ISSN 2304-0122 Ufa Mathematical Journal. Vol. 11. No 4 (2019). P. 131-139.

doi:10.13108/2019-11-4-131

SOME RELATIONS FOR UNIVERSAL BERNOULLI POLYNOMIALS

M.C. DAĞLI

Abstract. In this paper, we consider a generalization of the Bernoulli polynomials, which we call universal Bernoulli polynomials. They are related to the Lazard universal formal group. The corresponding numbers by construction coincide with the universal Bernoulli numbers. They turn out to have an important role in complex cobordism theory. They also obey generalizations of the celebrated Kummer and Clausen–von Staudt congruences.

We derive a formula on the integral of products of higher-order universal Bernoulli polynomials. As an application of this formula, the Laplace transform of periodic universal Bernoulli polynomials is presented. Moreover, we obtain the Fourier series expansion of higher-order universal Bernoulli function.

Keywords: Bernoulli polynomials and numbers, formal group, integrals, Fourier series.

Mathematics Subject Classification 2010: 11B68; 55N22; 42A16

1. Introduction

The higher-order Bernoulli polynomials $B_n^{(\alpha)}(x)$ are usually defined by means of the following generating function:

$$\left(\frac{t}{e^t-1}\right)^{\alpha} e^{xt} = \sum_{n=0}^{\infty} B_n^{(\alpha)}(x)\frac{t^n}{n!}.$$

For $\alpha = 1$, they are reduced to classical Bernoulli polynomials $B_n(x)$. The rational numbers $B_n = B_n(0)$ are called classical Bernoulli numbers.

As it is well known, the Bernoulli polynomials play important roles in different areas of mathematics such as number theory, combinatorics, special functions and analysis.

This paper is primarily concerned with a very large class of Bernoulli polynomials. Let us consider the formal logarithm group defined over the polynomial ring $\mathbb{Q}[c_1, c_2, \ldots]$ and the formal power series

$$F(s) = s + c_1\frac{s^2}{2} + c_2\frac{s^3}{3} + \ldots.$$

Let $G(t)$ be its compositional inverse (the formal exponential group):

$$G(t) = t - c_1\frac{t^2}{2} + \left(3c_1^2 - 2c_2\right)\frac{t^3}{6} + \ldots$$

Some relations for universal Bernoulli polynomials.

©Dağli M.C. 2019.

This work was supported by The Scientific Research Projects Coordination Unit of Akdeniz University. Project Number: FBA-2018-3593.

Submitted January 28, 2019.

The universal higher-order Bernoulli polynomials $B^G_{n,\alpha}(x)$ introduced in [13] and later discussed in [11] and [14] are defined as

$$\left(\frac{t}{G(t)}\right)^{\alpha} e^{xt} = \sum_{n=0}^{\infty} B^G_{n,\alpha}(x)\frac{t^n}{n!} \qquad x,\alpha \in \mathbb{R}, \qquad \alpha \neq 0. \tag{1.1}$$

As $\alpha = 1$ and $c_i = (-1)^i$, we have $F(s) = \log(1+s)$, $G(t) = e^t - 1$ and the universal higher-order Bernoulli polynomials and numbers reduce to the classical ones. The numbers $B^G_{n,1}(0)$ coincide with the universal Bernoulli numbers defined by Clarke in [4]. For brevity, we denote $B^G_{n,1}(x) = B^G_n(x)$ and $B^G_{n,\alpha}(0) = B^G_{n,\alpha}$.

Because of the generality of definition of universal Bernoulli polynomials, they do not provide many fundamental properties unlike classical case. For example, the higher-order Bernoulli polynomial $B^{(\alpha)}_n(x)$ is skew symmetric about $x = \frac{1}{2}$, but there is no symmetry in the universal case, and no obvious root as $n > 1$ and n is odd.

By construction, for any choice of the sequences $\{c_n\}_{n\in\mathbb{N}}$ they represent a class of Appell polynomials. It means that these polynomials possess the following differential property

$$\frac{d}{dx}B^G_{n,\alpha}(x) = nB^G_{n-1,\alpha}(x) \qquad \text{as} \quad n \in \mathbb{N}\setminus\{0\}. \tag{1.2}$$

The universal Bernoulli numbers provide extensions of the celebrated Kummer and Clausen–von Staudt congruences [1] and general Almkvist–Meurman–type congruences [14]. In [12], it was shown that interesting realizations of polynomials (1.1) can be constructed by means of finite operator theory introduced by G.C. Rota [10]. Relations between the universal formal group and the theory of L-series were established in [11, 13].

In this paper, we deduce an explicit formula for the following type integral

$$\int_0^x B^G_{n_1,\alpha_1}(b_1 z + y_1)\cdots B^G_{n_r,\alpha_r}(b_r z + y_r)\,dz.$$

Since this formula is also valid for the Appell polynomials, the earlier results obtained by Liu, Pan and Zhang [9], Hu, Kim and Kim [7], Agoh and Dilcher [3] are direct consequences of the derived formula. As an application of our formula, we present the Laplace transform of periodic universal Bernoulli polynomials. Furthermore, we obtain the Fourier series of higher-order universal Bernoulli function $\overline{B}^G_{m,r}(x)$.

2. Results

2.1. Integral of products of r universal Bernoulli polynomials. In this subsection we describe the integral of products of several universal Bernoulli polynomials.

Theorem 2.1. *Let $b_1,\ldots,b_r$, $y_1,\ldots,y_r$ be arbitrary real numbers obeying $b_s \neq 0$, $1 \leqslant s \leqslant r$, and*

$$\begin{aligned}
\widehat{I}_{n_1,\ldots,n_r}(x;b;y) &= \widehat{I}_{n_1,\ldots,n_r}(x;b_1,\ldots,b_r;y_1,\ldots,y_r) \\
&= \frac{1}{n_1!\cdots n_r!}\int_0^x \prod_{s=1}^{r} B^G_{n_s,\alpha_s}(b_s z + y_s)\,dz, \\
\widehat{C}_{n_1,\ldots,n_r}(x;b;y) &= \widehat{C}_{n_1,\ldots,n_r}(x;b_1,\ldots,b_r;y_1,\ldots,y_r) \\
&= \frac{1}{n_1!\cdots n_r!}\left(\prod_{s=1}^{r} B^G_{n_s,\alpha_s}(b_s x + y_s) - \prod_{s=1}^{r} B^G_{n_s,\alpha_s}(y_s)\right).
\end{aligned}$$

Then,

$$\begin{aligned}\widehat{I}_{n_1,\ldots,n_r}(x;b;y) =& \sum_{a=0}^{\mu}(-1)^a \sum_{j_1+\cdots+j_{r-1}=a}\binom{a}{j_1,\ldots,j_{r-1}} b_1^{j_1}\cdots b_{r-1}^{j_{r-1}} b_r^{-a-1} \\ & \cdot \widehat{C}_{n_1-j_1,\ldots,n_{r-1}-j_{r-1},n_r+a+1}(x;b;y) \\ & + \frac{(-1)^{\mu+1}}{(n+\mu+1)!}\int_0^x \left(\prod_{s=1}^{r-1} B^G_{n_s,\alpha_s}(b_s z+y_s)\right)^{(\mu+1)} B^G_{n_r+\mu+1,\alpha_r}(b_r z+y_r)\,dz,\end{aligned}$$

where $\binom{\mu}{n_1,\ldots,n_r}$ *are the multinomial coefficients defined by*

$$\binom{\mu}{n_1,\ldots,n_r} = \frac{\mu!}{n_1!\cdots n_r!}, \qquad n_1+\cdots+n_r=\mu \ \ and n_1,\ldots,n_r \geqslant 0.$$

In particular, if $\mu = n_1+\cdots+n_{r-1}$*, we have*

$$\begin{aligned}\widehat{I}_{n_1,\ldots,n_r}(x;b;y) =& \sum_{a=0}^{\mu}(-1)^a \sum_{j_1+\cdots+j_{r-1}=a}\binom{a}{j_1,\ldots,j_{r-1}} b_1^{j_1}\cdots b_{r-1}^{j_{r-1}} \\ & \cdot b_r^{-a-1}\widehat{C}_{n_1-j_1,\ldots,n_{r-1}-j_{r-1},n_r+a+1}(x;b;y).\end{aligned} \tag{2.1}$$

Proof. Let

$$f(z) = B^G_{n_1,\alpha_1}(b_1 z+y_1)\cdots B^G_{n_{r-1},\alpha_{r-1}}(b_{r-1}z+y_{r-1}).$$

We have

$$\begin{aligned}\frac{1}{n_r!}\int_0^x f(z)B^G_{n_r,\alpha_r}(b_r z+y_r)\,dz =& \left[\frac{1}{b_r(n_r+1)!} f(z)B^G_{n_r+1,\alpha_r}(b_r z+y_r)\right]_0^x \\ & - \frac{1}{(n_r+1)!}\int_0^x f'(z)B^G_{n_{r+1},\alpha_r}(b_r z+y_r)\,dz.\end{aligned}$$

Integrating μ times by parts, we get:

$$\begin{aligned}\frac{1}{n_r!}\int_0^x f(z)B^G_{n_r,\alpha_r}(b_r z+y_r)\,dz =& \sum_{a=0}^{\mu}\frac{(-1)^a}{(n_r+a+1)!}\left[f^{(a)}(z)B^G_{n_r+a+1,\alpha_r}(b_r z+y_r)\right]_0^x \\ & + \frac{(-1)^{\mu+1}}{(n_r+\mu+1)!}\int_0^x f^{(\mu+1)}(z)B^G_{n_r+\mu+1,\alpha_r}(b_r z+y_r)\,dz.\end{aligned} \tag{2.2}$$

By virtue of the property of a derivative

$$(f_1(z)\cdots f_m(z))^{(a)} = \sum_{j_1+\cdots+j_m=a}\binom{a}{j_1,\ldots,j_m} f_1{}^{(j_1)}(z)\cdots f_m^{(j_m)}(z),$$

and (1.2), we arrive at the statement of the theorem. The proof is complete. □

As a consequence of this theorem, we have the following reciprocity relation for Bernoulli polynomials, which generalizes [3, Prop. 2].

Corollary 1. *For all $n, m \geqslant 0$, we have*

$$\begin{aligned}
&\sum_{a=0}^{n}(-1)^{a}\binom{m+n+1}{n-a}b_1^a b_2^{-a-1}B^G_{n-a,\gamma}(b_1x+y_1)\,B^G_{m+a+1,\beta}(b_2x+y_2)\\
&-\sum_{a=0}^{m}(-1)^{a}\binom{m+n+1}{m-a}b_2^a b_1^{-a-1}B^G_{m-a,\gamma}(b_2x+y_2)\,B^G_{n+a+1,\beta}(b_1x+y_1)\\
=&\frac{1}{b_1^{m+1}b_2^{n+1}}\sum_{a=0}^{m+n+1}(-1)^{m+1-a}\binom{m+n+1}{a}b_1^a b_2^{m+n+1-a}B^G_{m+n+1-a,\gamma}(y_1)\,B^G_{a,\beta}(y_2),
\end{aligned}\tag{2.3}$$

where b_l $(b_l \neq 0)$ and y_l $(1 \leqslant l \leqslant r)$ are real numbers.

Proof. In view of the definition of the integral $\widehat{I}_{n_1,\ldots,n_r}(x;b;y)$, we see that the left hand side of (2.1) is invariant under interchanging the order of the integrands. We begin by writing the left hand side of (2.1) for $r=2$:

$$\begin{aligned}
&\sum_{a=0}^{n}(-1)^{a}\binom{m+n+1}{n-a}b_1^a b_2^{-a-1}\\
&\qquad\cdot\left(B^G_{n-a,\gamma}(b_1x+y_1)\,B^G_{m+a+1,\beta}(b_2x+y_2)-B^G_{n-a,\gamma}(y_1)\,B^G_{m+a+1,\beta}(y_2)\right)\\
&=\sum_{a=0}^{m}(-1)^{a}\binom{m+n+1}{m-a}b_2^a b_1^{-a-1}\\
&\qquad\cdot\left(B^G_{m-a,\gamma}(b_2x+y_2)\,B^G_{n+a+1,\beta}(b_1x+y_1)-B^G_{m-a,\gamma}(y_2)\,B^G_{n+a+1,\beta}(y_1)\right).
\end{aligned}\tag{2.4}$$

Now we can get the reciprocity relation for sums of products of universal Bernoulli polynomials as follows. Let

$$\begin{aligned}
T:=&\sum_{a=0}^{n}(-1)^{a}\binom{m+n+1}{n-a}b_1^a b_2^{-a-1}B^G_{n-a,\gamma}(y_1)\,B^G_{m+a+1,\beta}(y_2)\\
&-\sum_{a=0}^{m}(-1)^{a}\binom{m+n+1}{m-a}b_2^a b_1^{-a-1}B^G_{m-a,\gamma}(y_2)\,B^G_{n+a+1,\beta}(y_1).
\end{aligned}$$

This can be rewritten as

$$\begin{aligned}
T=&\sum_{a=0}^{n}(-1)^{n-a}\binom{m+n+1}{a}b_1^{n-a} b_2^{a-n-1}B^G_{a,\gamma}(y_1)\,B^G_{m+n+1-a,\beta}(y_2)\\
&-\sum_{a=0}^{m}(-1)^{m-a}\binom{m+n+1}{a}b_2^{m-a} b_1^{a-m-1}B^G_{a,\gamma}(y_2)\,B^G_{m+n+1-a,\beta}(y_1).
\end{aligned}\tag{2.5}$$

Without loss of generality we assume that $n \geqslant m$; in this case we split the first sum in (2.5) sum into two sums, one is from 0 to m and the other is from $m+1$ to n:

$$\begin{aligned}
&\sum_{a=0}^{m}(-1)^{n-a}\binom{m+n+1}{a}b_1^{n-a} b_2^{a-n-1}B^G_{a,\gamma}(y_1)\,B^G_{m+n+1-a,\beta}(y_2)\\
&\quad=\sum_{a=n+1}^{m+n+1}(-1)^{m+1-a}\binom{m+n+1}{a}b_1^{a-m-1} b_2^{m-a}B^G_{m+n+1-a,\gamma}(y_1)\,B^G_{a,\beta}(y_2)
\end{aligned}$$

and

$$\sum_{a=m+1}^{n}(-1)^{n-a}\binom{m+n+1}{a}b_1^{n-a} b_2^{a-n-1}B^G_{a,\gamma}(y_1)\,B^G_{m+n+1-a,\gamma}(y_2)$$

$$= \sum_{a=m+1}^{n} (-1)^{m+1-a} \binom{m+n+1}{a} b_1^{a-m-1} b_2^{m-a} B^G_{m+n+1-a,\gamma}(y_1) B^G_{a,\beta}(y_2).$$

Hence, we have

$$T = \frac{1}{b_1^{m+1} b_2^{n+1}} \sum_{a=0}^{m+n+1} (-1)^{m+1-a} \binom{m+n+1}{a} b_1^a b_2^{m+n+1-a} B^G_{m+n+1-a,\gamma}(y_1) B^G_{a,\beta}(y_2). \qquad (2.6)$$

The latter identity and (2.4) give the desired result, i.e., the reciprocity relation for sums of products of universal Bernoulli polynomials. The proof is complete. □

Remark 1. *If we start from the left-hand side of (2.3) and argue as in the proof of (2.6), the right hand side of (2.3) turns into*

$$\frac{1}{b_1^{m+1} b_2^{n+1}} \sum_{a=0}^{m+n+1} (-1)^{m+1-a} \binom{m+n+1}{a} b_1^a b_2^{m+n+1-a} B^G_{m+n+1-a,\gamma}(b_1 x + y_1) B^G_{a,\beta}(b_2 x + y_2).$$

This implies that for all x,

$$\sum_{a=0}^{m+n+1} (-1)^a m+n+1 a b_1^a b_2^{m+n+1-a} B^G_{m+n+1-a,\gamma}(b_1 x + y_1) B^G_{a,\beta}(b_2 x + y_2)$$
$$= \sum_{a=0}^{m+n+1} (-1)^a \binom{m+n+1}{a} b_1^a b_2^{m+n+1-a} B^G_{m+n+1-a,\gamma}(y_1) B^G_{a,\beta}(y_2).$$

Letting $\gamma = \beta = 1$, $y_1 = y_2 = 0$ in Corollary 1, we arrive at

$$\sum_{a=0}^{n} (-1)^a \binom{m+n+1}{n-a} b_1^a b_2^{-a-1} B^G_{n-a}(b_1 x) B^G_{m+a+1}(b_2 x)$$
$$- \sum_{a=0}^{m} (-1)^a \binom{m+n+1}{m-a} b_2^a b_1^{-a-1} B^G_{m-a}(b_2 x) B^G_{n+a+1}(b_1 x)$$
$$= \frac{1}{b_1^{m+1} b_2^{n+1}} \sum_{a=0}^{m+n+1} (-1)^{m+1-a} \binom{m+n+1}{a} b_1^a b_2^{m+n+1-a} B^G_{m+n+1-a} B^G_a.$$

Remark 2. *A relationship between the reciprocity relation of the Bernoulli polynomials or Euler polynomials and the reciprocity formulae for generalized Dedekind sum $T_r(c,d)$ or Hardy–Berndt sums $s_{3,r}(c,d)$ and $s_{4,r}(c,d)$ was established in* [5], [6]*. Furthermore, recently, the integral of the product of arbitrary many Euler polynomials was found and by using such formula, several reciprocity relations between the special values of Tornheim's multiple series were obtained by M.S. Kim* [8]*. In view of this, Corollary 1 may be related to generalized Dedekind sum of the form*

$$s_r^G(c,d) = \sum_{n=1}^{d} \overline{B}_1\left(\frac{n}{d}\right) \overline{B}_r^G\left(\frac{cn}{d}\right),$$

where $\overline{B}_r^G(x) = B_r^G(\{x\})$ and $\overline{B}_1(x)$ are universal Bernoulli function and ordinary Bernoulli function, respectively. However, this type of Dedekind sum has not been properly defined and considered yet, so it deserves a further study.

2.2. Laplace Transform. In this subsection, we find the Laplace transform of $\overline{B}_n^G(tu)$ by means of (2.2).

Let $\operatorname{Re} s) > 0$ and $|s/t| < M$, where M is constant. By setting $f(u) = e^{-su}$ and using $\overline{B}_n^G(tu)$ instead of $B_{n_r,\alpha_r}^G(u)$ in (2.2), we get

$$\begin{aligned}\frac{1}{n!}\int_0^x e^{-su}\overline{B}_n^G(tu)\,du =& \sum_{a=0}^{\mu}\frac{s^a t^{-a-1}}{(n+a+1)!}\left\{e^{-sx}\overline{B}_{n+a+1}^G(tx) - \overline{B}_{n+a+1}^G(0)\right\}\\ &+\left(\frac{s}{t}\right)^{\mu+1}\frac{1}{(n+\mu+1)!}\int_0^x e^{-su}\overline{B}_{n+\mu+1}^G(tu)\,du.\end{aligned} \tag{2.7}$$

Since the function $\overline{B}_m^G(u) = B_m^G(u-[u])$ is bounded, the integrals in (2.7) converge absolutely and $e^{-sx}\overline{B}_{n+a+1}^G(tx)$ tends zero as $x \to \infty$. Then, letting $x \to \infty$, we have

$$\begin{aligned}\frac{1}{n!}\int_0^\infty e^{-su}\overline{B}_n^G(tu)\,du =& -\frac{t^n}{s^{n+1}}\sum_{a=0}^{\mu}\frac{B_{n+a+1}^G(0)}{(n+a+1)!}\frac{s^{n+a+1}}{t^{n+a+1}}\\ &+\left(\frac{s}{t}\right)^{\mu+1}\frac{1}{(n+\mu+1)!}\int_0^\infty e^{-su}\overline{B}_{n+\mu+1}^G(tu)\,du.\end{aligned} \tag{2.8}$$

Observe that the sum in (2.8) converges absolutely for $|s/t| < M$ as $\mu \to \infty$. Also the sequence of the functions $s^\mu\overline{B}_\mu^G(tu)/\mu!t^\mu$ converges uniformly in u to zero under the assumption $G(t) \geqslant e^t - 1$. Thus, letting $\mu \to \infty$ and using (1.1), we obtain the Laplace transform of $\overline{B}_n^G(tu)$:

$$\begin{aligned}\frac{1}{n!}\int_0^\infty e^{-su}\overline{B}_n^G(tu)\,du =& -\frac{t^n}{s^{n+1}}\sum_{a=0}^{\infty}\frac{B_{n+a+1}^G(0)}{(n+a+1)!}\frac{s^{n+a+1}}{t^{n+a+1}}\\ =&\frac{t^n}{s^{n+1}}\left(\sum_{a=0}^{n}\frac{B_a^G(0)}{a!}\frac{s^a}{t^a} - \sum_{a=0}^{\infty}\frac{B_a^G(0)}{a!}\frac{s^a}{t^a}\right)\\ =&\frac{1}{s}\sum_{a=0}^{n}\frac{B_a^G(0)}{a!}\left(\frac{t}{s}\right)^{n-a} - \frac{t^n}{s^{n+1}}\frac{s/t}{G(s/t)}.\end{aligned}$$

2.3. Fourier series. This section is devoted to Fourier series of higher-order universal Bernoulli function $\overline{B}_{m,r}^G(x)$. From now on, J denotes the ideal generated by all $c_i - c_1^i$ in $\mathbb{Q}[r][c_1, c_2, \ldots, c_m]$.

The higher-order universal Bernoulli polynomials satisfy the following relation [2]

$$B_{m,r}^G(x+1) - B_{m,r}^G(x) \equiv -mc_1B_{m-1,r-1}^G(x) \pmod{J}. \tag{2.9}$$

Let us find the Fourier series expansion of $\overline{B}_{m,r}^G(x)$. We write

$$\overline{B}_{m,r}^G(x) = \sum_{n=-\infty}^{\infty} C_n^{(r,m)}e^{2\pi inx}.$$

The coefficients $C_n^{(r,m)}$ can be found as

$$C_n^{(r,m)} = \int_0^1 \overline{B}_{m,r}^G(x)e^{-2\pi inx} = \int_0^1 B_{m,r}^G(x)e^{-2\pi inx}$$

$$
\begin{aligned}
&= \left[\frac{1}{m+1} B^G_{m+1,r}(x) e^{-2\pi i n x}\right]_0^1 + \frac{2\pi i n}{m+1} \int_0^1 B^G_{m+1,r}(x) e^{-2\pi i n x} \\
&= \frac{1}{m+1}\left(B^G_{m+1,r}(1) - B^G_{m+1,r}(0)\right) + \frac{2\pi i n}{m+1} C_n^{(r,m+1)} \\
&\equiv -c_1 B^G_{m,r-1}(0) + \frac{2\pi i n}{m+1} C_n^{(r,m+1)} \qquad (\operatorname{mod} J)
\end{aligned}
$$

where we have used (2.9) for $x = 0$. Replacing m by $m-1$, we get

$$
C_n^{(r,m-1)} \equiv -c_1 B^G_{m-1,r-1}(0) + \frac{2\pi i n}{m} C_n^{(r,m)} \qquad (\operatorname{mod} J)\,.
$$

Suppose that $n \neq 0$. Then, we get a chain of equivalences mod J:

$$
\begin{aligned}
C_n^{(r,m)} &\equiv \frac{m}{2\pi i n} C_n^{(r,m-1)} + \frac{m c_1}{2\pi i n} B^G_{m-1,r-1}(0) \\
&\equiv \frac{m}{2\pi i n}\left(\frac{m-1}{2\pi i n} C_n^{(r,m-2)} + \frac{(m-1)\,c_1}{2\pi i n} B^G_{m-2,r-1}(0)\right) + \frac{m c_1}{2\pi i n} B^G_{m-1,r-1}(0) \\
&\equiv \frac{m\,(m-1)}{(2\pi i n)^2} C_n^{(r,m-2)} + \frac{m\,(m-1)\,c_1}{(2\pi i n)^2} B^G_{m-2,r-1}(0) + \frac{m c_1}{2\pi i n} B^G_{m-1,r-1}(0) \\
&\equiv \frac{m\,(m-1)}{(2\pi i n)^2}\left(\frac{m-2}{2\pi i n} C_n^{(r,m-3)} + \frac{(m-2)\,c_1}{2\pi i n} B^G_{m-3,r-1}(0)\right) \\
&+ \frac{m\,(m-1)\,c_1}{(2\pi i n)^2} B^G_{m-2,r-1}(0) + \frac{m c_1}{2\pi i n} B^G_{m-1,r-1}(0) \\
&\equiv \frac{m\,(m-1)\,(m-2)}{(2\pi i n)^3} C_n^{(r,m-3)} + \frac{m\,(m-1)\,(m-2)\,c_1}{(2\pi i n)^3} B^G_{m-3,r-1}(0) \\
&+ \frac{m\,(m-1)\,c_1}{(2\pi i n)^2} B^G_{m-2,r-1}(0) + \frac{m c_1}{2\pi i n} B^G_{m-1,r-1}(0).
\end{aligned}
$$

Proceeding as above, we obtain that

$$
C_n^{(r,m)} \equiv \frac{m\,(m-1)\,(m-2)\ldots 2}{(2\pi i n)^{m-1}} C_n^{(r,1)} + c_1 \sum_{k=1}^{m-1} \frac{(m)_k}{(2\pi i n)^k} B^G_{m-k,r-1}(0) \qquad (\operatorname{mod} J)\,, \tag{2.10}
$$

where $(x)_n = x\,(x-1)\ldots(x-n+1)$. Using the identity $B^G_{1,r}(x) = -c_1\left(x - \frac{r}{2}\right)$ given in [2], we can compute the coefficients $C_n^{(r,1)}$ in the last identity as follows:

$$
C_n^{(r,1)} = \int_0^1 \overline{B}^G_{1,r}(x) e^{-2\pi i n x} = -\int_0^1 c_1\left(x - \frac{r}{2}\right) e^{-2\pi i n x} = \frac{c_1}{2\pi i n}.
$$

Substituting this identity in (2.10) gives

$$
\begin{aligned}
C_n^{(r,m)} &\equiv \frac{m! c_1}{(2\pi i n)^m} + c_1 \sum_{k=1}^{m-1} \frac{(m)_k}{(2\pi i n)^k} B^G_{m-k,r-1}(0) \\
&\equiv c_1 \sum_{k=1}^{m} \frac{(m)_k}{(2\pi i n)^k} B^G_{m-k,r-1}(0) \qquad (\operatorname{mod} J)\,.
\end{aligned}
$$

As $n=0$, we readily see that

$$C_0^{(r,m)} = \int_0^1 \overline{B}_{m,r}^G(x) = \frac{1}{m+1}\left(B_{m+1,r}^G(1) - B_{m+1,r}^G(0)\right)$$
$$\equiv -c_1 B_{m,r-1}^G(x) \qquad (\operatorname{mod} J)\,.$$

This implies:

$$\begin{aligned}
\overline{B}_{m,r}^G(x) &= \sum_{n=-\infty}^{\infty} C_n^{(r,m)} e^{2\pi i n x} \\
&\equiv -c_1 B_{m,r-1}^G(x) + c_1 \sum_{n=-\infty}^{\infty}\left(\sum_{k=1}^{m}\frac{(m)_k}{(2\pi i n)^k} B_{m-k,r-1}^G(0)\right) e^{2\pi i n x} \\
&\equiv -c_1 B_{m,r-1}^G(x) - c_1 \sum_{k=1}^{m}\frac{(m)_k}{k!} B_{m-k,r-1}^G(0)\left(-k!\sum_{n=-\infty}^{\infty}\frac{e^{2\pi i n x}}{(2\pi i n)^k}\right) \\
&\equiv -c_1 B_{m,r-1}^G(x) - c_1 \sum_{k=2}^{m}\binom{m}{k} B_{m-k,r-1}^G(0)\overline{B}_k(x) \\
&- c_1 m B_{m-1,r-1}^G(0)\begin{cases}\overline{B}_1(x) & \text{if} x \notin \mathbb{Z}; \\ 0 & \text{if} x \in \mathbb{Z},\end{cases} \\
&\equiv \begin{cases} -c_1 \sum\limits_{k=0}^{m}\binom{m}{k} B_{m-k,r-1}^G(0)\overline{B}_k(x) & \text{if} x \notin \mathbb{Z}; \\ -c_1 \sum\limits_{\substack{k=0\\k\neq 1}}^{m}\binom{m}{k} B_{m-k,r-1}^G(0)\overline{B}_k(x) & \text{if} x \in \mathbb{Z},\end{cases} \qquad (\operatorname{mod} J)\,,
\end{aligned}$$

where we have used the Fourier series expansion of ordinary Bernoulli function $\overline{B}_m(x)$

$$\overline{B}_m(x) = -\frac{m!}{(2\pi i)^m}\sum_{\substack{n=-\infty\\n\neq 0}}^{\infty}\frac{e^{2\pi i n x}}{n^m}, \qquad m \geqslant 2,$$

and the fact

$$-\sum_{\substack{n=-\infty\\n\neq 0}}^{\infty}\frac{e^{2\pi i n x}}{2\pi i n} = \begin{cases}\overline{B}_1(x) & \text{if} x \notin \mathbb{Z}; \\ 0 & \text{if} x \in \mathbb{Z}.\end{cases}$$

Thus, we have proved the following theorem.

Theorem 2.2. *For $m, r \geqslant 2$ and $x \in (-\infty, +\infty)$, $\overline{B}_{m,r}^G(x)$ has the Fourier series expansion*

$$\overline{B}_{m,r}^G(x) \equiv -c_1 B_{m,r-1}^G(x) + c_1 \sum_{\substack{n=-\infty\\n\neq 0}}^{\infty}\left(\sum_{k=1}^{m}\frac{(m)_k}{(2\pi i n)^k} B_{m-k,r-1}^G(0)\right) e^{2\pi i n x} \qquad (\operatorname{mod} J)\,.$$

Moreover, for $x \in (-\infty, +\infty)$

$$\overline{B}_{m,r}^G(x) \equiv -c_1 \sum_{\substack{k=0\\k\neq 1}}^{m}\binom{m}{k} B_{m-k,r-1}^G(0)\overline{B}_k(x) \qquad (\operatorname{mod} J)\,,$$

where $\overline{B}_k(x)$ is the ordinary Bernoulli function.

BIBLIOGRAPHY

1. A. Adelberg. *Universal higher order Bernoulli numbers and Kummer and related congruences* // J. Number Theory. **84**:1, 119–135 (2000).
2. A. Adelberg. *Universal Bernoulli polynomials and p-adic congruences, applications of Fibonacci numbers* // in "Proceedings of the Tenth International research conference on Fibonacci numbers and their applications". Springer, Dordrecht. **9**, 1–8, (2004).
3. T. Agoh, K. Dilcher. *Integrals of products of Bernoulli polynomials* // J. Math. Anal. Appl. **381**:1, 10–16 (2011).
4. F. Clarke. *The universal von Staudt theorems* // Trans. Amer. Math. Soc. **315**:2, 591–603 (1989).
5. M.C. Dağlı, M. Can. *On reciprocity formula of character Dedekind sums and the integral of products of Bernoulli polynomials* // J. Number Theory **156**, 105–124 (2015).
6. M.C. Dağlı, M. Can. *On the integral of products of higher-order Bernoulli and Euler polynomials* // Funct. Approx. Comment. Math. **57**:1, 7–20 (2017).
7. S. Hu, D. Kim, M.S. Kim. *On the integral of the product of four and more Bernoulli polynomials* // Ramanujan J. **33**:2, 281–293 (2014).
8. M.S. Kim. *On the special values of Tornheim's multiple series* // J. Appl. Math. & Inform. **33**:3-4, 305–315 (2015).
9. J. Liu, H. Pan, Y. Zhang. *On the integral of the product of the Appell polynomials* // Integ. Transf. Spec. Funct. **25**:9, 680–685 (2014).
10. G.C. Rota. *Finite operator calculus.* Academic Press, New York (1975).
11. P. Tempesta. *L-series and Hurwitz zeta functions associated with the universal formal group.* Annali Scuola Normale Superiore, Classe di Scienze. **9**:5, 133–144 (2010).
12. P. Tempesta. *On Appell sequences of polynomials of Bernoulli and Euler type* // J. Math. Anal. Appl. **341**:2, 1295–1310 (2007).
13. P. Tempesta. *Formal groups, Bernoulli-type polynomials and L-series* // C. R. Math. Acad. Sci. Paris, Ser. I **345**:6, 303–306 (2007).
14. P. Tempesta. *The Lazard formal group, universal congruences and special values of zeta functions* // Trans. Amer. Math. Soc. **367**:10, 7015–7028 (2015).

Muhammet Cihat Dağlı,
Department of Mathematics,
Akdeniz University,
Dumlupınar Boulevard,
07058 Campus,
Antalya, Turkey
E-mail: `mcihatdagli@akdeniz.edu.tr`