4}\Bigg)\cos\gamma \\
&-\frac{2\sigma\left|y_1-x_1\right|}{2\pi}e^{-\sigma(-y_2^2+x_2^2+(y_1-x_1)^2)} \\
&\cdot\frac{(y_2-x_2)\cos\tau(y_1-x_1)+\left|y_1-x_1\right|\sin\tau\left|y_1-x_1\right|}{r^2}\sin\gamma \\
&-\frac{1}{2\pi}e^{-\sigma(-y_2^2+x_2^2+(y_1-x_1)^2)} \\
&\cdot\Bigg(\frac{(-\tau(y_2-x_2)\sin\tau(y_1-x_1)-\sin\tau(y_1-x_1)-\tau\left|y_1-x_1\right|\cos\tau\left|y_1-x_1\right|)}{r^2} \\
&+\frac{2(y_1-x_1)\left((y_2-x_2)\cos\tau(y_1-x_1)+2\left|y_1-x_1\right|^2\sin\tau\left|y_1-x_1\right|\right)}{r^4}\Bigg)\sin\gamma,
\end{aligned} \tag{2.14}$$

where $\tau = 2\sigma y_2$, and $\frac{\partial\Phi_\sigma(x,y)}{\partial y_1}$ and $\frac{\partial\Phi_\sigma(x,y)}{\partial y_2}$ are given by the following formulae [9]:

$$\begin{aligned}
\frac{\partial \Phi_\sigma(x,y)}{\partial y_1} &= -\frac{e^{\sigma(y_2^2-x_2^2-(y_1-x_1)^2)}}{2\pi} \\
&\quad\cdot\frac{(y_1-x_1)\cos 2\sigma y_2(y_1-x_1)+(y_2-x_2)\sin 2\sigma y_2(y_1-x_1)}{r^2},
\end{aligned} \tag{2.15}$$

$$\begin{aligned}
\frac{\partial \Phi_\sigma(x,y)}{\partial y_2} &= -\frac{e^{\sigma(y_2^2-x_2^2-|y_1-x_1|^2)}}{2\pi} \\
&\quad\cdot\frac{(y_2-x_2)\cos 2\sigma y_2(y_1-x_1)+\left|y_1-x_1\right|\sin 2\sigma y_2\left|y_1-x_1\right|}{r^2}.
\end{aligned} \tag{2.16}$$

This implies:

$$\int_T \left| \frac{\partial}{\partial x_1}\left(\frac{\partial \Phi_\sigma(x,y)}{\partial n}\right)\Big|_{y_2=0}\right| dS_y = \int_T \Bigg| \Bigg(\frac{\sigma(y_1-x_1)}{\pi} \frac{e^{-\sigma x_2^2-\sigma(y_1-x_1)^2}(y_1-x_1)}{(y_1-x_1)^2+x_2^2}\cos\gamma$$
$$+\frac{e^{-\sigma x_2^2-\sigma(y_1-x_1)^2}}{2\pi}\left(\frac{1}{(y_1-x_1)^2+x_2^2}-\frac{2(y_1-x_1)^2}{((y_1-x_1)^2+x_2^2)^2}\right)\cos\gamma \tag{2.17}$$
$$+\frac{1}{\pi}e^{-\sigma x_2^2-\sigma|y_1-x_1|^2}\left(\frac{\sigma\,|y_1-x_1|\,x_2}{(y_1-x_1)^2+x_2^2}+\frac{(y_1-x_1)x_2}{((y_1-x_1)^2+x_2^2)^2}\right)\sin\gamma\Bigg)\Bigg| dS_y.$$

Since $\cos\gamma$ and $\sin\gamma$ are the coordinates of the unit outward normal n at the point y of the boundary ∂G, we can estimate (2.17).

In view of the estimate

$$\frac{(y_1-x_1)}{(y_1-x_1)^2+x_2^2}\leqslant\frac{1}{2x_2},$$

we begin with estimating the first integral:

$$\frac{1}{\pi}e^{-\sigma x_2^2}\int_{a_1}^{b_1}\frac{\sigma\,|y_1-x_1|\,x_2}{(y_1-x_1)^2+x_2^2}e^{-\sigma|y_1-x_1|^2}dy_1 \leqslant \frac{\sigma x_2}{\pi}e^{-\sigma x_2^2}\int_{-\infty}^{+\infty}\frac{|y_1-x_1|}{(y_1-x_1)^2+x_2^2}e^{-\sigma|y_1-x_1|^2}dy_1$$
$$\leqslant \frac{\sigma x_2}{\pi}e^{-\sigma x_2^2}\frac{1}{2x_2}\int_{-\infty}^{+\infty}e^{-\sigma|y_1-x_1|^2}dy_1=\frac{\sqrt{\sigma}}{2\sqrt{\pi}}e^{-\sigma x_2^2}.$$

In the same way we estimate the second integral:

$$\frac{1}{\pi}e^{-\sigma x_2^2}\int_{a_1}^{b_1}\frac{|y_1-x_1|\,x_2}{((y_1-x_1)^2+x_2^2)^2}e^{-\sigma|y_1-x_1|^2}dy_1 \leqslant \frac{1}{2\pi}e^{-\sigma x_2^2}\int_{-\infty}^{+\infty}\frac{1}{(y_1-x_1)^2+x_2^2}e^{-\sigma|y_1-x_1|^2}dy_1$$
$$\leqslant \frac{1}{2\pi x_2^2}e^{-\sigma x_2^2}\int_{-\infty}^{+\infty}e^{-\sigma|y_1-x_1|^2}dy_1=\frac{1}{2x_2^2\sqrt{\pi\sigma}}e^{-\sigma x_2^2}.$$

In view of the obtained estimates we have:

$$\int_T\left|\frac{\partial}{\partial x_1}\frac{\partial\Phi_\sigma(x,y)}{\partial n}\right|dS_y\leqslant e^{-\sigma x_2^2}\left(\frac{\sqrt{\sigma}}{2\sqrt{\pi}}+\frac{1}{2x_2^2\sqrt{\pi\sigma}}\right). \tag{2.18}$$

Taking into consideration (2.13) and (2.18), we arrive at inequality (2.9). This proves inequality (2.4) for $i=1$.

Now we are going to prove inequality (2.4) for $i=2$. By (1.6) and (2.1) we find the derivative with respect to x_2:

$$\frac{\partial U(x)}{\partial x_2}=\int_S\left(\frac{\partial U}{\partial n}\frac{\partial\Phi_\sigma(x,y)}{\partial x_2}-U(y)\frac{\partial}{\partial x_2}\frac{\partial\Phi_\sigma(x,y)}{\partial n}\right)dS_y$$
$$+\int_T\left(\frac{\partial U}{\partial n}\frac{\partial\Phi_\sigma(x,y)}{\partial x_2}-U(y)\frac{\partial}{\partial x_2}\frac{\partial\Phi_\sigma(x,y)}{\partial n}\right)dS_y,$$
$$\frac{\partial U_\sigma(x)}{\partial x_2}=\int_S\left(g(y)\frac{\partial\Phi_\sigma(x,y)}{\partial x_2}-f(y)\frac{\partial}{\partial x_2}\frac{\partial\Phi_\sigma(x,y)}{\partial n}\right)dS_y.$$

We denote by $I_{2\sigma}(x)$ the difference of the derivatives:

$$I_{2\sigma}(x)=\frac{\partial U(x)}{\partial x_2}-\frac{\partial U_\sigma(x)}{\partial x_2}=\int\limits_T\left(\frac{\partial U}{\partial n}\frac{\partial\Phi_\sigma(x,y)}{\partial x_2}-U(y)\frac{\partial}{\partial x_2}\frac{\partial\Phi_\sigma(x,y)}{\partial n}\right)dS_y.$$

By (2.2) this yields:

$$|I_{2\sigma}(x)|=\left|\int\limits_T\left(\frac{\partial U}{\partial n}\frac{\partial\Phi_\sigma(x,y)}{\partial x_2}-U(y)\frac{\partial}{\partial x_2}\frac{\partial\Phi_\sigma(x,y)}{\partial n}\right)dS_y\right|\leqslant MP_\sigma(x),$$

where

$$P_\sigma(x)=\int\limits_T\left(\left|\frac{\partial\Phi_\sigma(x,y)}{\partial x_2}\right|+\left|\frac{\partial}{\partial x_2}\frac{\partial\Phi_\sigma(x,y)}{\partial n}\right|\right)dS_y.$$

Let us prove the inequality

$$P_\sigma(x)\leqslant\varphi_2(\sigma,x_2)e^{-\sigma x_2^2},\qquad \sigma>0. \tag{2.19}$$

By (1.4) we have:

$$\begin{aligned}\int\limits_T\left|\frac{\partial\Phi_\sigma(x,y)}{\partial x_2}\right|dS_y=\int\limits_T\Bigg|\Bigg(&-2\sigma x_2\Phi_\sigma(x,y)+\frac{1}{2\pi}e^{-\sigma(\alpha^2+x_2^2-y_2^2)}\\ &\cdot\Bigg(\int\limits_0^\infty\frac{2(y_2-x_2)ue^{-\sigma u^2}\cos 2\sigma y_2\sqrt{u^2+\alpha^2}}{(u^2+r^2)^2}du\\ &+\int\limits_0^\infty\frac{e^{-\sigma u^2}\sin 2\sigma y_2\sqrt{u^2+\alpha^2}}{u^2+r^2}\frac{udu}{\sqrt{u^2+\alpha^2}}\\ &-\int\limits_0^\infty\frac{2(y_2-x_2)^2e^{-\sigma u^2}\sin 2\sigma y_2\sqrt{u^2+\alpha^2}}{(u^2+r^2)^2}\frac{udu}{\sqrt{u^2+\alpha^2}}\Bigg)\Bigg)\Bigg|dS_y.\end{aligned} \tag{2.20}$$

Letting $y_2=0$ in (2.20), we find:

$$\begin{aligned}\int\limits_T\left|\frac{\partial\Phi_\sigma(x,y)}{\partial x_2}\right|dS_y\Bigg|_{y_2=0}=\int\limits_T\Bigg|\Bigg(&-\frac{x_2}{\pi}e^{-\sigma(\alpha^2+x_2^2)}\Bigg(\int\limits_0^\infty\frac{\sigma ue^{-\sigma u^2}}{u^2+(y_1-x_1)^2+x_2^2}du\\ &+\int\limits_0^\infty\frac{ue^{-\sigma u^2}}{(u^2+(y_1-x_1)^2+x_2^2)^2}du\Bigg)\Bigg)\Bigg|dS_y.\end{aligned}$$

Passing to the polar coordinates in the first integral and estimating it, we obtain:

$$\begin{aligned}&\frac{x_2}{\pi}e^{-\sigma(\alpha^2+x_2^2)}\int\limits_{a_1}^{b_1}dy_1\int\limits_0^\infty\frac{\sigma\,|u|\,e^{-\sigma u^2}}{u^2+(y_1-x_1)^2+x_2^2}du\leqslant\frac{\sigma x_2}{2\pi}e^{-\sigma x_2^2}\int\limits_{-\infty}^{+\infty}dy_1\int\limits_{-\infty}^{+\infty}\frac{|u|\,e^{-\sigma(u^2+(y_1-x_1)^2)}}{u^2+(y_1-x_1)^2+x_2^2}du\\ &=\frac{\sigma x_2}{2\pi}e^{-\sigma x_2^2}\int\limits_0^{2\pi}d\varphi\int\limits_0^{+\infty}\frac{t^2\,|\cos\varphi|\,e^{-\sigma t^2}}{t^2+x_2^2}dt\leqslant\sigma x_2e^{-\sigma x_2^2}\int\limits_0^{+\infty}\frac{t^2e^{-\sigma t^2}}{t^2+x_2^2}dt\\ &\leqslant\sigma x_2e^{-\sigma x_2^2}\int\limits_0^{+\infty}e^{-\sigma t^2}dt=\frac{\sqrt{\sigma\pi}x_2}{2}e^{-\sigma x_2^2}.\end{aligned}$$

In the same way we estimate the second integral:

$$\frac{x_2}{\pi}e^{-\sigma(\alpha^2+x_2^2)}\int\limits_{a_1}^{b_1}dy_1\int\limits_0^{+\infty}\frac{\sigma\,|u|\,e^{-\sigma u^2}}{(u^2+(y_1-x_1)^2+x_2^2)^2}du\leqslant\frac{x_2}{2\pi}e^{-\sigma x_2^2}\int\limits_{-\infty}^{+\infty}dy_1\int\limits_{-\infty}^{+\infty}\frac{|u|\,e^{-\sigma(u^2+(y_1-x_1)^2)}}{(u^2+(y_1-x_1)^2+x_2^2)^2}du$$

$$=\frac{x_2}{2\pi}e^{-\sigma x_2^2}\int\limits_0^{2\pi}d\varphi\int\limits_0^{+\infty}\frac{t^2\,|\cos\varphi|\,e^{-\sigma t^2}}{(t^2+x_2^2)^2}dt\leqslant e^{-\sigma x_2^2}\int\limits_0^{+\infty}\frac{t^2x_2e^{-\sigma t^2}}{(t^2+x_2^2)^2}dt$$

$$\leqslant e^{-\sigma x_2^2}\int\limits_0^{+\infty}\frac{x_2e^{-\sigma t^2}}{t^2+x_2^2}dt\leqslant\frac{e^{-\sigma x_2^2}}{x_2}\int\limits_0^{+\infty}e^{-\sigma t^2}dt=\frac{\sqrt{\pi}}{2x_2\sqrt{\sigma}}e^{-\sigma x_2^2}.$$

By these estimates we obtain:

$$\int\limits_T\left|\frac{\partial\Phi_\sigma(x,y)}{\partial x_2}\bigg|_{y_2=0}\right|dS_y\leqslant e^{-\sigma x_2^2}\left(\frac{\sqrt{\pi\sigma}x_2}{2}+\frac{\sqrt{\pi}}{2x_2\sqrt{\sigma}}\right).\tag{2.21}$$

We calculate the integral:

$$\int\limits_T\frac{\partial}{\partial x_2}\frac{\partial\Phi_\sigma(x,y)}{\partial n}dS_y=\int\limits_T\Bigg(-\frac{e^{\sigma(y_2^2-x_2^2-(y_1-x_1)^2)}}{\pi}\Bigg(-\frac{1}{2}\frac{\sin 2\sigma y_2(y_1-x_1)}{r^2}$$

$$+\frac{\sigma x_2(y_1-x_1)\cos 2\sigma y_2(y_1-x_1)}{r^2}-\frac{(y_1-x_1)(y_2-x_2)\cos 2\sigma y_2(y_1-x_1)}{r^4}$$

$$-\frac{\sigma x_2(y_2-x_2)\sin 2\sigma y_2(y_1-x_1)}{r^2}+\frac{(y_2-x_2)^2\sin 2\sigma y_2(y_1-x_1)}{r^4}\Bigg)\cos\gamma$$

$$-\frac{e^{\sigma(y_2^2-x_2^2-(y_1-x_1)^2}}{\pi}\Bigg(\frac{1}{2}\frac{\cos 2\sigma y_2(y_1-x_1)}{r^2}$$

$$+\frac{\sigma x_2(y_2-x_2)\cos 2\sigma y_2(y_1-x_1)+\sigma x_2\,|y_1-x_1|\sin 2\sigma y_2\,|y_1-x_1|}{r^2}$$

$$+\frac{(y_2-x_2)^2\cos 2\sigma y_2(y_1-x_1)-|y_1-x_1(y_2-x_2)\sin 2\sigma y_2\,|y_1-x_1||}{r^4}\Bigg)\sin\gamma\Bigg)dS_y$$

As $y_2=0$, this implies:

$$\int\limits_T\left|\frac{\partial}{\partial x_2}\frac{\partial\Phi_\sigma(x,y)}{\partial n}\bigg|_{y_2=0}\right|dS_y$$

$$=\int\limits_T\Bigg|\Bigg(\frac{e^{-\sigma x_2^2-\sigma(y_1-x_1)^2}}{\pi}\frac{x_2(y_1-x_1)}{(y_1-x_1)^2+x_2^2}\left(\sigma+\frac{1}{(y_1-x_1)^2+x_2^2}\right)\cos\gamma$$

$$+\frac{e^{-\sigma x_2^2-\sigma|y_1-x_1|^2}}{\pi}\Bigg(\frac{-\sigma x_2^2}{(y_1-x_1)^2+x_2^2}+\frac{1}{2(y_1-x_1)^2+x_2^2}\tag{2.22}$$

$$+\frac{x_2^2}{((y_1-x_1)^2+x_2^2)^2}\Bigg)\sin\gamma\Bigg)\Bigg|dS_y\leqslant\frac{\sigma x_2^2e^{-\sigma x_2^2}}{\pi}\int\limits_{a_1}^{b_1}\frac{e^{-\sigma|y_1-x_1|^2}}{(y_1-x_1)^2+x_2^2}dy_1$$

$$+\frac{e^{-\sigma x_2^2}}{2\pi}\int\limits_{a_1}^{b_1}\frac{e^{-\sigma|y_1-x_1|^2}}{(y_1-x_1)^2+x_2^2}dy_1+\frac{x_2^2e^{-\sigma x_2^2}}{\pi}\int\limits_{a_1}^{b_1}\frac{e^{-\sigma|y_1-x_1|^2}}{((y_1-x_1)^2+x_2^2)^2}dy_1.$$

Here we have taken into consideration that $\cos\gamma$, $\sin\gamma$ are the coordinates of the unit outward normal n at the point y of the boundary ∂G. We estimate these integrals as follows:

$$\frac{\sigma x_2^2 e^{-\sigma x_2^2}}{\pi}\int\limits_{a_1}^{b_1}\frac{e^{-\sigma|y_1-x_1|^2}}{(y_1-x_1)^2+x_2^2}dy \leqslant \frac{\sigma e^{-\sigma x_2^2}}{\pi}\int\limits_{-\infty}^{+\infty} e^{-\sigma|y_1-x_1|^2}dy_1 = \frac{\sqrt{\sigma}e^{-\sigma x_2^2}}{\sqrt{\pi}},$$

$$\frac{e^{-\sigma x_2^2}}{2\pi}\int\limits_{a_1}^{b_1}\frac{e^{-\sigma|y_1-x_1|^2}}{(y_1-x_1)^2+x_2^2}dy_1 \leqslant \frac{e^{-\sigma x_2^2}}{2\pi}\frac{1}{x_2^2}\int\limits_{-\infty}^{+\infty} e^{-\sigma|y_1-x_1|^2}dy_1 = \frac{e^{-\sigma x_2^2}}{2x_2^2\sqrt{\pi\sigma}},$$

$$\frac{x_2^2 e^{-\sigma x_2^2}}{\pi}\int\limits_{a_1}^{b_1}\frac{e^{-\sigma|y_1-x_1|^2}}{((y_1-x_1)^2+x_2^2)^2}dy_1 \leqslant \frac{e^{-\sigma x_2^2}}{\pi}\frac{1}{x_2^2}\int\limits_{-\infty}^{+\infty} e^{-\sigma|y_1-x_1|^2}dy_1 = \frac{e^{-\sigma x_2^2}}{x_2^2\sqrt{\pi\sigma}}.$$

In view of the obtained estimates we have:

$$\int\limits_T \left|\frac{\partial}{\partial x_2}\frac{\partial \Phi_\sigma(x,y)}{\partial n}\right| dS_y \leqslant e^{-\sigma x_2^2}\left(\frac{\sqrt{\sigma}}{\sqrt{\pi}}+\frac{3}{2x_2^2\sqrt{\pi\sigma}}\right). \tag{2.23}$$

Inequalities (2.21) and (2.23) imply (2.19). The proof is complete. □

Corollary 2.1. *For each $x \in G$, the identity holds:*

$$\lim_{\sigma\to\infty} U_\sigma(x) = U(x), \qquad \lim_{\sigma\to\infty}\frac{\partial U_\sigma(x)}{\partial x_i} = \frac{\partial U(x)}{\partial x_i}, \qquad i = 1,2.$$

We denote

$$\overline{G}_\varepsilon = \left\{(x_1,x_2)\in G,\ a > x_2 \geqslant \varepsilon,\ a = \max_T h(x_1),\ 0<\varepsilon<a\right\}.$$

It is easy to see that the set $\overline{G}_\varepsilon \subset G$ is compact.

Corollary 2.2. *If $x \in \overline{G}_\varepsilon$, then the family of the functions $\{U_\sigma(x)\}$ and $\left\{\frac{\partial U_\sigma(x)}{\partial x_i}\right\}$ converges uniformly as $\sigma\to\infty$:*

$$U_\sigma(x) \rightrightarrows U(x), \qquad \frac{\partial U_\sigma(x)}{\partial x_i} \rightrightarrows \frac{\partial U(x)}{\partial x_i}, \qquad i=1,2.$$

It should be noted that the sets $\Pi_\varepsilon = G\setminus \overline{G}_\varepsilon$ serve as a boundary layer in the considered problem, similar to the theory of singular perturbation, where usually the uniform convergence fails.

3. Stability of solutions to Cauchy problems

We consider the set

$$E = \left\{U \in C^2(G)\cap C^1(\overline{G}) :\ |U(y)| + |\mathrm{grad}\, U| \leqslant M, M>0, y\in T\right\}.$$

We let

$$a = \max_T h(y_1), \qquad b = \max_T \sqrt{1+\left(\frac{dh}{dy_1}\right)^2},$$

where the curve S is given by the equation $y_2 = h(y_1)$.

Theorem 3.1. *Let the function $U(y) \in E$ satisfies Laplace equation* (1.1) *and on the part S of the boundary of the domain G the inequality holds:*

$$|U(y)| + \left|\frac{\partial U(y)}{\partial n}\right| < \delta, \qquad y \in S. \tag{3.1}$$

Then for each $x \in G$ and $\sigma > 0$ the estimates hold true:

$$|U(x)| \leqslant 2\psi(\sigma, x_2) M^{1-\frac{x_2^2}{a^2}} \delta^{\frac{x_2^2}{a^2}}, \tag{3.2}$$

$$\left|\frac{\partial U(x)}{\partial x_i}\right| \leqslant 2\mu_i(\sigma, x_2) M^{1-\frac{x_2^2}{a^2}} \delta^{\frac{x_2^2}{a^2}}, \qquad i = 1, 2, \qquad 0 < \delta \leqslant Me^{-\sigma a^2}, \tag{3.3}$$

where

$$\psi(\sigma, x_2) = \max_S(\psi^2(\sigma, x_2), \psi_2(\sigma)),$$
$$\psi^2(\sigma, x_2) = \frac{b\sqrt{\pi}}{4\sqrt{\sigma}} + ab + \frac{b}{2\sqrt{\pi\sigma}(a - x_2)} + \frac{2\sqrt{\sigma}ab}{\sqrt{\pi}},$$
$$\mu_1(\sigma, x_2) = \max_S(\nu_1(\sigma, x_2), \varphi_1(\sigma, x_2)),$$
$$\nu_1(\sigma, x_2) = \left(\frac{b + 3ab\sqrt{\pi\sigma}}{4} + \frac{2\sigma ab + 4b\sqrt{\sigma} + a^2 b\sigma\sqrt{\pi}}{\sqrt{\pi}} + \frac{b\sqrt{\pi}}{4\sqrt{\sigma}(a - x_2)} + \frac{ab\sqrt{\pi\sigma}}{(a - x_2)^2} + \frac{2ab\sqrt{\sigma}}{\sqrt{\pi}(a - x_2)} + \frac{5b}{\sqrt{\pi\sigma}(a - x_2)^2}\right),$$
$$\mu_2(\sigma, x_2) = \max_S(\nu_2(\sigma, x_2), \varphi_2(\sigma, x_2)),$$
$$\nu_2(\sigma, x_2) = \left(\frac{bx_2\sqrt{\sigma\pi} + 2\sigma abx_2}{2} + \frac{b\sqrt{\pi}}{4\sqrt{\sigma}(a - x_2)} + 3ab\sqrt{\sigma\pi} + \frac{2b\sqrt{\sigma}x_2}{\sqrt{\pi}(a - x_2)} + \frac{4b}{\sqrt{\pi\sigma}(a - x_2)^2}\right),$$

and $\varphi_1(\sigma, x_2))$ and $\varphi_2(\sigma, x_2))$ are given by formulae (2.6) and (2.7).

Proof. By the Green formula we have:

$$U(x) = \int_S \left(\frac{\partial U}{\partial n}\Phi_\sigma(x, y) - U(y)\frac{\partial \Phi_\sigma(x, y)}{\partial n}\right) dS_y + \int_T \left(\frac{\partial U}{\partial n}\Phi_\sigma(x, y) - U(y)\frac{\partial \Phi_\sigma(x, y)}{\partial n}\right) dS_y. \tag{3.4}$$

Condition (1.2) and inequality (3.1) imply

$$\begin{aligned}|U(x)| \leqslant &\left|\int_S \left(g(y)\Phi_\sigma(x, y) - f(y)\frac{\partial \Phi_\sigma(x, y)}{\partial n}\right) dS_y\right| \\ &+ \left|\int_T \left(\frac{\partial U}{\partial n}\Phi_\sigma(x, y) - U(y)\frac{\partial \Phi_\sigma(x, y)}{\partial n}\right) dS_y\right| \\ \leqslant &\delta\, |U_\sigma(x)| + M\left(\int_T |\Phi_\sigma(x, y)|\, dS_y + \int_T \left|\frac{\partial \Phi_\sigma(x, y)}{\partial n}\right| dS_y\right) \\ \leqslant &\delta\, |U_\sigma(x)| + M\psi_2(\sigma)e^{-\sigma x_2^2}.\end{aligned}$$

Here we have employed the estimate

$$M\left(\int_T |\Phi_\sigma(x, y)|\, dS_y + \int_T \left|\frac{\partial \Phi_\sigma(x, y)}{\partial n}\right| dS_y\right) \leqslant M\psi_2(\sigma)e^{-\sigma x_2^2},$$

which was proved in work [9], where $\psi_2(\sigma)$ is defined by formula (2.5). Taking into consideration (1.4) and passing to the polar coordinates, we find:

$$\int\limits_S |\Phi_\sigma(x,y)|\,dS_y = \int\limits_S \left| \frac{1}{2\pi} e^{-\sigma(\alpha^2+x_2^2-y_2^2)} \left(\int\limits_0^{+\infty} \frac{e^{-\sigma u^2}\cos 2\sigma y_2\sqrt{u^2+\alpha^2}udu}{u^2+r^2} \right.\right.$$

$$\left.\left. - \int\limits_0^{+\infty} \frac{e^{-\sigma u^2}(y_2-x_2)\sin 2\sigma y_2\sqrt{u^2+\alpha^2}}{u^2+r^2} \frac{udu}{\sqrt{u^2+\alpha^2}} \right) \right| dS_y$$

$$\leqslant \frac{b}{2\pi} e^{\sigma a^2-\sigma x_2^2} \int\limits_{a_1}^{b_1} dy_1 \int\limits_0^{+\infty} \frac{ue^{-\sigma(u^2+(y_1-x_1)^2)}}{u^2+r^2} du$$

$$+ \frac{b}{2\pi} e^{\sigma a^2-\sigma x_2^2} \int\limits_{a_1}^{b_1} dy_1 \int\limits_0^{+\infty} \frac{u\,|y_2-x_2|\,\left|\sin 2\sigma y_2\sqrt{u^2+\alpha^2}\right| e^{-\sigma(u^2+(y_1-x_1)^2)}}{(u^2+r^2)\sqrt{u^2+\alpha^2}} du$$

$$\leqslant \frac{b}{4\pi} e^{\sigma a^2-\sigma x_2^2} \int\limits_{-\infty}^{+\infty} dy_1 \int\limits_{-\infty}^{+\infty} \frac{e^{-\sigma(u^2+(y_1-x_1)^2)}}{\sqrt{u^2+(y_1-x_1)^2}} du$$

$$+ \frac{ab\sigma}{\pi} e^{\sigma a^2-\sigma x_2^2} \int\limits_{-\infty}^{+\infty} dy_1 \int\limits_{-\infty}^{+\infty} e^{-\sigma(u^2+(y_1-x_1)^2)} du \leqslant e^{\sigma a^2-\sigma x_2^2} \left(\frac{b\sqrt{\pi}}{4\sqrt{\sigma}} + ab \right).$$

Since

$$|\sin x| \leqslant \frac{2|x|}{1+|x|}, \qquad x \geqslant 0,$$

we have

$$\left|\sin 2\sigma y_2\sqrt{u^2+\alpha^2}\right| \leqslant \frac{4|\sigma y_2\sqrt{u^2+\alpha^2}|}{1+|2\sigma y_2\sqrt{u^2+\alpha^2}|}.$$

Taking into consideration the formulae

$$\frac{\partial\Phi_\sigma(x,y)}{\partial n} = \frac{\partial\Phi_\sigma(x,y)}{\partial y_1}\cos\gamma + \frac{\partial\Phi_\sigma(x,y)}{\partial y_2}\sin\gamma$$

and (2.15), (2.16) and reproducing the arguing in the proof of Theorem 2.1, we obtain:

$$\int\limits_S \left|\frac{\partial\Phi_\sigma(x,y)}{\partial y_1}\right| dS_y \leqslant \left(\frac{b}{4\sqrt{\pi\sigma}(a-x_2)} + \frac{\sqrt{\sigma}ab}{\sqrt{\pi}} \right) e^{\sigma a^2-\sigma x_2^2},$$

$$\int\limits_S \left|\frac{\partial\Phi_\sigma(x,y)}{\partial y_2}\right| dS_y \leqslant \left(\frac{b}{4\sqrt{\pi\sigma}(a-x_2)} + \frac{\sqrt{\sigma}ab}{\sqrt{\pi}} \right) e^{\sigma a^2-\sigma x_2^2}.$$

Summing up the obtained estimate, we get:

$$\int\limits_S \left|\frac{\partial\Phi_\sigma(x,y)}{\partial n}\right| dS_y \leqslant \left(\frac{b}{2\sqrt{\pi\sigma}(a-x_2)} + \frac{2\sqrt{\sigma}ab}{\sqrt{\pi}} \right) e^{\sigma a^2-\sigma x_2^2}.$$

It follows from integral formula (3.4) and condition (2.2), we obtain:

$$\begin{aligned}|U(x)| \leqslant & \delta \int\limits_S \left(|\Phi_\sigma(x,y)| + \left|\frac{\partial \Phi_\sigma(x,y)}{\partial n}\right|\right) dS_y + M \int\limits_T \left(|\Phi_\sigma(x,y)| + \left|\frac{\partial \Phi_\sigma(x,y)}{\partial n}\right|\right) dS_y \\ \leqslant & \delta e^{\sigma a^2 - \sigma x_2^2} \left(\frac{b\sqrt{\pi}}{4\sqrt{\sigma}} + ab + \frac{b}{2\sqrt{\pi\sigma}(a-x_2)} + \frac{\sqrt{\sigma}ab}{2\sqrt{\pi}}\right) \\ & + Me^{-\sigma x_2^2}\left(\frac{1}{2} + \frac{\sqrt{\pi}}{4\sqrt{\sigma}}\right) = \psi(\sigma)(Me^{-\sigma x_2^2} + \delta e^{\sigma a^2 - \sigma x_2^2}),\end{aligned} \tag{3.5}$$

The best possible estimate for the function $|U(x)|$ is obtained in the case, when

$$Me^{-\sigma x_2^2} = \delta e^{\sigma a^2 - \sigma x_2^2}$$

or

$$\sigma = \frac{1}{a^2} \ln \frac{M}{\delta}. \tag{3.6}$$

Substituting the expression for σ from identity (3.6) into (3.5), we arrive at inequality (3.2), see [6].

We proceed to proving inequality (3.3) for $i = 1$. In order to do this, we find the derivative with respect to the variable x_1 from integral formula (3.4):

$$\begin{aligned}\frac{\partial U(x)}{\partial x_1} = & \int\limits_S \left(\frac{\partial U}{\partial n}\frac{\partial \Phi_\sigma(x,y)}{\partial x_1} - U(y)\frac{\partial}{\partial x_1}\frac{\partial \Phi_\sigma(x,y)}{\partial n}\right) dS_y \\ & + \int\limits_T \left(\frac{\partial U}{\partial n}\frac{\partial \Phi_\sigma(x,y)}{\partial x_1} - U(y)\frac{\partial}{\partial x_1}\frac{\partial \Phi_\sigma(x,y)}{\partial n}\right) dS_y \\ = & \frac{\partial U_\sigma(x)}{\partial x_1} + \int\limits_T \left(\frac{\partial U}{\partial n}\frac{\partial \Phi_\sigma(x,y)}{\partial x_1} - U(y)\frac{\partial}{\partial x_1}\frac{\partial \Phi_\sigma(x,y)}{\partial n}\right) dS_y.\end{aligned} \tag{3.7}$$

Here

$$\frac{\partial U_\sigma(x)}{\partial x_1} = \int_S \left(\frac{\partial U}{\partial n}\frac{\partial \Phi_\sigma(x,y)}{\partial x_1} - U(y)\frac{\partial}{\partial x_1}\frac{\partial \Phi_\sigma(x,y)}{\partial n}\right) dS_y. \tag{3.8}$$

Proceeding with estimating, we obtain:

$$\begin{aligned}\left|\frac{\partial U(x)}{\partial x_1}\right| \leqslant & \left|\frac{\partial U_\sigma(x)}{\partial x_1}\right| + M \int\limits_T \left(\left|\frac{\partial \Phi_\sigma(x,y)}{\partial x_1}\right| + \left|\frac{\partial}{\partial x_1}\left(\frac{\partial \Phi_\sigma(x,y)}{\partial n}\right)\right|\right) dS_y \\ \leqslant & \left|\frac{\partial U_\sigma(x)}{\partial x_1}\right| + M\varphi_1(\sigma, x_2)e^{-\sigma x_2^2}.\end{aligned}$$

This estimate follows Theorem 2.1, where $\varphi_1(\sigma, x_2)$ is defined by formula (2.6). In view of (2.10), in identity (3.8), on the part S of the boundary of the domain G we have:

$$\begin{aligned}& \int\limits_S \left|\frac{\partial \Phi_\sigma(x,y)}{\partial x_1}\right| dS_y \\ & \leqslant e^{\sigma a^2 - \sigma x_2^2}\left(\frac{b + 3ab\sqrt{\pi\sigma}}{4} + \frac{2\sigma ab + a^2 b\sigma\sqrt{\pi}}{\sqrt{\pi}} + \frac{b\sqrt{\pi}}{4\sqrt{\sigma}(a-x_2)} + \frac{ab\sqrt{\pi\sigma}}{(a-x_2)^2}\right),\end{aligned} \tag{3.9}$$

$$\int\limits_S \left|\frac{\partial}{\partial x_1}\left(\frac{\partial \Phi_\sigma(x,y)}{\partial n}\right)\right| dS_y \leqslant \frac{2b}{\pi} e^{\sigma a^2 - \sigma x_2^2} \int\limits_{a_1}^{b_1} \frac{\sigma\,|y_1 - x_1|^2}{r^2 e^{\sigma(y_1-x_1)^2}} dy_1$$
$$+ \frac{b}{\pi} e^{\sigma a^2 - \sigma x_2^2} \left(\int\limits_{a_1}^{b_1} \frac{2\sigma\,|y_1 - x_1|\,|y_2 - x_2|}{r^2 e^{\sigma(y_1-x_1)^2}} dy_1 + \int\limits_{a_1}^{b_1} \frac{e^{-\sigma(y_1-x_1)^2}}{r^2} dy_1\right)$$
$$+ \frac{2b\sigma a}{\pi} e^{\sigma a^2 - \sigma x_2^2} \left(\int\limits_{a_1}^{b_1} \frac{|y_1 - x_1|}{r^2 e^{\sigma(y_1-x_1)^2}} dy_1 + \int\limits_{a_1}^{b_1} \frac{|y_2 - x_2|}{r^2 e^{-\sigma(y_1-x_1)^2}} dy_1\right) \qquad (3.10)$$
$$+ \frac{2b}{\pi} e^{\sigma a^2 - \sigma x_2^2} \left(\int\limits_{a_1}^{b_1} \frac{|y_1 - x_1|^2}{r^4 e^{\sigma(y_1-x_1)^2}} dy_1 + \int\limits_{a_1}^{b_1} \frac{|y_1 - x_1|\,|y_2 - x_2|}{r^4 e^{\sigma(y_1-x_1)^2}} dy_1\right)$$
$$\leqslant e^{\sigma a^2 - \sigma x_2^2}\left(\frac{4b\sqrt{\sigma}}{\sqrt{\pi}} + \frac{2ab\sqrt{\sigma}}{\sqrt{\pi}(a-x_2)} + \frac{5b}{\sqrt{\pi\sigma}(a-x_2)^2}\right).$$

By the assumptions of Theorems 2.1 and 3.1, as well as by (3.9), (3.10), it follows from integral formula (3.7) that

$$\left|\frac{\partial U(x)}{\partial x_1}\right| \leqslant \delta \int\limits_S \left(\left|\frac{\partial \Phi_\sigma(x,y)}{\partial x_1}\right| + \left|\frac{\partial}{\partial x_1}\left(\frac{\partial \Phi_\sigma(x,y)}{\partial n}\right)\right|\right) dS_y$$
$$+ M \int\limits_T \left(\left|\frac{\partial \Phi_\sigma(x,y)}{\partial x_1}\right| + \left|\frac{\partial}{\partial x_1}\left(\frac{\partial \Phi_\sigma(x,y)}{\partial n}\right)\right|\right) dS_y$$
$$\leqslant \delta \left(e^{\sigma a^2 - \sigma x_2^2}\left(\frac{b + 3ab\sqrt{\pi\sigma}}{4} + \frac{2\sigma ab + a^2 b\sigma\sqrt{\pi}}{\sqrt{\pi}}\right.\right.$$
$$+ \frac{b\sqrt{\pi}}{4\sqrt{\sigma}(a-x_2)} + \frac{ab\sqrt{\pi\sigma}}{(a-x_2)^2} + \frac{4b\sqrt{\sigma}}{\sqrt{\pi}}$$
$$\left.\left.+ \frac{2ab\sqrt{\sigma}}{\sqrt{\pi}(a-x_2)} + \frac{5b}{\sqrt{\pi\sigma}(a-x_2)^2}\right)\right)$$
$$+ M\left(e^{-\sigma x_2^2}\left(\frac{1}{2} + \frac{\sqrt{\pi}}{4\sqrt{\sigma}x_2} + \frac{\sqrt{\sigma}}{2\sqrt{\pi}} + \frac{1}{2x_2^2\sqrt{\pi\sigma}}\right)\right)$$
$$\leqslant \mu_1(\sigma, x_2)(\delta e^{\sigma a^2 - \sigma x_2^2} + M e^{-\sigma x_2^2}).$$

Choosing here $\sigma = \frac{1}{a^2} \ln \frac{M}{\delta}$, we arrive at inequality (3.3) for $i = 1$:

$$\left|\frac{\partial U(x)}{\partial x_1}\right| \leqslant 2\mu_1(\sigma, x_2) M^{1 - \frac{x_2^2}{a^2}} \delta^{\frac{x_2^2}{a^2}}.$$

We proceed to proving inequality (3.3) for $i = 2$. In order to do this, we find the derivative with respect to the variable x_2 from integral formula (3.4) and we estimate it as follows:

$$\frac{\partial U(x)}{\partial x_2} = \int\limits_S \left(\frac{\partial U}{\partial n}\frac{\partial \Phi_\sigma(x,y)}{\partial x_2} - U(y)\frac{\partial}{\partial x_2}\frac{\partial \Phi_\sigma(x,y)}{\partial n}\right) dS_y$$
$$+ \int\limits_T \left(\frac{\partial U}{\partial n}\frac{\partial \Phi_\sigma(x,y)}{\partial x_2} - U(y)\frac{\partial}{\partial x_2}\frac{\partial \Phi_\sigma(x,y)}{\partial n}\right) dS_y \qquad (3.11)$$

$$\left|\frac{\partial U(x)}{\partial x_2}\right| \leqslant \int\limits_S \left|\left(\frac{\partial U}{\partial n}\frac{\partial \Phi_\sigma(x,y)}{\partial x_2} - U(y)\frac{\partial}{\partial x_2}\frac{\partial \Phi_\sigma(x,y)}{\partial n}\right)\right| dS_y + M\int\limits_T \left(\left|\frac{\partial \Phi_\sigma(x,y)}{\partial x_2}\right| + \left|\frac{\partial}{\partial x_2}\frac{\partial \Phi_\sigma(x,y)}{\partial n}\right|\right) dS_y \leqslant \left|\frac{\partial U_\sigma(x)}{\partial x_2}\right| + M\varphi_2(\sigma,x_2)e^{-\sigma x_2^2}.$$

Here

$$\left|\frac{\partial U(x)}{\partial x_2}\right| = \int\limits_S \left|\left(\frac{\partial U}{\partial n}\frac{\partial \Phi_\sigma(x,y)}{\partial x_2}\right| + \left|U(y)\frac{\partial}{\partial x_2}\frac{\partial \Phi_\sigma(x,y)}{\partial n}\right|\right) dS_y.$$

The estimates for the second integrals are implied by Theorem 2.1, where $\varphi_2(\sigma, x_2)$ is defined by formula (2.7). Taking (2.20) into consideration on the part S of the boundary of the domain G, we obtain:

$$\int\limits_S \left|\frac{\partial \Phi_\sigma(x,y)}{\partial x_2}\right| dS_y \leqslant b e^{\sigma a^2 - \sigma x_2^2}\left(\frac{x_2\sqrt{\sigma\pi} + 2\sigma a x_2}{2} + \frac{\sqrt{\pi}}{4\sqrt{\sigma}(a-x_2)} + 3a\sqrt{\sigma\pi}\right). \tag{3.12}$$

In the same way we obtain:

$$\begin{aligned}\int\limits_S \left|\frac{\partial}{\partial x_2}\frac{\partial \Phi_\sigma(x,y)}{\partial n}\right| dS_y \leqslant{}& \frac{2bx_2}{\pi}e^{\sigma a^2-\sigma x_2^2}\int\limits_{a_1}^{b_1}\frac{\sigma\,|y_1-x_1|}{r^2 e^{\sigma(y_1-x_1)^2}}dy_1 \\ &+ \frac{2b}{\pi}e^{\sigma a^2-\sigma x_2^2}\left(\int\limits_{a_1}^{b_1}\frac{|y_1-x_1|\,|y_2-x_2|}{r^4 e^{\sigma(y_1-x_1)^2}}dy_1 + x_2\int\limits_{a_1}^{b_1}\frac{\sigma\,|y_2-x_2|}{r^2 e^{\sigma(y_1-x_1)^2}}dy_1\right) \\ &+ \frac{b}{\pi}e^{\sigma a^2-\sigma x_2^2}\left(\int\limits_{a_1}^{b_1}\frac{e^{-\sigma(y_1-x_1)^2}}{r^2}dy_1 + 2\int\limits_{a_1}^{b_1}\frac{|y_2-x_2|^2}{r^4 e^{\sigma(y_1-x_1)^2}}dy_1\right).\end{aligned} \tag{3.13}$$

For (3.13), an estimate holds:

$$\int\limits_S \left|\frac{\partial}{\partial x_2}\frac{\partial \Phi_\sigma(x,y)}{\partial n}\right| dS_y \leqslant e^{\sigma a^2-\sigma x_2^2}\left(\frac{2b\sqrt{\sigma}x_2}{\sqrt{\pi}(a-x_2)} + \frac{4b}{\sqrt{\pi\sigma}(a-x_2)^2}\right). \tag{3.14}$$

Now inequality (3.3) for $i = 2$ follows estimate for integral formula (3.11) in view of the assumptions of Theorems 2.1 and 3.1 and (3.12), (3.14):

$$\begin{aligned}\left|\frac{\partial U(x)}{\partial x_2}\right| \leqslant{}& \delta\int\limits_S \left(\left|\frac{\partial \Phi_\sigma(x,y)}{\partial x_2}\right| + \left|\frac{\partial}{\partial x_2}\frac{\partial \Phi_\sigma(x,y)}{\partial n}\right|\right) dS_y \\ &+ M\int\limits_T \left(\left|\frac{\partial \Phi_\sigma(x,y)}{\partial x_2}\right| + \left|\frac{\partial}{\partial x_2}\frac{\partial \Phi_\sigma(x,y)}{\partial n}\right|\right) dS_y \\ \leqslant{}& \delta e^{\sigma a^2-\sigma x_2^2}\left(\frac{bx_2\sqrt{\sigma\pi} + 2\sigma a b x_2}{2} + \frac{b\sqrt{\pi}}{4\sqrt{\sigma}(a-x_2)}\right. \\ &\left. + 3ab\sqrt{\sigma\pi} + \frac{2b\sqrt{\sigma}x_2}{\sqrt{\pi}(a-x_2)} + \frac{4b}{\sqrt{\pi\sigma}(a-x_2)^2}\right) \\ &+ Me^{-\sigma x_2^2}\left(\frac{\sqrt{\pi\sigma}x_2}{2} + \frac{\sqrt{\pi}}{2x_2\sqrt{\sigma}} + \frac{\sqrt{\sigma}}{\sqrt{\pi}} + \frac{3}{2x_2^2\sqrt{\pi\sigma}}\right)\end{aligned}$$

$$\leqslant \mu_2(\sigma, x_2)(\delta e^{\sigma a^2 - \sigma x_2^2} + M e^{-\sigma x_2^2}).$$

Choosing here $\sigma = \frac{1}{a^2} \ln \frac{M}{\delta}$, we arrive at inequality (3.3), that is,

$$\left| \frac{\partial U(x)}{\partial x_2} \right| \leqslant 2\mu_2(\sigma, x_2) M^{1-\frac{x_2^2}{a^2}} \delta^{\frac{x_2^2}{a^2}}.$$

The proof is complete. □

We let

$$U_{\sigma\delta}(x) = \int\limits_S \left(g_\delta(y)\Phi_\sigma(x,y) - f_\delta(y) \frac{\partial \Phi_\sigma(x,y)}{\partial n} \right) dS_y. \tag{3.15}$$

Theorem 3.2. *Let the function $U(y) \in E$ satisfies condition (1.2) on S and instead of the functions $f(y)$, $g(y)$, their approximations $f_\delta(y)$ and $g_\delta(y)$ are given with a prescribed error $\delta > 0$:*

$$\max_S |f(y) - f_\delta(y)| < \delta, \qquad \max_S |g(y) - g_\delta(y)| < \delta. \tag{3.16}$$

Then for each $x \in G$ and $\sigma > 0$ the estimate hold:

$$|U(x) - U_{\sigma\delta}(x)| \leqslant 2\psi(\sigma) M^{1-\frac{x_2^2}{a^2}} \delta^{\frac{x_2^2}{a^2}}, \tag{3.17}$$

$$\left| \frac{\partial U(x)}{\partial x_i} - \frac{\partial U_{\sigma\delta}(x)}{\partial x_i} \right| \leqslant 2\mu_i(\sigma, x_2) M^{1-\frac{x_2^2}{a^2}} \delta^{\frac{x_2^2}{a^2}}, \qquad 0 < \delta \leqslant M e^{-\sigma a^2}, \qquad i = 1, 2. \tag{3.18}$$

Proof. By (3.4), (3.7), (3.11) and (3.15) we get:

$$|U(x) - U_{\sigma\delta}(x)| \leqslant |I_\sigma(x)| + \delta \int\limits_S \left(|\Phi_\sigma(x,y)| + \left| \frac{\partial \Phi_\sigma(x,y)}{\partial n} \right| \right) dS_y,$$

$$\left| \frac{\partial U(x)}{\partial x_i} - \frac{\partial U_{\sigma\delta}(x)}{\partial x_i} \right| \leqslant |I_{i\sigma}(x)| + \delta \int\limits_S \left(\left| \frac{\partial \Phi_\sigma(x,y)}{\partial x_i} \right| + \left| \frac{\partial}{\partial x_i} \frac{\partial \Phi_\sigma(x,y)}{\partial n} \right| \right) dS_y, \quad i = 1, 2.$$

Taking into consideration the estimates

$$|U(x)| \leqslant \psi(\sigma)(M e^{-\sigma x_2^2} + \delta e^{\sigma a^2 - \sigma x_2^2}),$$

$$\left| \frac{\partial U(x)}{\partial x_1} \right| \leqslant \mu_1(\sigma, x_2)(M e^{-\sigma x_2^2} + \delta e^{\sigma a^2 - \sigma x_2^2}),$$

$$\left| \frac{\partial U(x)}{\partial x_2} \right| \leqslant \mu_2(\sigma, x_2)(M e^{-\sigma x_2^2} + \delta e^{\sigma a^2 \sigma x_2^2}),$$

and choosing $\sigma = \frac{1}{a^2} \ln \frac{M}{\delta}$, we apply Theorems 2.1 and 3.1 and this complete the proof. □

Corollary 3.1. *For each $x \in G$, the identity holds:*

$$\lim_{\delta \to 0} U_{\sigma\delta}(x) = U(x), \qquad \lim_{\delta \to 0} \frac{\partial U_{\sigma\delta}(x)}{\partial x_i} = \frac{\partial U(x)}{\partial x_i}, \qquad i = 1, 2.$$

Corollary 3.2. *If $x \in \overline{G}_\varepsilon$, then the family of the functions $\{U_{\sigma\delta}(x)\}$ and $\left\{ \frac{\partial U_{\sigma\delta}(x)}{\partial x_i} \right\}$ converges uniformly as $\delta \to 0$:*

$$U_{\sigma\delta}(x) \rightrightarrows U(x), \qquad \frac{\partial U_{\sigma\delta}(x)}{\partial x_i} \rightrightarrows \frac{\partial U(x)}{\partial x_i}, \qquad i = 1, 2.$$

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