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FRACTIONAL INTEGRODIFFERENTIAL EQUATIONS WITH NONLOCAL CONDITIONS AND GENERALIZED HILFER FRACTIONAL DERIVATIVE

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Abstract. We study some basic properties of the qualitative theory such as existence, uniqueness, and stability of solutions to the first-order of weighted Cauchy-type problem for nonlinear fractional integro-differential equation with nonlocal conditions involving a general form of Hilfer fractional derivative. The fractional integral and derivative of different orders are involved in the given problem and the classical integral is involved in nonlinear terms. We establish the equivalence between the weighted Cauchy-type problem and its mixed type integral equation by employing various tools and properties of fractional calculus in weighted spaces of continuous functions. The Krasnoselskii's fixed point theorem and the Banach fixed point theorem are used to obtain the existence and uniqueness of solutions of a given problem, and also the results of nonlinear analysis such as Arzila-Ascoli theorem and some special functions like Gamma function, Beta function, and Mittag-Leffler function serves as tools in our analysis. Further, the generalized Gronwall inequality is used to obtain the Ulam-Hyers, generalized Ulam-Hyers, Ulam-Hyers-Rassias, and generalized Ulam-Hyers-Rassias stability of solutions of the weighted Cauchy-type problem. In the end, we provide two examples demonstrating our main results.

Keywords: fractional integro-differential equations, nonlocal conditions ,$\psi-$Hilfer fractional derivative, existence and Ulam-Hyers stability, fixed point theorem.

Mathematics Subject Classification: 34K37, 26A33, 34A12, 47H10.

1. Introduction

A fractional calculus is an extension of the ordinary calculus to non-integer orders. The fractional calculus is more than three centuries old, but it attracted much attention in recent decades due to it is a solid and growing employing both in the theoretical and applied concepts, see [15], [17], [19], [25]. The fractional derivatives were developed in the past epoch by Riemann-Liouville, Grunwald-Letnikov, Riesz, Erdlyi-Kober, Caputo, Hadamard, Hilfer and others. In the past years, fractional differential equations appeared as rich and nice field to be studied due to their applications to the physical and life sciences and to engineering as is witnessed by blossoming literature. Many researchers worked with the fractional derivatives and the results can be found in [1], [4], [5], [8]–[13], [18], [23], [24], see also the references therein.

The properties of fractional integrals and fractional derivatives of a function with respect to another function have been introduced by Kilbas with co-authors in [17]. However, recently, in [6], Almeida have introduced a fractional differentiation operator, a so-called ψ-Caputo fractional operator. On the other hand, Hilfer [15] introduced a fractional derivative, which in particular gives the Riemann–Liouville and the Caputo fractional derivative operator. In

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[13], Furati with co-authors considered a nonlinear fractional differential equation involving Hilfer fractional derivative:

$$D^{\alpha,\beta}_{a^+}u(t) = f(t,u(t)), \qquad t>a, \quad 0<\alpha<1, \quad 0\leqslant\beta\leqslant 1, \tag{1.1}$$

$$I^{1-\gamma}_{a^+}u(a^+) = u_a, \qquad \gamma=\alpha+\beta-\alpha\beta, \tag{1.2}$$

where $D^{\alpha,\beta}_{a^+}(\cdot)$ and $I^{1-\gamma}_{a^+}(\cdot)$ are Hilfer fractional derivative of order α and type β and Riemann-Liouville fractional integral of order $1-\gamma$, respectively, $u_a\in\mathbb{R}$. The authors used some fixed point theorems to study the existence and uniqueness of global solutions in the weighted space to problem (1.1)–(1.2). The stability of the solution of a weighted Cauchy-type problem was also analyzed. In [24], Wang and Zhang proved the existence of the solutions to equation (1.1) with the nonlocal condition

$$I^{1-\gamma}_{a^+}u(a^+) = \sum_{k=1}^{m}\lambda_k u(\tau_k), \qquad \tau_k\in(a,b], \quad \gamma=\alpha+\beta(1-\alpha), \tag{1.3}$$

by using Krasnoselskii and Schauder fixed point theorems. Vivek et al. [23] established the existence, uniqueness and Ulam stability results for an implicit differential equations of Hilfer-type fractional order:

$$D^{\alpha,\beta}_{0^+}u(t) = f(t,u(t),D^{\alpha,\beta}_{a^+}u(t)), \qquad t>0, \qquad 0<\alpha<1 \qquad ,0\leqslant\beta\leqslant1,$$

$$I^{1-\gamma}_{0^+}u(0^+) = \sum_{k=1}^{m}\lambda_k u(\tau_k), \qquad \tau_k\in[0,b], \quad \gamma=\alpha+\beta(1-\alpha),$$

via Schaefer fixed point theorem and Banach contraction principle.

Lately, Sousa and Oliveira [21] have recently proposed a ψ-Hilfer fractional operator and extended the results of few previous works [13], [15]. In [22], Sousa and Oliveira disccused the existence, uniqueness, Ulam–Hyers and Ulam–Hyers–Rassias stabilities of the implicit fractional differential equation involving ψ-Hilfer fractional derivative. Very recently, in [20], Sousa with co-authors proposed a generalized Grönwall inequality for a fractional integral with respect to another function ψ. They also considered Cauchy-type problem (1.1)–(1.2) involving the ψ-Hilfer fractional derivative $D^{\alpha,\beta;\psi}_{a^+}(\cdot)$ introduced by Sousa and Oliveira in [21] and they established results on existence, uniqueness, and continuous dependence.

Motivated by the above works, we prove the existence, uniqueness, and Ulam–Hyers and Ulam–Hyers-Rassias stabilities for a nonlinear fractional integro-differntial equation with nonlocal condition and ψ-Hilfer fractional derivatives of the form:

$$D^{\alpha,\beta;\psi}_{a^+}u(t) = f\left(t,u(t),\chi u(t)\right), \qquad 0<\alpha<1, \qquad 0\leqslant\beta\leqslant1, \qquad t\in(a,b], \tag{1.4}$$

$$I^{1-\gamma;\psi}_{a^+}u(t)\mid_{t=a} = u_a+\sum_{k=1}^{m}c_k u(\tau_k), \qquad \tau_k\in(a,b),\ \alpha\leqslant\gamma=\alpha+\beta-\alpha\beta, \tag{1.5}$$

where $u_a\in\mathbb{R}$, $D^{\alpha,\beta;\psi}_{a^+}(\cdot)$ is the generalized Hilfer fractional derivative introduced by Sousa and de Oliveira in [21], $I^{1-\gamma;\psi}_{a^+}(\cdot)$ is the generalized fractional integral in the sense of Riemann-Liouville, for $\chi:\mathbb{D}\times\mathbb{R}\to\mathbb{R}$, $\chi u(t) := \int_0^t h(t,s,u(s))ds$. Here $\mathbb{D}: = \{(t,s): a\leqslant s\leqslant t\leqslant b\}$, $f:(a,b]\times\mathbb{R}\times\mathbb{R}\longrightarrow\mathbb{R}$ is appropriate function, τ_k, $k=1,2,\ldots,m$ are given points satisfying $a<\tau_1<\tau_2<\ldots<\tau_m<b$ and c_k are real numbers.

This paper is organized as follows. In Section 2 we introduce some notations, basic definitions, and preliminary facts, which will be used in the paper. In Section 3, we list the hypotheses and we also show that problem (1.4)–(1.5) is equivalent a the mixed type integral equation. We also prove the existence and uniqueness of solution to problem (1.4)–(1.5). The Ulam–Hyers and Ulam–Hyers-Rassias stabilities in a weighted space for such equations is discussed in Section 4.

In Section 5 we provide examples demonstrating our main results. Finally, the conclusion is given in the last section.

2. Preliminaries

In this section, we gather some essential facts, definitions, and lemmata concerning fractional calculus and fractional differential equations.

Let $J=[a,b]$, $-\infty<a<b<\infty$, be a finite interval in $\mathbb{R}$. We denote by $C(J,\mathbb{R})$ and $C^n(J,\mathbb{R})$, $n\in\mathbb{Z}_+:=\mathbb{N}\cup\{0\})$ the spaces of continuous and n-times continuously differentiable functions on J with the norms

$$\|f\|_C=\max_{t\in J}|f(t)|,\qquad \|f\|_{C^n}=\sum_{i=1}^{n}\left\|f^{(i)}\right\|_C=\sum_{i=0}^{n}\qquad\max_{t\in J}\left|f^{(i)}(t)\right|,$$

respectively, where $C(J,\mathbb{R})=C^0(J,\mathbb{R})$. And $L^p(J,\mathbb{R})$ $(p\geqslant 1)$ is the space of measurable functions $f:J\to\mathbb{R}$ with the norm

$$\|f\|_{L^p}=\left(\int_a^b|f(t)|^p\right)^{\frac{1}{p}}dt.$$

We introduce the following weighted spaces of continuous functions:

$$C_{\gamma;\psi}(J,\mathbb{R})=\{f:(a,b]\to\mathbb{R}:(\psi(t)-\psi(a))^\gamma f(t)\in C(J,\mathbb{R})\},\qquad 0\leqslant\gamma<1,$$
$$C^n_{\gamma;\psi}(J,\mathbb{R})=\left\{f\in C^{n-1}(J,\mathbb{R});f^{(n)}\in C_{\gamma;\psi}(J,\mathbb{R})\right\},\qquad 0\leqslant\gamma<1,\qquad n\in\mathbb{N}.$$

Obviously, $C_{\gamma;\psi}(J,\mathbb{R})$ and $C^n_{\gamma;\psi}(J,\mathbb{R})$ are the Banach spaces with the norms

$$\|f\|_{C_{\gamma;\psi}}=\|(\psi(t)-\psi(a))^\gamma f(t)\|_C=\max_{t\in I}|(\psi(t)-\psi(a))^\gamma f(t)|,$$

and

$$\|f\|_{C^n_{\gamma;\psi}}=\sum_{k=0}^{n-1}\left\|f^{(k)}\right\|_C+\left\|f^{(n)}\right\|_{C_{\gamma;\psi}},$$

respectively, where $C_{\gamma;\psi}(J,\mathbb{R})=C^0_{\gamma;\psi}(J,\mathbb{R})$.

A well-known function frequently used in the solution of fractional differential equations is the Mittag-Leffler function given by

$$E_{\alpha,\beta}(z)=\sum_{k=0}^{\infty}\frac{z^k}{\Gamma(\alpha k+\beta)},\qquad \operatorname{Re}(\alpha),\ \operatorname{Re}(\beta)>0,\qquad z\in\mathbb{C},$$

where $\Gamma(z)=\int\limits_0^\infty e^{-x}x^{z-1}dx$, $z>0$, is the Euler gamma function. Moreover, if $\alpha=\frac{1}{2}$ and $\beta=1$, we have

$$E_{\frac{1}{2}}(z)\leqslant e^{z^2}\left(1+\operatorname{erf}(z)\right),$$

where

$$\operatorname{erf}(z)=\frac{2}{\sqrt{\pi}}\int_0^z e^{-t^2}dt$$

is the error function.

Definition 2.1. [17] *Let f be an integrable function defined on (a,b) and ψ be an increasing function having a continuous derivative ψ' on (a,b) such that $\psi'(t)\neq 0$ for all $t\in J$ and $\alpha>0$ is a constant. The left-sided fractional integral of order α of function f with respect to ψ is defined by*

$$I^{\alpha,\psi}_{a^+}f(t)=\frac{1}{\Gamma(\alpha)}\int_a^t\psi'(s)(\psi(t)-\psi(s))^{\alpha-1}f(s)ds, \tag{2.1}$$

In particular, if $\psi(t) = t$, we obtain the known classical Riemann-Liouville fractional integral.

Definition 2.2. [7], [17] *Let $n-1 < \alpha < n$, and $f, \psi \in C^n(J, \mathbb{R})$ be two functions such that ψ is increasing and $\psi'(t) \neq 0$ for all $t \in J$. The left-sided fractional derivatives of order α of f with respect to ψ in the sense of Riemann-Liouville and Caputo are given by*

$$D_{a^+}^{\alpha;\psi} f(t) = \left[\frac{1}{\psi'(t)}\frac{d}{dt}\right]^n I_{a^+}^{n-\alpha,\psi} f(t),$$

and

$${}^C D_{a^+}^{\alpha;\psi} f(t) = I_{a^+}^{n-\alpha;\psi} \left[\frac{1}{\psi'(t)}\frac{d}{dt}\right]^n f(t),$$

respectively, where $n = [\alpha] + 1$ and $[\alpha]$ denotes the integer part of a real number α.

The fractional derivative we deal with is a ψ-Hilfer type operator defined as follows.

Definition 2.3. [21] *Let $n-1 < \alpha < n \in \mathbb{N}$, $0 \leqslant \beta \leqslant 1$ and $f, \psi \in C^n(J, \mathbb{R})$ two functions such that ψ is an increasing and $\psi'(t) \neq 0$ for all $t \in J$. The left-sided ψ-Hilfer fractional derivative of order α and type β of function f is determined as*

$$D_{a^+}^{\alpha,\beta;\psi} f(t) = I_{a^+}^{\beta(n-\alpha);\psi} \left[\frac{1}{\psi'(t)}\frac{d}{dt}\right]^n I_{a^+}^{(1-\beta)(n-\alpha);\psi} f(t).$$

On the other hand, we have

$$D_{a^+}^{\alpha,\beta;\psi} f(t) = I_{a^+}^{\beta(n-\alpha);\psi} D_{a^+}^{\gamma;\psi} f(t), \qquad \gamma = \alpha + \beta(n-\alpha), \tag{2.2}$$

where

$$D_{a^+}^{\gamma;\psi} f(t) = \left[\frac{1}{\psi'(t)}\frac{d}{dt}\right]^n I_{a^+}^{(1-\beta)(n-\alpha);\psi} f(t).$$

In particular, The ψ-Hilfer fractional derivative of order $0 < \alpha < 1$ and type $0 \leqslant \beta \leqslant 1$ can be written in the following form

$$D_{a^+}^{\alpha,\beta;\psi} f(t) = \frac{1}{\Gamma(\gamma-\alpha)} \int_a^t \psi'(s)(\psi(t)-\psi(s))^{\gamma-\alpha-1} D_{a^+}^{\gamma;\psi} f(s) ds = I_{a^+}^{\gamma-\alpha;\psi} D_{a^+}^{\gamma;\psi} f(t),$$

where $\gamma = \alpha + \beta(1-\alpha)$, $I_{a^+}^{\gamma-\alpha;\psi}(\cdot)$ are defined by equation (2.1) and

$$D_{a^+}^{\gamma;\psi} f(t) = \left[\frac{1}{\psi'(t)}\frac{d}{dt}\right] I_{a^+}^{1-\gamma;\psi} f(t).$$

Lemma 2.1. [3, 14] *Let $\alpha > 0$ and $\beta > 0$. The following semigroup property holds:*

(i): *If $f \in L^p(J, \mathbb{R})$ $(p \geqslant 1)$, then $I_{a^+}^{\alpha;\psi} I_{a^+}^{\beta;\psi} f(t) = I_{a^+}^{\alpha+\beta;\psi} f(t)$, a.e. $t \in J$.*
(ii): *If $f \in C_{\gamma;\psi}(J, \mathbb{R})$, then $I_{a^+}^{\alpha;\psi} I_{a^+}^{\beta;\psi} f(t) = I_{a^+}^{\alpha+\beta;\psi} f(t)$, $t \in (a, b]$, $0 \leqslant \gamma < 1$.*
(iii): *If $f \in C(J, \mathbb{R})$, then $I_{a^+}^{\alpha;\psi} I_{a^+}^{\beta;\psi} f(t) = I_{a^+}^{\alpha+\beta;\psi} f(t)$, $t \in J$.*
As $\alpha + \beta > 1$, Statement (i) holds at each point in J.

Lemma 2.2. [21] *Let $0 < \alpha < 1$, $0 \leqslant \beta \leqslant 1$ and $0 \leqslant \gamma < 1$. If $f \in L^1(J, \mathbb{R})$ and $D_{a^+}^{\beta(1-\alpha);\psi} f$ is well-defined as an element of $L^1(J, \mathbb{R})$, then*

$$D_{a^+}^{\alpha,\beta;\psi} I_{a^+}^{\alpha;\psi} f(t) = I_{a^+}^{\beta(1-\alpha);\psi} D_{a^+}^{\beta(1-\alpha);\psi} f(t).$$

Moreover, if $f \in C^1(J, \mathbb{R})$, then

$$D_{a^+}^{\alpha,\beta;\psi} I_{a^+}^{\alpha;\psi} h(t) = h(t).$$

For $0 < \alpha < 1$, $0 \leqslant \beta \leqslant 1$ and $\gamma = \alpha + \beta(1-\alpha)$ we introduce the weighted spaces

$$C^{\alpha,\beta}_{1-\gamma;\psi}(J,\mathbb{R}) = \{f \in C_{1-\gamma;\psi}(J,\mathbb{R}); \qquad D^{\alpha,\beta;\psi}_{a^+} f \in C_{1-\gamma;\psi}(J,\mathbb{R})\},$$

and

$$C^{\gamma}_{1-\gamma;\psi}(J,\mathbb{R}) = \{f \in C_{1-\gamma;\psi}(J,\mathbb{R}), D^{\gamma;\psi}_{a^+} f \in C_{1-\gamma;\psi}(J,\mathbb{R})\}. \tag{2.3}$$

Since $D^{\alpha,\beta;\psi}_{a^+} f = I^{\beta(1-\alpha);\psi}_{a^+} D^{\gamma;\psi}_{a^+} f$, it is obvious that $C^{\gamma}_{1-\gamma;\psi}(J,\mathbb{R}) \subset C^{\alpha,\beta}_{1-\gamma;\psi}(J,\mathbb{R})$.

Lemma 2.3. [2] *Let $0 < \alpha < 1$, $0 \leqslant \beta \leqslant 1$ and $\gamma = \alpha + \beta(1-\alpha)$. If $f \in C^{\gamma}_{1-\gamma;\psi}(J,\mathbb{R})$, then*

$$I^{\gamma;\psi}_{a^+} D^{\gamma;\psi}_{a^+} f(t) = I^{\alpha;\psi}_{a^+} D^{\alpha,\beta;\psi}_{a^+} f(t) \tag{2.4}$$

and

$$D^{\gamma;\psi}_{a^+} I^{\alpha;\psi}_{a^+} f(t) = D^{\beta(1-\alpha);\psi}_{a^+} f(t). \tag{2.5}$$

Lemma 2.4. [17] *If $\alpha > 0$ and $\delta > 0$, then ψ-fractional integral and derivative of a power function are given by*

$$I^{\alpha;\psi}_{a^+} (\psi(t) - \psi(a))^{\delta-1} = \frac{\Gamma(\delta)}{\Gamma(\delta+\alpha)} (\psi(t) - \psi(a))^{\alpha+\delta-1},$$

and

$$D^{\alpha;\psi}_{a^+} (\psi(t) - \psi(a))^{\alpha-1} = 0, \qquad 0 < \alpha < 1.$$

Lemma 2.5. [20] *Let $\alpha > 0$, and $0 \leqslant \gamma < 1$. Then $I^{\alpha;\psi}_{a^+}(\cdot)$ is bounded from $C_{1-\gamma;\psi}(J,\mathbb{R})$ into $C_{1-\gamma;\psi}(J,\mathbb{R})$. In particular, if $\gamma \leqslant \alpha$, then, $I^{\alpha;\psi}_{a^+}(\cdot)$ is bounded from $C_{1-\gamma;\psi}(J,\mathbb{R})$ into $C(J,\mathbb{R})$.*

Lemma 2.6. [21] *Let $\alpha > 0$, $0 \leqslant \gamma < 1$, and $f \in C_{1-\gamma;\psi}(J,\mathbb{R})$. If $\alpha > \gamma$, then $I^{\alpha;\psi}_{a^+} f \in C(J,\mathbb{R})$ and*

$$I^{\alpha;\psi}_{a^+} f(a) = \lim_{t \to a^+} I^{\alpha;\psi}_{a^+} f(t) = 0.$$

Theorem 2.1. [21] *Let $0 < \alpha < 1$, $0 \leqslant \beta \leqslant 1$. If $f \in C_{1-\gamma}(J,\mathbb{R})$, then*

$$I^{\alpha;\psi}_{a^+} D^{\alpha,\beta;\psi}_{a^+} f(t) = f(t) - \frac{I^{(1-\beta)(1-\alpha);\psi}_{a^+} f(a)}{\Gamma(\alpha+\beta(1-\alpha))} \left[\psi(t) - \psi(a)\right]^{\alpha+\beta(1-\alpha)-1},$$

Moreover, if $\gamma = \alpha + \beta(1-\alpha)$, $f \in C^{\gamma}_{1-\gamma;\psi}(J,\mathbb{R})$ and $I^{1-\gamma;\psi}_{a^+} f \in C^{1}_{1-\gamma;\psi}(J,\mathbb{R})$, then

$$I^{\gamma;\psi}_{a^+} D^{\gamma;\psi}_{a^+} f(t) = f(t) - \frac{I^{1-\gamma;\psi}_{a^+} f(a)}{\Gamma(\gamma)} \left[\psi(t) - \psi(a)\right]^{\gamma-1}.$$

Theorem 2.2. [25] *(Banach fixed point theorem) Let (X,d) be a nonempty complete metric space with $T : X \to X$ is a contraction mapping. Then map T has a fixed point $x^* \in X$ such that $Tx^* = x^*$.*

Theorem 2.3. [25] *(Krasnoselskii's fixed point theorem) Let X be a Banach space, let $\mathcal{S}$ be a bounded closed convex subset of X and let T_1, T_2 be mapping from $\mathcal{S}$ into X such that $T_1 x + T_2 y \in \mathcal{S}$ for every pair $x, y \in \mathcal{S}$. If T_1 is contraction and T_2 is completely continuous, then the equation $T_1 x + T_2 x = x$ has a solution in $\mathcal{S}$.*

3. Main results

In this section, we show that problem (1.4)-(1.5) is equivalent to a mixed type integral equation. We also prove the unique solvability of this problem. To this aim, we shall apply the fixed point theorems by Krasnoselskii and Banach. First we make the following assumptions.

(A1) $f : (a,b] \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ is a function such that $f(\cdot, u, \chi u) \in C^{\beta(1-\alpha)}_{1-\gamma;\psi}(J,\mathbb{R})$, for any $u \in C_{1-\gamma;\psi}(J,\mathbb{R})$ there exists $M > 0$ such that

$$|f(t,u,\chi u) - f(t,v,\chi v)| \leqslant M\left[|u-v| + |\chi u - \chi v|\right], \qquad \forall t \in (a,b], u, v \in \mathbb{R}. \tag{3.1}$$

(A2) $h : \mathbb{D} \times \mathbb{R} \longrightarrow \mathbb{R}$ is continuous on $\mathbb{D}$ and there exists $L^* > 0$ such that

$$|\chi u - \chi v| \leqslant L^* |u-v|, \qquad u, v \in G \subset \mathbb{R},$$

where

$$\chi u(t) = \int_0^t h(t,s,u(s))ds, \qquad \mathbb{D} = \{(t,s) : a \leqslant s \leqslant t \leqslant b\}.$$

(A3) The inequality

$$\Omega := \frac{M + ML^*}{1-B} \sum_{k=1}^{m} c_k \Phi_\psi^{\alpha+\gamma}(\tau_k, a) < 1$$

holds, where

$$B = \sum_{k=1}^{m} c_k \Phi_\psi^{\gamma}(\tau_k, a) \neq 1, \qquad \Phi_\psi^{\gamma}(\tau_k, a) := \frac{1}{\Gamma(\gamma)}\big(\psi(\tau_k) - \psi(a)\big)^{\gamma-1},$$

and $\mathcal{B}(\cdot,\cdot)$ is the Beta function.

Lemma 3.1. [2] *Let $0 < \alpha < 1$ and $0 \leqslant \beta \leqslant 1$. Then a function u solves the Cauchy problem*

$$D_{a^+}^{\alpha,\beta;\psi} u(t) = f\left(t, u(t), \chi u(t)\right), \qquad t \in (a,b],$$
$$I_{a^+}^{1-\gamma;\psi} u(t)\, |_{t=a} = u_a, \qquad \gamma = \alpha + \beta(1-\alpha),$$

if and only if u solves the following Volterra integral equation

$$u(t) = \frac{u_a}{\Gamma(\gamma)}(\psi(t)-\psi(a))^{\gamma-1} + \frac{1}{\Gamma(\alpha)}\int_a^t \psi'(s)(\psi(t)-\psi(s))^{\alpha-1} f\left(s, u(s), \chi u(s)\right) ds,$$

where

$$\chi u(s) = \int_0^t h(t,s,u(s))ds.$$

Lemma 3.2. *Let $0 < \alpha < 1$, $0 \leqslant \beta \leqslant 1$ and $\gamma = \alpha + \beta(1-\alpha)$. Assume that $f\left(\cdot, u(\cdot), \chi u(\cdot)\right) \in C_{1-\gamma;\psi}(J,\mathbb{R})$. If $u \in C^{\gamma}_{1-\gamma;\psi}(J,\mathbb{R})$ then u satisfies the problem (1.4)-(1.5) if and only if u satisfies the mixed type integral equation*

$$\begin{aligned} u(t) = &\frac{\Phi_\psi^{\gamma}(t,a)}{1-B}\left[\sum_{k=1}^{m} \frac{c_k}{\Gamma(\alpha)} \int_a^{\tau_k} \Phi_\psi^{\alpha}(\tau_k, s) f\left(s, u(s), \chi u(s)\right) ds + u_a\right] \\ &+ \frac{1}{\Gamma(\alpha)}\int_a^t \Phi_\psi^{\alpha}(t,s) f\left(s, u(s), \chi u(s)\right) ds, \end{aligned} \tag{3.2}$$

where

$$\Phi_\psi^{\gamma}(t,a) := \frac{[\psi(t)-\psi(a)]^{\gamma-1}}{\Gamma(\gamma)}, \qquad \Phi_\psi^{\alpha}(\tau_k, s) = \psi'(s)\left[\psi(\tau_k) - \psi(s)\right]^{\alpha-1}$$

and

$$B = \sum_{k=1}^{m} c_k \Phi_\psi^{\gamma}(\tau_k, a) \neq 1. \tag{3.3}$$

Proof. First, we prove the necessary condition. According to Lemma 3.1, a solution to equation (1.4) can be expressed as

$$\begin{aligned} u(t) =&\frac{[\psi(t)-\psi(a)]^{\gamma-1}}{\Gamma(\gamma)} I_{a^+}^{1-\gamma;\psi} u(a) \\ &+ \frac{1}{\Gamma(\alpha)} \int_a^t \psi'(s)(\psi(t)-\psi(s))^{\alpha-1} f\left(s, u(s), \chi u(s)\right) ds \end{aligned} \tag{3.4}$$

We substitute $t = \tau_k$ into the above equation, then multiply it by c_k to obtain

$$\begin{aligned} c_k u(\tau_k) =&\frac{c_k\left[\psi(\tau_k)-\psi(a)\right]^{\gamma-1}}{\Gamma(\gamma)} I_{a^+}^{1-\gamma;\psi} u(a) \\ &+ \frac{c_k}{\Gamma(\alpha)} \int_a^{\tau_k} \psi'(s)\left[\psi(\tau_k)-\psi(s)\right]^{\alpha-1} f\left(s, u(s), \chi u(s)\right) ds. \end{aligned} \tag{3.5}$$

Hence,

$$\begin{aligned} I_{a^+}^{1-\gamma;\psi} u(a) =&\sum_{k=1}^{m} c_k u(\tau_k) + u_a \\ =&\frac{1}{\Gamma(\gamma)} I_{a^+}^{1-\gamma;\psi} u(a) \sum_{k=1}^{m} c_k \left[\psi(\tau_k)-\psi(a)\right]^{\gamma-1} \\ &+ \sum_{k=1}^{m} \frac{c_k}{\Gamma(\alpha)} \int_a^{\tau_k} \psi'(s)\left[\psi(\tau_k)-\psi(s)\right]^{\alpha-1} f\left(s, u(s), \chi u(s)\right) ds + u_a, \end{aligned}$$

which implies

$$I_{a^+}^{1-\gamma;\psi} u(a) = \frac{1}{1-B}\left[\sum_{k=1}^{m} \frac{c_k}{\Gamma(\alpha)} \int_a^{\tau_k} \Phi_\psi^\alpha(\tau_k, s) f\left(s, u(s), \chi u(s)\right) ds + u_a\right], \tag{3.6}$$

Substituting equation (3.6) into equation (3.4), we get

$$\begin{aligned} u(t) =&\frac{\Phi_\psi^\gamma(t,a)}{1-B}\left[\sum_{k=1}^{m} \frac{c_k}{\Gamma(\alpha)} \int_a^{\tau_k} \Phi_\psi^\alpha(\tau_k, s) f\left(s, u(s), \chi u(s)\right) ds + u_a\right] \\ &+ \frac{1}{\Gamma(\alpha)} \int_a^t \Phi_\psi^\alpha(t,s) f\left(s, u(s), \chi u(s)\right) ds. \end{aligned}$$

We proceed to proving the sufficient condition. Applying the operator $I_{a^+}^{1-\gamma;\psi}$ to both sides of (3.2) and employing Lemmata 2.4, 2.1, we obtain:

$$\begin{aligned} I_{a^+}^{1-\gamma;\psi} u(t) =&\frac{1}{1-B}\left[\sum_{k=1}^{m} \frac{c_k}{\Gamma(\alpha)} \int_a^{\tau_k} \Phi_\psi^\alpha(\tau_k, s) f\left(s, u(s), \chi u(s)\right) ds + u_a\right] \\ &+ \frac{1}{\Gamma(1-\gamma+\alpha)} \int_a^t \Phi_\psi^\alpha(t,s) f\left(s, u(s), \chi u(s)\right) ds. \end{aligned}$$

We pass to the limit as $t \to a$; since $1-\gamma < 1-\gamma+\alpha$, Lemma 2.6 implies

$$I_{a^+}^{1-\gamma;\psi} u(a) = \frac{1}{1-B}\left[\sum_{k=1}^{m} \frac{c_k}{\Gamma(\alpha)} \int_a^{\tau_k} \Phi_\psi^\alpha(\tau_k, s) f\left(s, u(s), \chi u(s)\right) ds + u_a\right]. \tag{3.7}$$

Substituting $t = \tau_k$ and multiplying equation (3.2) by c_k we get

$$\begin{aligned}\sum_{k=1}^{m} c_k u(\tau_k) =& \sum_{k=1}^{m} c_k \frac{\Phi_{\psi}^{\gamma}(\tau_k, a)}{1-B}\left[\sum_{k=1}^{m} \frac{c_k}{\Gamma(\alpha)} \int_a^{\tau_k} \Phi_{\psi}^{\alpha}(\tau_k, s) f\left(s, u(s), \chi u(s)\right) ds + u_a\right] \\ &+ \sum_{k=1}^{m} \frac{c_k}{\Gamma(\alpha)} \int_a^{\tau_k} \Phi_{\psi}^{\alpha}(\tau_k, s) f\left(s, u(s), \chi u(s)\right) ds \\ =& \frac{1}{1-B} \sum_{k=1}^{m} \frac{c_k}{\Gamma(\alpha)} \int_a^{\tau_k} \Phi_{\psi}^{\alpha}(\tau_k, s) f\left(s, u(s), \chi u(s)\right) ds + \frac{B}{1-B} u_a,\end{aligned}$$

which implies

$$\sum_{k=1}^{m} c_k u(\tau_k) + u_a = \frac{1}{1-B}\left[\sum_{k=1}^{m} \frac{c_k}{\Gamma(\alpha)} \int_a^{\tau_k} \Phi_{\psi}^{\alpha}(\tau_k, s) f\left(s, u(s), \chi u(s)\right) ds + u_a\right]. \tag{3.8}$$

Comparing equation (3.7) and equation (3.8), we see that

$$I_{a^+}^{1-\gamma;\psi} u(a) = \sum_{k=1}^{m} c_k u(\tau_k) + u_a.$$

Now we apply the operator $D_{a^+}^{\gamma;\psi}$ to both sides of equation (3.2) and by Lemmata 2.4, 2.6, and 2.3 we obtain that

$$\begin{aligned}D_{a^+}^{\gamma;\psi} u(t) =& D_{a^+}^{\gamma;\psi} I_{a^+}^{\alpha;\psi} f\left(t, u(t), \chi u(t)\right) \\ =& D_{a^+}^{\beta(1-\alpha);\psi} f\left(t, u(t), \chi u(t)\right).\end{aligned} \tag{3.9}$$

Since $u \in C_{1-\gamma;\psi}^{\gamma}(J, \mathbb{R})$, by the definition of $C_{1-\gamma;\psi}^{\gamma}(J, \mathbb{R})$ we have $D_{a^+}^{\gamma;\psi} u \in C_{1-\gamma;\psi}(J, \mathbb{R})$. Hence,

$$D_{a^+}^{\beta(1-\alpha);\psi} f\left(t, u(t), \chi u(t)\right) = D I_{a^+}^{1-\beta(1-\alpha);\psi} f\left(t, u(t), \chi u(t)\right) \in C_{1-\gamma;\psi}(J, \mathbb{R}).$$

For each $f(\cdot, u(\cdot), \chi u(\cdot)) \in C_{1-\gamma;\psi}(J, \mathbb{R})$, by Lemma 2.5 we have

$$I_{a^+}^{1-\beta(1-\alpha);\psi} f(\cdot, u(\cdot), \chi u(\cdot) \in C_{1-\gamma;\psi}(J, \mathbb{R}). \tag{3.10}$$

Equation (3.10) and the definition of $C_{1-\gamma;\psi}^{\gamma}(J, \mathbb{R})$ yield that

$$I_{a^+}^{1-\beta(1-\alpha);\psi} f(\cdot, u(\cdot), \chi u(\cdot)) \in C_{1-\gamma;\psi}^{1}(J, \mathbb{R}).$$

Therefore, f and $I_{a^+}^{1-\beta(1-\alpha);\psi} f$ satisfy the assumptions of Theorem 2.1. We apply the operator $I_{a^+}^{\beta(1-\alpha);\psi}$ to the both sides of equation (3.9) and by Lemmata 2.1, 2.2, 2.6 we get

$$\begin{aligned}I_{a^+}^{\beta(1-\alpha);\psi} D_{a^+}^{\gamma;\psi} u(t) =& I_{a^+}^{\beta(1-\alpha);\psi} D_{a^+}^{\beta(1-\alpha);\psi} f\left(t, u(t), \chi u(t)\right) \\ =& f\left(t, u(t), \chi u(t)\right) \\ &- \frac{I_{a^+}^{1-\beta(1-\alpha);\psi} f\left(a, u(a), \chi u(a)\right)}{\Gamma(\beta(1-\alpha))} \left[\psi(t) - \psi(a)\right]^{\beta(1-\alpha)-1} \\ =& f\left(t, u(t), \chi u(t)\right).\end{aligned} \tag{3.11}$$

Comparing equation (3.11) with equation (2.2) when $n = 1$, we get

$$D_{a^+}^{\alpha,\beta;\psi} u(t) = f\left(t, u(t), \chi u(t)\right).$$

This means that equation (1.4) holds true. The proof is complete. □

We proceed to proving the solvability of problem (1.4)-(1.5) in the weighted space $C_{1-\gamma;\psi}^{\alpha,\beta}(J, \mathbb{R})$; we shall do this by means of Krasnoselskii fixed point theorems.

Theorem 3.1. *Let*

$$0 < \alpha < 1, \qquad 0 \leqslant \beta \leqslant 1, \qquad \gamma = \alpha + \beta - \alpha\beta$$

and Assumptions (A1), (A2) and (A3) are satisfied. Then problem (1.4)-(1.5) is solvable in the space $C^{\gamma}_{1-\gamma;\psi}(J,\mathbb{R}) \subset C^{\alpha,\beta}_{1-\gamma;\psi}(J,\mathbb{R})$.

Proof. We are going to reduce problem (1.4)-(1.5) into a fixed point problem. Consider the operator $T : C_{1-\gamma;\psi}(J,\mathbb{R}) \to C_{1-\gamma;\psi}(J,\mathbb{R})$ defined by

$$\begin{aligned}(Tu)(t) =& \frac{\Phi^{\gamma}_{\psi}(t,a)}{1-B}\left[\sum_{k=1}^{m}\frac{c_k}{\Gamma(\alpha)}\int_a^{\tau_k}\Phi^{\alpha}_{\psi}(\tau_k,s)f\left(s,u(s),\chi u(s)\right)ds + u_a\right] \\ &+ \frac{1}{\Gamma(\alpha)}\int_a^t \Phi^{\alpha}_{\psi}(t,s)f\left(s,u(s),\chi u(s)\right)ds, \qquad t \in (a,b].\end{aligned} \tag{3.12}$$

We define

$$\begin{aligned} r :=& 1 + \frac{u_a}{1-B} \\ &+ \frac{\mu\mathcal{B}(\alpha,\gamma)}{\Gamma(\alpha)}\left[\frac{1}{1-B}\sum_{k=1}^{m} c_k(\psi(\tau_k)-\psi(a))^{\alpha} + [\psi(b)-\psi(a)]^{\alpha}\right], \\ \mathbb{B}_r =& \{u \in C_{1-\gamma;\psi}((J,\mathbb{R}) : \|u\|_{C_{1-\gamma;\psi}} \leqslant r\}, \\ \mu :=& \max_{(s,u,\chi u)\in J\times\mathbb{B}_r\times\mathbb{B}_r}\left|[\psi(s)-\psi(a)]^{1-\gamma} f(s,u,\chi u)\right|, \end{aligned}$$

and we introduce operators T_1 and T_2 on $\mathbb{B}_r$ as follows:

$$T_1u(t) = \frac{\Phi^{\gamma}_{\psi}(t,a)}{1-B}\left[\sum_{k=1}^{m}\frac{c_k}{\Gamma(\alpha)}\int_a^{\tau_k}\Phi^{\alpha}_{\psi}(\tau_k,s)f\left(s,u(s),\chi u(s)\right)ds + u_a\right],$$

and

$$T_2u(t) = \frac{1}{\Gamma(\alpha)}\int_a^t \Phi^{\alpha}_{\psi}(t,s)f\left(s,u(s),\chi u(s)\right)ds.$$

Observe that $T_1 + T_2 = T$, where the operator $T : C_{1-\gamma;\psi}(J,\mathbb{R}) \to C_{1-\gamma;\psi}(J,\mathbb{R})$ is defined by equation (3.12). The rest of the proof is split into several steps.

Step 1: Let us prove that $T_1u + T_2v \in \mathbb{B}_r$ for each $u, v \in \mathbb{B}_r$.

For $u \in \mathbb{B}_r$ and $t \in (a,b]$ we have

$$\begin{aligned}\left|T_1u(t)\left[\psi(t)-\psi(a)\right]^{1-\gamma}\right| \leqslant& \frac{1}{\Gamma(\gamma)}\frac{1}{1-B}\left[\sum_{k=1}^{m}\frac{c_k}{\Gamma(\alpha)}\int_a^{\tau_k}\Phi^{\alpha}_{\psi}(\tau_k,s)\left|f\left(s,u(s),\chi u(s)\right)\right|ds + u_a\right] \\ \leqslant& \frac{1}{1-B}\left[\sum_{k=1}^{m}\frac{\mu c_k}{\Gamma(\alpha)}\int_a^{\tau_k}\Phi^{\alpha}_{\psi}(\tau_k,s)\Phi^{\gamma}_{\psi}(s,a)ds + u_a\right] \\ =& \frac{1}{1-B}\left[\sum_{k=1}^{m} c_k\frac{\mu\mathcal{B}(\alpha,\gamma)}{\Gamma(\alpha)}(\psi(\tau_k)-\psi(a))^{\alpha+\gamma-1} + u_a\right] \\ \leqslant& \frac{1}{1-B}\left[\sum_{k=1}^{m} c_k\frac{\mu\mathcal{B}(\alpha,\gamma)}{\Gamma(\alpha)}(\psi(\tau_k)-\psi(a))^{\alpha} + u_a\right],\end{aligned}$$

where we have employed that

$$\frac{1}{\Gamma(\alpha)}\int_a^{\tau_k}\Phi^{\alpha}_{\psi}(\tau_k,s)\Phi^{\gamma}_{\psi}(s,a)ds = \frac{\mathcal{B}(\alpha,\gamma)}{\Gamma(\alpha)}(\psi(\tau_k)-\psi(a))^{\alpha+\gamma-1},$$

and

$$\frac{[\psi(\tau_k)-\psi(a)]^{\gamma}}{[\psi(\tau_k)-\psi(a)]}<1.$$

Hence, it follows that

$$\|T_1u\|_{C_{1-\gamma;\psi}}\leqslant\frac{1}{1-B}\left[\sum_{k=1}^{m}c_k\frac{\mu\mathcal{B}(\alpha,\gamma)}{\Gamma(\alpha)}(\psi(\tau_k)-\psi(a))^{\alpha}+u_a\right]. \tag{3.13}$$

In the same way, for the operator T_2 we have

$$\begin{aligned}\left|T_2v(t)\,[\psi(t)-\psi(a)]^{1-\gamma}\right|&\leqslant[\psi(t)-\psi(a)]^{1-\gamma}\,\frac{1}{\Gamma(\alpha)}\int_a^t\Phi_{\psi}^{\alpha}(t,s)\,|f\,(s,u(s),\chi u(s))|\,ds\\&=\frac{\mu\mathcal{B}(\alpha,\gamma)}{\Gamma(\alpha)}\,[\psi(t)-\psi(a)]^{\alpha}\,,\ \text{for } v\in\mathbb{B}_r.\end{aligned}$$

Therefore,

$$\|T_2v\|_{C_{1-\gamma;\psi}}\leqslant\frac{\mu\mathcal{B}(\alpha,\gamma)}{\Gamma(\alpha)}\,[\psi(b)-\psi(a)]^{\alpha}\,. \tag{3.14}$$

By equations (3.13), (3.14) we get

$$\begin{aligned}\|T_1u+T_2v\|_{C_{1-\gamma;\psi}}&\leqslant\|T_1u\|_{C_{1-\gamma;\psi}}+\|T_2v\|_{C_{1-\gamma;\psi}}\\&\leqslant\frac{1}{1-B}\left[\sum_{k=1}^{m}c_k\frac{\mu\mathcal{B}(\alpha,\gamma)}{\Gamma(\alpha)}(\psi(\tau_k)-\psi(a))^{\alpha}+u_a\right]+\frac{\mu\mathcal{B}(\alpha,\gamma)}{\Gamma(\alpha)}\,[\psi(b)-\psi(a)]^{\alpha}\\&=\frac{u_a}{1-B}+\frac{\mu\mathcal{B}(\alpha,\gamma)}{\Gamma(\alpha)}\left[\frac{1}{1-B}\sum_{k=1}^{m}c_k(\psi(\tau_k)-\psi(a))^{\alpha}+[\psi(b)-\psi(a)]^{\alpha}\right]<r.\end{aligned}$$

This proves that $T_1u+T_2v\in\mathbb{B}_r$ for each $u,v\in\mathbb{B}_r$.

Step 2: At this step, we are going to show that the operator T_1 is a contracting mapping on $\mathbb{B}_r$.

For each $u,v\in\mathbb{B}_r$, and for each $t\in(a,b]$, it follows from Assumptions (A1) and (A2) that

$$\begin{aligned}&\left|[\psi(t)-\psi(a)]^{1-\gamma}T_1u(t)-[\psi(t)-\psi(a)]^{1-\gamma}T_1v(t)\right|\\&\leqslant\frac{1}{\Gamma(\gamma)}\frac{1}{1-B}\sum_{k=1}^{m}\frac{c_k}{\Gamma(\alpha)}\int_a^{\tau_k}\Phi_{\psi}^{\alpha}(\tau_k,s)\,|f(s,u(s),\chi u(s))-f(s,v(s),\chi v(s))|\,ds\\&\leqslant\frac{1}{\Gamma(\gamma)}\frac{1}{1-B}\sum_{k=1}^{m}\frac{c_k}{\Gamma(\alpha)}\int_a^{\tau_k}\Phi_{\psi}^{\alpha}(\tau_k,s)M\,[|u(s)-v(s)|+|\chi u(s)-\chi v(s)|]\,ds\\&\leqslant\frac{1}{\Gamma(\gamma)}\frac{1}{1-B}\sum_{k=1}^{m}\frac{c_k}{\Gamma(\alpha)}\int_a^{\tau_k}\Phi_{\psi}^{\alpha}(\tau_k,s)\,(M+ML^{*})\,|u-v|\,ds\\&\leqslant\frac{(M+ML^{*})}{1-B}\sum_{k=1}^{m}\frac{c_k}{\Gamma(\alpha)}\int_a^{\tau_k}\Phi_{\psi}^{\alpha}(\tau_k,s)\Phi_{\psi}^{\gamma}(s,a)\,\|u-v\|_{C_{1-\gamma;\psi}}\,ds\\&\leqslant\frac{(M+ML^{*})}{1-B}\sum_{k=1}^{m}c_k\Phi_{\psi}^{\alpha+\gamma}(\tau_k,a)\,\|u-v\|_{C_{1-\gamma;\psi}}\,,\end{aligned}$$

which implies

$$\|T_1u-T_1v\|_{C_{1-\gamma;\psi}}\leqslant\Omega\,\|u-v\|_{C_{1-\gamma;\psi}}\,.$$

Due to Assumption (A3), we conclude that the operator T_1 is contracting.

Step 3: Here we prove the operator T_2 is completely continuous on $\mathbb{B}_r$.

The continuity of f implies that operator T_2 is continuous. Also, T_2 is uniformly bounded on $\mathbb{B}_r$. Indeed, by Step 1, for $v \in \mathbb{B}_r$ we have

$$\|T_2 v\|_{C_{1-\gamma;\psi}} \leqslant \frac{\mu \mathcal{B}(\alpha,\gamma)}{\Gamma(\alpha)} [\psi(b)-\psi(a)]^{\alpha} := \ell.$$

This shows that for each $r > 0$ there exists a positive constant ℓ such that $\|T_2 v\|_{C_{1-\gamma;\psi}} \leqslant \ell$ for $v \in \mathbb{B}_r$.

In order to prove the compactness of the operator T_2, we take $u \in \mathbb{B}_r$ and $t_1, t_2 \in (a, b]$ with $t_1 < t_2$ and we have

$$\begin{aligned}
&\left|[\psi(t_2)-\psi(a)]^{1-\gamma} T_2 u(t_2) - [\psi(t_1)-\psi(a)]^{1-\gamma} T_2 u(t_1)\right| \\
&= \left| \frac{[\psi(t_2)-\psi(a)]^{1-\gamma}}{\Gamma(\alpha)} \int_a^{t_2} \Phi_\psi^\alpha(t_2,s) f(s,u(s),\chi u(s)) ds \right. \\
&\quad \left. - \frac{[\psi(t_1)-\psi(a)]^{1-\gamma}}{\Gamma(\alpha)} \int_a^{t_1} \Phi_\psi^\alpha(t_1,s) f(s,u(s),\chi u(s)) ds \right| \\
&\leqslant \left| \frac{[\psi(t_2)-\psi(a)]^{1-\gamma} \mu}{\Gamma(\alpha)} \int_a^{t_2} \Phi_\psi^\alpha(t_2,s) \Phi_\psi^\gamma(s,a) ds \right. \\
&\quad \left. - \frac{[\psi(t_1)-\psi(a)]^{1-\gamma} \mu}{\Gamma(\alpha)} \int_a^{t_1} \Phi_\psi^\alpha(t_1,s) \Phi_\psi^\gamma(s,a) ds \right| \\
&= \frac{\mu \mathcal{B}(\alpha,\gamma)}{\Gamma(\alpha)} \left| [\psi(t_2)-\psi(a)]^{1-\gamma} \Phi_\psi^{\gamma+\alpha}(t_2,a) - [\psi(t_1)-\psi(a)]^{1-\gamma} \Phi_\psi^{\gamma+\alpha}(t_1,a) \right| \\
&= \frac{\mu \mathcal{B}(\alpha,\gamma)}{\Gamma(\alpha)} \left| [\psi(t_2)-\psi(a)]^{\alpha} - [\psi(t_1)-\psi(a)]^{\alpha} \right|.
\end{aligned}$$

The right hand side of the above inequality is independent of u. The continuity of ψ as $t_2 \to t_1$ implies that

$$\left|[\psi(t_2)-\psi(a)]^{1-\gamma} T_2 u(t_2) - [\psi(t_1)-\psi(a)]^{1-\gamma} T_2 u(t_1)\right| \to 0 \quad \text{as} \quad t_2 \to t_1.$$

This proves that T_2 is equicontinuous. Hence, T_2 is relatively compact on $\mathbb{B}_r$. By the Arzela–Ascoli theorem, T_2 is compact on $\mathbb{B}_r$. Thus, all assumptions of Theorem 2.3 are satisfied and Hilfer problem (1.4)-(1.5) is solvable in $C_{1-\gamma;\psi}(J,\mathbb{R})$.

Finally, we show that such a solution is in $C^{\gamma}_{1-\gamma;\psi}(J,\mathbb{R})$. By applying $D^{\gamma;\psi}_{a^+}$ on both sides of equation (3.2), and using Lemmata 2.4, 2.3 we obtain

$$D^{\gamma;\psi}_{a^+} u(t) = D^{\gamma;\psi}_{a^+} I^{\alpha;\psi}_{a^+} f(t,u(t),\chi u(t) = D^{\beta(1-\alpha);\psi}_{a^+} f(t,u(t),\chi u(t).$$

Since $f(\cdot,u(\cdot),\chi u(\cdot)) \in C^{\beta(1-\alpha)}_{1-\gamma;\psi}(J,\mathbb{R})$, it follows from the definition of the space $C^{\beta(1-\alpha)}_{1-\gamma;\psi}(J,\mathbb{R})$ that $D^{\gamma;\psi}_{a^+} u(t) \in C_{1-\gamma;\psi}(J,\mathbb{R})$ and this $u(t) \in C^{\gamma}_{1-\gamma;\psi}(J,\mathbb{R})$. The proof is complete. □

4. Ulam–Hyers–Rassias stability

In this section, we study Ulam–Hyers and Ulam–Hyers–Rassias stabilities for nonlocal fractional integro-differntial equation (1.4)-(1.5). The stability results are based on the Banach fixed point theorem.

Theorem 4.1. *Let Assumptions (A1) and (A2) be satisfied. If*

$$\Lambda := \left[\frac{1}{1-B} \sum_{k=1}^{m} \frac{c_k}{\Gamma(\gamma)} \Phi_\psi^\alpha(\tau_k,a) + \Phi_\psi^\alpha(b,a) \right] (M + ML^*) \mathcal{B}(\alpha,\gamma) < 1, \tag{4.1}$$

then problem (1.4)-(1.5) has a unique solution.

Proof. We consider the operator $T : C_{1-\gamma;\psi}(J,\mathbb{R}) \to C_{1-\gamma;\psi}(J,\mathbb{R})$ defined by equation (3.12). In view of Theorem 3.1, we know that the fixed points of T are solutions of problem (1.4)-(1.5).

Let us prove that T has a unique fixed point, which is a solution of problem (1.4)-(1.5). Indeed, for each $u, v \in C_{1-\gamma;\psi}(J,\mathbb{R})$ and each $t \in (a,b]$, we have

$$\begin{aligned}
&\left|[\psi(t)-\psi(a)]^{1-\gamma}Tu(t)-[\psi(t)-\psi(a)]^{1-\gamma}Tv(t)\right| \\
&\leqslant \frac{1}{\Gamma(\gamma)(1-B)}\sum_{k=1}^{m}\frac{c_k}{\Gamma(\alpha)}\int_a^{\tau_k}\Phi_\psi^\alpha(\tau_k,s)\left|f\left(s,u(s),\chi u(s)\right)-f\left(s,v(s),\chi v(s)\right)\right|ds \\
&\quad+\frac{[\psi(t)-\psi(a)]^{1-\gamma}}{\Gamma(\alpha)}\int_a^t\Phi_\psi^\alpha(t,s)\left|f\left(s,u(s),\chi u(s)\right)-f\left(s,v(s),\chi v(s)\right)\right|ds \\
&\leqslant \frac{(M+ML^*)}{1-B}\sum_{k=1}^{m}\frac{c_k}{\Gamma(\alpha)}\int_a^{\tau_k}\Phi_\psi^\alpha(\tau_k,s)\Phi_\psi^\gamma(s,a)\left\|u-v\right\|_{C_{1-\gamma;\psi}}ds \\
&\quad+\frac{\Gamma(\gamma)\left[\psi(t)-\psi(a)\right]^{1-\gamma}(M+ML^*)}{\Gamma(\alpha)}\int_a^t\Phi_\psi^\alpha(t,s)\Phi_\psi^\gamma(s,a)\left\|u-v\right\|_{C_{1-\gamma;\psi}}ds \\
&\leqslant \frac{(M+ML^*)}{1-B}\sum_{k=1}^{m}c_k\frac{\mathcal{B}(\alpha,\gamma)}{\Gamma(\gamma)}\Phi_\psi^\alpha(\tau_k,a)\left\|u-v\right\|_{C_{1-\gamma;\psi}} \\
&\quad+(M+ML^*)\,\mathcal{B}(\alpha,\gamma)\Phi_\psi^\alpha(t,a)\left\|u-v\right\|_{C_{1-\gamma;\psi}} \\
&=\left[\frac{1}{1-B}\sum_{k=1}^{m}\frac{c_k}{\Gamma(\gamma)}\Phi_\psi^\alpha(\tau_k,a)+\Phi_\psi^\alpha(t,a)\right](M+ML^*)\,\mathcal{B}(\alpha,\gamma)\left\|u-v\right\|_{C_{1-\gamma;\psi}}
\end{aligned}$$

This gives:

$$\left\|Tu-Tv\right\|_{C_{1-\gamma;\psi}} \leqslant \Lambda\left\|u-v\right\|_{C_{1-\gamma;\psi}}.$$

By inequality (4.1), the operator $T : C_{1-\gamma;\psi}(J,\mathbb{R}) \to C_{1-\gamma;\psi}(J,\mathbb{R})$ is a contracting mapping. Hence, we conclude that the operator T has a unique fixed point $u \in C_{1-\gamma;\psi}(J,\mathbb{R})$ given by Banach fixed point theorem. □

We proceed to studying the Ulam–Hyers stability and Ulam–Hyers–Rassias stability. For $\epsilon > 0$ and for each $\varphi \in C_{1-\gamma;\psi}(J,\mathbb{R})$ we consider the following inequalities:

$$\left|D_{a^+}^{\alpha,\beta,\psi}\widetilde{u}(t)-f(t,\widetilde{u}(t),\chi\widetilde{u}(t))\right| \leqslant \epsilon, \qquad t \in (a,b], \tag{4.2}$$

$$\left|D_{a^+}^{\alpha,\beta,\psi}\widetilde{u}(t)-f(t,\widetilde{u}(t),\chi\widetilde{u}(t))\right| \leqslant \epsilon\varphi(t), \qquad t \in (a,b], \tag{4.3}$$

$$\left|D_{a^+}^{\alpha,\beta,\psi}\widetilde{u}(t)-f(t,\widetilde{u}(t),\chi\widetilde{u}(t))\right| \leqslant \varphi(t), \qquad t \in (a,b]. \tag{4.4}$$

Definition 4.1. *Problem (1.4)-(1.5) is Ulam–Hyers stable if there exists a real number* $C_f > 0$ *such that for each* $\epsilon > 0$ *and for each function* $\widetilde{u} \in C_{1-\gamma;\psi}^{\gamma}(J,\mathbb{R})$ *satisfying inequality (4.2), there exists a solution* $u \in C_{1-\gamma;\psi}^{\gamma}(J,\mathbb{R})$ *of equation (1.4) obeying*

$$|u(t)-\widetilde{u}(t)| \leqslant C_f\epsilon, \qquad t \in (a,b].$$

Definition 4.2. *Problem (1.4)-(1.5) is generalized Ulam–Hyers stable if there exists* $\phi_f \in C(\mathbb{R}^+,\mathbb{R}^+)$ *with* $\phi_f(0) = 0$ *such that for each function* $\widetilde{u} \in C_{1-\gamma;\psi}^{\gamma}(J,\mathbb{R})$ *satisfying inequality (4.2), there exists a solution* $u \in C_{1-\gamma;\psi}^{\gamma}(J,\mathbb{R})$ *of equation (1.4) obeying*

$$|u(t)-\widetilde{u}(t)| \leqslant \phi_f(\epsilon), \qquad t \in (a,b].$$

Definition 4.3. *Problem (1.4)-(1.5) is Ulam–Hyers–Rassias stable with respect to* $\varphi \in C_{1-\gamma;\psi}(J,\mathbb{R})$ *if there exists a real number* $C_{f,\varphi} > 0$ *such that for each* $\epsilon > 0$ *and for each function* $\widetilde{u} \in C^{\gamma}_{1-\gamma;\psi}(J,\mathbb{R})$ *satisfying inequality (4.3), there exists a solution* $u \in C^{\gamma}_{1-\gamma;\psi}(J,\mathbb{R})$ *of equation (1.4) obeying*

$$|u(t) - \widetilde{u}(t)| \leqslant C_{f,\varphi}\epsilon\varphi(t), \qquad t \in (a,b].$$

Definition 4.4. *Problem (1.4)-(1.5) is generalized Ulam–Hyers–Rassias stable with respect to* $\varphi \in C_{1-\gamma;\psi}(J,\mathbb{R})$ *if there exists a real number* $C_{f,\varphi} > 0$ *such that for each function* $\widetilde{u} \in C^{\gamma}_{1-\gamma;\psi}(J,\mathbb{R})$ *satisfying inequality (4.4) there exists a solution* $u \in C^{\gamma}_{1-\gamma;\psi}(J,\mathbb{R})$ *of equation (1.4) obeying*

$$|u(t) - \widetilde{u}(t)| \leqslant C_{f,\varphi}\varphi(t), \qquad t \in (a,b].$$

The next lemma is a generalization of Grönwall lemma.

Lemma 4.1. [20] *Let* $\alpha < 0$ *and* $\psi \in C^1[a,b]$ *be an increasing function such that* $\psi'(t) \neq 0$ *for all* $t \in [a,b]$. *Assume that* h *is nonnegative and non-decreasing, and* y *is a nonnegative function locally integrable on* $[a,b]$ *and suppose also that* x *is nonnegative and locally integrable on* $[a,b]$ *obeying*

$$x(t) \leqslant y(t) + h(t)\int_a^t \Phi^{\alpha}_{\psi}(t,s)x(s)ds, \qquad t \in [a,b],$$

then, for all $t \in [a,b]$, *we have*

$$x(t) \leqslant y(t) + \int_a^t \sum_{k=1}^{\infty} \frac{[h(t)\Gamma(\alpha)]^k}{\Gamma(\alpha k)} \Phi^{\alpha k}_{\psi}(t,s)y(s)ds.$$

Moreover, if $y(t)$ *is a nondecreasing function on* $[a,b]$, *then*

$$x(t) \leqslant y(t)E_{\alpha}\left[h(t)\Gamma(\alpha)\Phi^{\alpha}_{\psi}(b,a)\right], \qquad t \in [a,b].$$

Now, we are ready to prove Ulam-Hyers and Ulam-Hyers-Rassias stability for problem (1.4)-(1.5).

Theorem 4.2. *Let Assumptions (A1) and (A2) be satisfied. Then problem (1.4)-(1.5) is Ulam–Hyers stable.*

Proof. Let $\widetilde{u} \in C^{\gamma}_{1-\gamma,\psi}(J,\mathbb{R})$ be a function satisfying inequality (4.2). Applying operator $I^{\alpha;\psi}_{a^+}$ to the both sides of inequality (4.2) and using Theorem 2.1, we have

$$\left| I^{\alpha;\psi}_{a^+} D^{\alpha,\beta,\psi}_{0^+}\widetilde{u}(t) - I^{\alpha;\psi}_{a^+} f(t,\widetilde{u}(t),\chi\widetilde{u}(t)) \right| \leqslant I^{\alpha;\psi}_{a^+}\epsilon.$$

This implies that

$$\left| \widetilde{u}(t) - \mathcal{H}_{\widetilde{u}} - \frac{1}{\Gamma(\alpha)}\int_a^t \Phi^{\alpha}_{\psi}(t,s) f\left(s,\widetilde{u}(s),\chi\widetilde{u}(s)\right) ds \right| \leqslant \frac{\epsilon}{\alpha}\Psi^{\alpha}_{\psi}(b,a), \tag{4.5}$$

where

$$\Psi^{\alpha}_{\psi}(b,a) := \frac{[\psi(b) - \psi(a)]^{\alpha}}{\Gamma(\alpha)}$$

and

$$\mathcal{H}_{\widetilde{u}} := \frac{\Phi^{\gamma}_{\psi}(t,a)}{1-B}\left[\sum_{k=1}^{m} \frac{c_k}{\Gamma(\alpha)}\int_a^{\tau_k} \Phi^{\alpha}_{\psi}(\tau_k,s) f\left(s,\widetilde{u}(s),\chi\widetilde{u}(s)\right) ds + u_a\right].$$

We denote by $u \in C^{\gamma}_{1-\gamma;\psi}(J,\mathbb{R})$ the unique solution of the following problem

$$D^{\alpha,\beta;\psi}_{a^+}u(t) = f\left(t,u(t),\chi u(s)\right), \qquad 0<\alpha<1, \qquad 0\leqslant\beta\leqslant 1, \qquad t\in(a,b],$$
$$I^{1-\gamma;\psi}_{a^+}u(t)\mid_{t=a} = I^{1-\gamma;\psi}_{a^+}\widetilde{u}(t)\mid_{t=a} = u_a + \sum_{k=1}^{m} c_k u(\tau_k), \qquad \tau_k\in(a,b],$$

where $\gamma=\alpha+\beta-\alpha\beta$. Using Lemma 3.2, we obtain

$$u(t) = \mathcal{H}_u + \frac{1}{\Gamma(\alpha)}\int_a^t \Phi^{\alpha}_{\psi}(t,s) f\left(s,u(s),\chi u(s)\right) ds, \tag{4.6}$$

where

$$\mathcal{H}_u := \frac{\Phi^{\gamma}_{\psi}(t,a)}{1-B}\left[\sum_{k=1}^{m}\frac{c_k}{\Gamma(\alpha)}\int_a^{\tau_k}\Phi^{\alpha}_{\psi}(\tau_k,s) f\left(s,u(s),\chi u(s)\right) ds + u_a\right].$$

On the other hand, if

$$u(\tau_k) = \widetilde{u}(\tau_k) \qquad \text{and} \qquad I^{1-\gamma;\psi}_{a^+}u(t)\mid_{t=a} = I^{1-\gamma;\psi}_{a^+}\widetilde{u}(t)\mid_{t=a},$$

it is easy to see that $\mathcal{H}_u = \mathcal{H}_{\widetilde{u}}$. Hence, by Assumptions (A1) and (A2) and equations (4.5), (4.6), for each $t\in(a,b]$ we have

$$\begin{aligned}
|\widetilde{u}(t)-u(t)| \leqslant& \left|\widetilde{u}(t)-\mathcal{H}_{\widetilde{u}}-\frac{1}{\Gamma(\alpha)}\int_a^t\Phi^{\alpha}_{\psi}(t,s) f\left(s,\widetilde{u}(s),\chi\widetilde{u}(s)\right) ds\right| \\
&+ |\mathcal{H}_{\widetilde{u}}-\mathcal{H}_u| + \frac{1}{\Gamma(\alpha)}\int_a^t\Phi^{\alpha}_{\psi}(t,s)\left|f(t,\widetilde{u}(t),\chi\widetilde{u}(t)) - f(t,u(t),\chi u(t))\right| ds \\
\leqslant& \frac{\epsilon}{\alpha}\Psi^{\alpha}_{\psi}(b,a) + \frac{1}{\Gamma(\alpha)}\int_a^t\Phi^{\alpha}_{\psi}(t,s) M\left[|\widetilde{u}(s)-u(s)| + |\chi\widetilde{u}(s)-\chi u(s)|\right] ds \\
\leqslant& \frac{\epsilon}{\alpha}\Psi^{\alpha}_{\psi}(b,a) + \frac{(M+ML^*)}{\Gamma(\alpha)}\int_a^t\Phi^{\alpha}_{\psi}(t,s)\left|\widetilde{u}(s)-u(s)\right| ds.
\end{aligned} \tag{4.7}$$

We apply Lemma 4.1 to obtain

$$\begin{aligned}
|\widetilde{u}(t)-u(t)| \leqslant& \frac{\epsilon}{\alpha}\Psi^{\alpha}_{\psi}(b,a)\left[1+\int_a^t\sum_{k=1}^{\infty}\frac{(M+ML^*)^k}{\Gamma(\alpha k)}\Phi^{\alpha k}_{\psi}(t,s)\right] ds \\
\leqslant& \frac{\epsilon}{\alpha}\Psi^{\alpha}_{\psi}(b,a)\left[1+\sum_{k=1}^{\infty}\frac{\left[(M+ML^*)\Phi^{\alpha}_{\psi}(b,a)\right]^k}{\Gamma(\alpha k+1)}\right] \\
=& \frac{\epsilon}{\alpha}\Psi^{\alpha}_{\psi}(b,a)\left[\sum_{k=0}^{\infty}\frac{\left[(M+ML^*)\Phi^{\alpha}_{\psi}(b,a)\right]^k}{\Gamma(\alpha k+1)}\right] \\
=& \frac{\epsilon}{\alpha}\Psi^{\alpha}_{\psi}(b,a)E_{\alpha}\left[(M+ML^*)\Phi^{\alpha}_{\psi}(b,a)\right].
\end{aligned}$$

Then for

$$C_f = \frac{\Psi^{\alpha}_{\psi}(b,a)}{\alpha}E_{\alpha}\left[(M+ML^*)\Phi^{\alpha}_{\psi}(b,a)\right]$$

we get:

$$|\widetilde{u}(t)-u(t)| \leqslant C_f\epsilon.$$

This means that problem (1.4)-(1.5) is Ulam–Hyers stable. The proof is complete. □

Corollary 4.1. *Under the assumptions Theorem 4.2, if there exists $\phi_f\in C(\mathbb{R}^+,\mathbb{R}^+)$ with $\phi_f(0)=0$, the problem (1.4)–(1.5) is generalized Ulam-Hyers stabile.*

Proof. Following the lines of the proof of Theorem 4.2, we choose $\phi_f(\epsilon) = C_f\epsilon$ and $\phi_f(0) = 0$ and we obtain $|\widetilde{u}(t) - u(t)| \leqslant \phi_f(\epsilon)$. Hence, problem (1.4)-(1.5) is generalized Ulam–Hyers stable. □

Theorem 4.3. *Let Assumptions (A1) and (A2) hold and the following condition be satisfied*

(A4) *There exists an increasing function $\varphi \in C_{1-\gamma,\psi}(J,\mathbb{R})$ and $\lambda_\varphi > 0$ such that for each $t \in (a,b]$, the inequality holds:*

$$I_{a^+}^{\alpha,\psi}\varphi(t) \leqslant \lambda_\varphi \varphi(t).$$

Then problem (1.4)-(1.5) is Ulam–Hyers–Rassias stable.

Proof. Let $\epsilon > 0$ and let $\widetilde{u} \in C^{\gamma}_{1-\gamma,\psi}(J,\mathbb{R})$ satisfty inequality (4.3). Solving equation (4.3) and taking into consideration Assumption (A4), we get

$$\left|\widetilde{u}(t) - \mathcal{H}_{\widetilde{u}} - \frac{1}{\Gamma(\alpha)}\int_a^t \Phi_\psi^\alpha(t,s) f\left(s,\widetilde{u}(s),\chi\widetilde{u}(s)\right) ds\right| \leqslant \epsilon\lambda_\varphi\varphi(t). \tag{4.8}$$

On the other hand, let $u \in C^{\gamma}_{1-\gamma,\psi}(J,\mathbb{R})$ be the unique solution of problem (1.4)-(1.5), that is

$$u(t) = \mathcal{H}_u + \frac{1}{\Gamma(\alpha)}\int_a^t \Phi_\psi^\alpha(t,s) f\left(s,u(s),\chi u(s)\right) ds, \tag{4.9}$$

Then, as in (4.7), for each $t \in (a,b]$ we have

$$\begin{aligned}
|\widetilde{u}(t) - u(t)| \leqslant& \left|\widetilde{u}(t) - \mathcal{H}_{\widetilde{u}} - \frac{1}{\Gamma(\alpha)}\int_a^t \Phi_\psi^\alpha(t,s) f\left(s,\widetilde{u}(s),\chi\widetilde{u}(s)\right) ds\right| \\
&+ |\mathcal{H}_{\widetilde{u}} - \mathcal{H}_u| + \frac{1}{\Gamma(\alpha)}\int_a^t \Phi_\psi^\alpha(t,s)\left|f(t,\widetilde{u}(t),\chi\widetilde{u}(t)) - f(t,u(t),\chi u(t))\right| ds \\
\leqslant& \epsilon\lambda_\varphi\varphi(t) + \frac{1}{\Gamma(\alpha)}\int_a^t \Phi_\psi^\alpha(t,s) M\left[|\widetilde{u}(s) - u(s)| + |\chi\widetilde{u}(s) - \chi u(s)|\right] ds \\
\leqslant& \epsilon\lambda_\varphi\varphi(t) + \frac{(M + ML^*)}{\Gamma(\alpha)}\int_a^t \Phi_\psi^\alpha(t,s)\left|\widetilde{u}(s) - u(s)\right| ds.
\end{aligned}$$

We apply Lemma 4.1 and we get:

$$\begin{aligned}
|\widetilde{u}(t) - u(t)| \leqslant& \epsilon\lambda_\varphi\varphi(t) + \epsilon\lambda_\varphi\int_a^t \sum_{k=1}^{\infty}\frac{(M + ML^*)^k}{\Gamma(\alpha k)}\Phi_\psi^{\alpha k}(t,s)\varphi(s) ds \\
=& \epsilon\lambda_\varphi\varphi(t) + \epsilon\lambda_\varphi\left[\int_a^t \frac{(M + ML^*)}{\Gamma(\alpha)}\Phi_\psi^{\alpha}(t,s)\varphi(s) ds\right. \\
&\left. + \int_a^t \frac{(M + ML^*)^2}{\Gamma(2\alpha)}\Phi_\psi^{2\alpha}(t,s)\varphi(s) ds + \dots\right] \\
=& \epsilon\lambda_\varphi\varphi(t) + \epsilon\lambda_\varphi\left[(M + ML^*)\, I_{a^+}^{\alpha;\psi}\varphi(t) + (M + ML^*)^2\, I_{a^+}^{2\alpha;\psi}\varphi(t) + \dots\right] \\
\leqslant& \epsilon\lambda_\varphi\varphi(t) + \epsilon\lambda_\varphi\left[(M + ML^*)\,\lambda_\varphi\varphi(t) + (M + ML^*)^2\,(\lambda_\varphi)^2\,\varphi(t) + \dots\right] \\
=& \epsilon\lambda_\varphi\varphi(t)\left[1 + \sum_{k=1}^{\infty}(M + ML^*)^k\,(\lambda_\varphi)^k\right].
\end{aligned}$$

Then, for

$$C_{f,\varphi} = \lambda_\varphi\left[1 + \sum_{k=1}^{\infty}(M + ML^*)^k\,(\lambda_\varphi)^k\right]$$

we get that

$$|\widetilde{u}(t) - u(t)| \leqslant C_{f,\varphi}\epsilon\varphi(t).$$

This proves that problem (1.4)-(1.5) is Ulam–Hyers-Rassias stable. The proof is complete. □

Corollary 4.2. *Under the assumptions of Theorem 4.3 problem (1.4)-(1.5) is generalized Ulam-Hyers-Rassias stable.*

Proof. We follow the lines of the proof of Theorem 4.3 and choosing $\epsilon = 1$, we get

$$|\widetilde{u}(t) - u(t)| \leqslant C_{f,\varphi}\epsilon\varphi(t).$$

□

5. Examples

5.1. Example 1. Consider the nonlocal fractional integro-differential equation involving the ψ-Hilfer fractional derivative:

$$D_{0^+}^{\frac{1}{2},\frac{1}{3},t}u(t) = \frac{1}{3E_1(t+2)(1+|u|)} + \frac{1}{3E_1(2)}\int_0^t e^{\frac{-1}{2}u(s)}ds, \qquad (0,1], \tag{5.1}$$

$$I_{0^+}^{\frac{1}{3},t}u(0) = \frac{1}{2}u(\frac{2}{3}). \tag{5.2}$$

Here

$$\alpha = \frac{1}{2}, \qquad \beta = \frac{1}{3}, \qquad \gamma = \frac{2}{3}, \qquad u_0 = 0, \qquad c_1 = \frac{1}{2}, \qquad \tau_1 = \frac{2}{3},$$

and $\psi : [0,1] \to \mathbb{R}$ is such that $\psi(t) = t$ for all $t \in [0,1]$ and

$$f(t, u(t), \chi u(t)) = t^{\frac{-1}{6}} + \frac{1}{3E_1(t+2)(1+|u|)} + \frac{1}{3E_1(2)}\int_0^t e^{\frac{-1}{2}u(s)}ds \qquad t \in (0,1],$$

$$\chi u(t) = \int_0^t h(t,s,u(s))ds = \int_0^t e^{\frac{-1}{2}u(s)}ds.$$

Clearly,

$$f(t, u(t), \chi u(t)) \in C_{\frac{1}{3};t}^{\frac{2}{3}}([0,1], \mathbb{R}^+)$$

since

$$t^{\frac{1}{3}}f(t, u(t), \chi u(t)) \in C([0,1], \mathbb{R}^+).$$

Let $u, v \in \mathbb{R}^+$ and $t \in (0,1]$; then it is easy to see that

$$|f(t, u(t), \chi u(t)) - f(t, v(t), \chi v(t))| \leqslant \frac{1}{3E_1(2)}\left(|u - v| + |\chi u - \chi v|\right),$$

and

$$|\chi u - \chi v| \leqslant \int_0^t e^{\frac{-1}{2}|u(s)-v(s)|}ds \leqslant \frac{1}{2}|u - v|.$$

Hence, Assumptions (A1) and (A2) hold with

$$M = \frac{1}{3E_1(2)}, \qquad L^* = \frac{1}{2}.$$

We are goint to check that Assumption (A3) hold as well.

Indeed, by simple calculations we see that

$$B = c_1\Phi_\psi^\gamma(\tau_1, 0) = c_1\frac{[\psi(\tau_1) - \psi(0)]^{\gamma-1}}{\Gamma(\gamma)} \simeq 0.67 \neq 1,$$

and

$$\Omega \simeq 0.10 < 1.$$

And since all assumptions of Theorem 3.1 are satisfied, problem (5.1)-(5.2) has at least one solution in $C^{\frac{2}{3}}_{\frac{1}{3};t}([0,1],\mathbb{R}^+)$. For $t\in(0,1]$ and $u\in\mathbb{R}^+$ we have $\Lambda\simeq 0.42<1$, therefore, condition (4.1) holds. Hence, by Theorem 4.1, problem (5.1)-(5.2) has a unique solution in $C^{\frac{2}{3}}_{\frac{1}{3};t}([0,1],\mathbb{R}^+)$.

5.2. Example 2. We consider the nonlocal fractional integro-differential equation involving the ψ-Hilfer-Hadamard fractional derivative

$$D^{\frac{1}{2},0,\ln t}_{1^+}u(t)=\frac{1}{20}\ln(\sqrt{t})\cos(t)u(t)+\frac{1}{20}\int_1^t e^{\frac{-1}{2}u(s)}ds,\qquad t\in(1,e], \tag{5.3}$$

$$I^{\frac{1}{2},\ln t}_{1^+}u(1)=\frac{1}{2}u\left(\frac{2}{3}\right), \tag{5.4}$$

where

$$\alpha=\frac{1}{2},\qquad \beta=0,\qquad \gamma=\frac{1}{2},\qquad u_0=0,\qquad c_1=\frac{1}{2},\qquad \tau_1=\frac{3}{2},\qquad \psi(t)=\ln t,\quad t\in[1,e],$$

and

$$f(t,u(t),\chi u(t))=\frac{1}{20}\ln(\sqrt{t})\cos(t)u(t)+\frac{1}{20}\int_1^t e^{\frac{-1}{2}u(s)}ds\qquad t\in(1,e],$$

$$\chi u(t)=\int_1^t e^{\frac{-1}{2}u(s)}ds.$$

Clearly,

$$f(t,u(t),\chi u(t))\in C^{\frac{1}{2}}_{\frac{1}{2};\ln t}([1,e],\mathbb{R})$$

since

$$(\ln t)^{\frac{1}{2}}f(t,u(t),\chi u(t))\in C([1,e],\mathbb{R}).$$

Let $u,v\in\mathbb{R}$ and $t\in(1,e]$, then it is easy to see that

$$|f(t,u(t),\chi u(t))-f(t,v(t),\chi v(t))|\leqslant\frac{1}{20}\left(|u-v|+|\chi u-\chi v|\right),$$

and

$$|\chi u-\chi v|\leqslant\int_1^t e^{-\frac{1}{2}|u(s)-v(s)|}ds\leqslant\frac{1}{2}|u-v|.$$

Hence, Assumptions (A1) and (A2) hold with

$$M=\frac{1}{20},\qquad L^*=\frac{1}{2}.$$

Let us check that condition (4.1) is satisfied.

Indeed, by simple calculations we see that for $t\in(1,e]$,

$$B=c_1\Phi^{\gamma}_{\psi}(\tau_1,0)=c_1\frac{[\ln(\tau_1)-\ln(1)]^{\gamma-1}}{\Gamma(\gamma)}=\frac{1}{2\sqrt{\pi\log(\frac{3}{2})}}\simeq 0.44\neq 1,$$

and

$$\Lambda\simeq 0.29<1.$$

Then by Theorem 4.1 problem (5.3)-(5.4) has a unique solution in $C^{\frac{1}{2}}_{\frac{1}{2};\ln t}([1,e],\mathbb{R})$. As it has been shown in Theorem 4.2, for $\epsilon=\frac{1}{2}>0$, if $\widetilde{u}\in C^{\frac{1}{2}}_{\frac{1}{2};\ln t}([1,e],\mathbb{R})$ satisfies

$$\left|D^{\alpha,\beta;\psi}_{1^+}\widetilde{u}(t)-f(t,\widetilde{u}(t),\chi\widetilde{u}(t))\right|\leqslant\frac{1}{2},\qquad t\in(1,e],$$

there exists a unique solution $u \in C^{\frac{1}{2}}_{\frac{1}{2};\ln t}([1,e],\mathbb{R})$ such that

$$|\widetilde{u}(t)-u(t)| \leqslant \frac{1}{2}C_f,$$

where

$$C_f = \frac{1}{2}\Psi^{\alpha}_{\psi}(e,1)E_{\frac{1}{2}}\left[\frac{3}{40}\Phi^{\alpha}_{\psi}(e,1)\right] = \frac{1}{\Gamma(\frac{3}{2})}e^{\left(\frac{3}{40}\right)^2}\left[1+\operatorname{erf}\left(\frac{3}{40}\right)\right] \simeq 1.23 > 0.$$

Here

$$\Psi^{\alpha}_{\psi}(e,1) := \frac{[\ln(e)-\ln(1)]^{\alpha}}{\Gamma(\alpha)}, \qquad \Phi^{\alpha}_{\psi}(e,1) := \ln'(t)\mid_{t=1}[\ln(e)-\ln(1)]^{\alpha-1}.$$

Hence, problem (5.3)-(5.4) is Ulam-Hyers stable.

Finally, we consider $\varphi(t) = (\ln t)^{\frac{1}{2}}$, then

$$(\ln t)^{\frac{1}{2}}\varphi(t) = \ln(t) \in C([1,e],\mathbb{R}),$$

i.e., $\varphi(t) \in C_{\frac{1}{2},\ln t}([1,e],\mathbb{R})$.

In order to verify the condition

$$I^{\alpha;\ln t}_{1^+}\varphi(t) \leqslant \lambda_{\varphi}\varphi(t), \qquad \lambda_{\varphi} > 0,$$

by employing the Hadamard fractional integral and simple computations we get

$$I^{\frac{1}{2},\ln t}_{1^+}\ln t = \frac{1}{\Gamma(\frac{1}{2})}\int_1^t\left(\ln\frac{t}{s}\right)^{-\frac{1}{2}}(\ln s)^{\frac{1}{2}}\frac{ds}{s} \leqslant \frac{1}{\Gamma\left(\frac{1}{2}\right)}\int_1^t\left(\ln\frac{t}{s}\right)^{-\frac{1}{2}}\frac{ds}{s} \leqslant \frac{2}{\sqrt{\pi}}\ln^{\frac{1}{2}}(t).$$

Thus, Assumption (A4) is satisfied with

$$\lambda_{\varphi} = \frac{2}{\sqrt{\pi}} > 0.$$

And for $\epsilon = \frac{1}{2} > 0$, if $\widetilde{u} \in C^{\frac{1}{2}}_{\frac{1}{2};\ln t}([1,e],\mathbb{R})$ satisfies

$$\left|D^{\alpha,\beta;\psi}_{1^+}\widetilde{u}(t) - f(t,\widetilde{u}(t),\chi\widetilde{u}(t))\right| \leqslant \frac{1}{2}(\ln t)^{\frac{1}{2}}, \qquad t \in (1,e],$$

there exists a unique solution $u \in C^{\frac{1}{2}}_{\frac{1}{2};\ln t}([1,e],\mathbb{R})$ such that

$$|\widetilde{u}(t)-u(t)| \leqslant C_{f,\varphi}\frac{1}{2}(\ln t)^{\frac{1}{2}} = \frac{2}{\sqrt{\pi}}\left[1+\sum_{k=1}^{\infty}\left(\frac{3}{40}\right)^k\left(\frac{2}{\sqrt{\pi}}\right)^k\right]\frac{1}{2}(\ln t)^{\frac{1}{2}}.$$

Hence, problem (5.3)-(5.4) is Ulam–Hyers–Rassias stable. Finally, taking $\epsilon = 1$, we get

$$|\widetilde{y}(t)-y(t)| \leqslant \frac{2}{\sqrt{\pi}}\left[1+\sum_{k=1}^{\infty}\left(\frac{1}{20}+\frac{1}{40}\right)^k\left(\frac{2}{\sqrt{\pi}}\right)^k\right](\ln t)^{\frac{1}{2}}.$$

Therefore, problem (5.3)-(5.4) is generalized Ulam–Hyers–Rassias stable.

Conclusions

The main results of this article have been successfully achieved by employing Krasnoselskii and Banach fixed point theorems and our most important results in the made nonlinear analysis is the study of the existence and uniqueness of solutions tothe Cauchy-type problem for a nonlinear fractional integro-differential equation introduced by the left ψ-Hilfer fractional derivative. We discussed the Ulam-Hyers, generalized Ulam–Hyers, Ulam-Hyers-Rassias, and generalized Ulam-Hyers-Rassias stabilities. This paper contributes to the growth of the fractional calculus, especially in the case of fractional differential equations involving a general formulation of Hilfer fractional derivative with respect to another function.

There are some articles that carried out a brief study on existence, uniqueness, and stability of solutions of fractional differential equations, however, there are just a few of them devoted to Hilfer type operator and one of our aims was to contribute in this field. We expect that our results can be extended to some other fractional differential equations involving Hilfer derivative with respect to another function ψ.

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