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CONSERVATION LAWS FOR VOLTERRA CHAIN WITH INITIAL STEP-LIKE CONDITION

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Abstract. In the present work we study a system of equations in the Volterra chain with initial step-like condition. The solutions to the Cauchy problem are sought in the class of positive functions. The nature of the problem is in some sense close to the problem on collapse of a discontinuity for the Korteweg-de-Vries equation. We show that the solution to the Cauchy problem for the Volterra chani can be constructed as a Taylor series. For bounded initial conditions, we obtain estimates implying that the convergence series exceeds zero. We formulate a local existence and uniqueness theorem for the solution to the Cauchy problem with bounded initial conditions.

We consider a special condition of the break of the Volterra chain: $b_n b_{n+1} = 1$, $n \geqslant N \geqslant 2$. We provide specified estimates for solutions of the break of the chain. We prove that under the break, the solutions to the chain are defined for all positive time. We also establish two conservation laws for the broken chain. One of the laws follows the break condition, while the other is implied by the Lagrange property.

Keywords: Volterra chain, Langmuir chain, integrable systems, conservation laws, problem on collapse of an initial discontinuity.

Mathematics Subject Classification: 34A12, 34A55, 35Q53, 37K40

Introduction

In the present work wee study equations of Volterra chain [1]:

$$\dot{b}_n = b_n(b_{n+1} - b_{n-1}), \quad b_n = b_n(t), \quad t \in \mathbb{R}, \quad n \in \mathbb{Z}. \tag{1}$$

Many works are devoted to system (1) (see, for instance, [2]–[5] and the references therein). Such interest is not only due to various applications, but also due to the fact that the Volterra chain is an interesting example of an integrable differential-difference model, which can be studied in the framework of the method of the inverse scattering problem. It is well-known that the method of the inverse scattering problem allows one to study in details the Cauchy problem for an infinite Volterra chain in the case of a fast decaying initial data (see [4]) and in a periodic case (see [6]).

In the present work, the main attention is paid to studying the properties of *positive* solutions to the Cauchy problem for the Volterra chain with step-like initial data:

$$b_n(0) = \begin{cases} a, & n < 0, \\ c, & n = 0, \\ b, & n > 0, \end{cases} \tag{2}$$

where a, b, c are given non-negative numbers, and we also study a particular case on the half-line:

$$b_0(t) = 0, \qquad \dot{b}_n = b_n(b_{n+1} - b_{n-1}), \quad t > 0, \qquad b_n(0) = 1, \quad n \geqslant 1, \tag{3}$$

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corresponding to the choice $a = c = 0$. We discuss solvability and breaking conditions for the Volterra chain. We note that the Volterra chain is a difference analogue of the Korteweg-de-Vries equation and in the continuous limit, this chain becomes this equation [7, Ch. 1, Sect. 7]. The Cauchy problem for the Korteweg-de-Vries equation with a step-like initial data was considered in work by A.V. Gurevich and L.P. Pitaevskii [8] in the framework of studying a non-stationary structure of collisionless shock wave. Further developing and the results in this theory can be found in work by B.A. Dubrovin and S.P. Novikov [9] in a section devoted to the discontinuity collapse in the KdV theory and to asymptotics as $t \gg 1$ for KdV solution with a step-like initial condition.

Considering positive solutions of system (1), we take into consideration that as numerical experiments show, omitting the positivity condition gives rise to breakup of the solution to Cauchy problem (1), (2) (or to problem (3)) in a finite time. The positivity of solutions of system (1) allows us to introduce new dynamical variables $y_n(t) = \log b_n(t)$ and write the Volterra chain as

$$\dot{y}_n = e^{y_{n+1}} - e^{y_{n-1}}, \qquad n \in \mathbb{Z}. \tag{4}$$

The solution to the Cauchy problem for equations (4) with given initial conditions $y_n(0)$ can be represented as a power series[1]

$$y_n(t) = y_n(0) + \frac{y_n^{(1)}(0)}{1!}t + \cdots + \frac{y_n^{(k)}(0)}{k!}t^k + \ldots, \tag{5}$$

but the issue on the convergence radius for this series, it analytic continuation and a character of singular points at the boundary of its convergence circle is still open. This is why a more appropriate way of proving the solvability of the Cauchy problem is the successive approximations method. The structure of right hand sides in differential equations of chain (4) allows us to formaly the theorem on local solvability of the Cauchy problem for infinite system (4) with initial conditions $\{y_n(0)\} \in l^\infty$.

Nowadays there are various break conditions for chain (1) keeping the integrability properties. For instance, in work [3], there was considered a break compatible with the conservations laws, while in [10] a break was compatible with higher symmetries. In the present paper we consider the break of the chain (3) determined by the conditions

$$b_n b_{n+1} = 1, \qquad n \geqslant N,$$

for some fixed $N \geqslant 2$. For instance,in the simplest case $N = 2$, system of equations (1) becomes

$$\dot{b}_1 = b_1 b_2, \qquad \dot{b}_2 = 1 - b_1 b_2, \qquad t > 0,$$

and is reduced to Ricatti equation

$$\dot{b}_1 + b_1^2 = (t + 2)b_1,$$

whose solution is well-defined on the entire half-line and is expressed via the probability integral. As $N > 2$, for broken chain (3) we establish two conservation laws, one being implied the break condition, while the other follows the Lagrangian of the broken system. Moreover, the first conservation law ensures the possibility of continuing the solutions of closed chain (3) on the entire half-line in the general case $N \geqslant 2$. Altogether, as numerical experiments show, in problem (3) we can select three time intervals: 1) an initial interval $0 \leqslant t < R$, where R is the convergence radius of Taylor series (5); 2) a period of linear growth $b_1(t)$; 3) a segment of quasi-stationary growth as $\dot{b}_1(t) \approx 0$ and $b_1(t) \approx 4$, see Section 13.

1. Solvability of Cauchy problem

In the present section we consider the solvability of the Cauchy problem

$$\dot{y}_n = e^{y_{n+1}} - e^{y_{n-1}}, \quad y_n(0) = y_{n,0}, \quad n \in \mathbb{Z}. \tag{6}$$

In introduce a function of two variables $f(x) = f(x_1, x_2) = e^{x_1} - e^{x_2}$ defined in the square $\Pi = \{x : -\rho \leqslant x_1, x_2 \leqslant \rho\}$. It is obvious that

$$M = \max_{x \in \Pi} |f(x)| = e^\rho - e^{-\rho}. \tag{7}$$

[1]The superscripts stands for the order of the derivative

Also, since

$$\max_{x\in\Pi} |\nabla f(x)| = \sqrt{2}e^{\rho},$$

the function f satisfies the Lipschitz condition in

$$|f(x^*) - f(x)| \leqslant \sqrt{2}e^{\rho} \left(|x_1^* - x_1| + |x_2^* - x_2|\right)$$

in Π. Returning back to problem (6), we introduce the notations

$$\mathbf{y}(t) = \{y_n(t)\}_{n\in\mathbb{Z}}, \qquad \mathbf{y}_k = \{y_{k,n}(t)\}_{n\in\mathbb{Z}}, \qquad F(\mathbf{y}) = \{F_n(\mathbf{y})\}_{n\in\mathbb{Z}} = \{f(y_{n+1}, y_{n-1})\}_{n\in\mathbb{Z}}.$$

Then Cauchy problem for system (6) becomes

$$\dot{\mathbf{y}}(t) = F(\mathbf{y}(t)), \qquad \mathbf{y}(0) = \mathbf{y}_0. \tag{8}$$

By $C[(\xi,\eta); l^\infty]$ we denote the set of the functions $\mathbf{y} : (\xi,\eta) \to l^\infty$ such that $y_n(t) \in C(\xi,\eta)$ for all $n \in \mathbb{Z}$. Here l^∞ stands for the space of bounded sequences of real numbers. To each function $\mathbf{y} \in C[(\xi,\eta); l^\infty]$, we associate a real-valued function

$$\|\mathbf{y}(t)\| := \sup_{n\in\mathbb{Z}} |y_n(t)| < \infty$$

defined on the same segment (ξ,η).

It is easy to see that the Cauchy problem (8) is equivalent to the integral equation

$$\mathbf{y}(t) = \mathbf{y}_0 + \int_0^t F(\mathbf{y}(s))ds. \tag{9}$$

We assume that a solution to integral equation (9) exists if there exists an interval $(\xi,\eta) \ni 0$ and a function $\mathbf{y} \in C[(\xi,\eta); l^\infty]$ satisfying equation (9).

Theorem 1. *For all $\rho > 0$ and $\mathbf{y}_0 \in l^\infty$ there exists a unique solution of integral equation* (9) *defined at least on the interval $(-h,h)$, where $h = \frac{\rho}{M}$, and such that $\sup\limits_{|t|<h} \|\mathbf{y}(t) - \mathbf{y}_0\| \leqslant \rho$.*

Proof. The theorem is proved by the standard scheme by means of the successive approximations method. Cauchy problem (8) is reduced to a problem with zero initial conditions. This is why, thanks to the condition $\mathbf{y}_0 \in l^\infty$, without loss of generality we can suppose that $\mathbf{y}_0 = \mathbf{0}$.

Taking into consideration the properties of the function F, it is easy to see that if $\sup\limits_t \|\mathbf{y}^*(t)\| \leqslant \rho$, $\sup\limits_t \|\mathbf{y}(t)\| \leqslant \rho$, then for each $n \in \mathbb{Z}$, the Lipschitz condition

$$|F_n(\mathbf{y}^*(t)) - F_n(\mathbf{y}(t))| \leqslant L\left(|y_{n+1}^*(t) - y_{n+1}(t)| + |y_{n-1}^*(t) - y_{n-1}(t)|\right), \quad L = \sqrt{2}e^{\rho}$$

holds true. This allows to introduce the sequence of approximations

$$\mathbf{y}_k(t) = \int_0^t F(\mathbf{y}_{k-1}(s))ds, \quad k = 1, 2, \ldots$$

and to reproduce "classical" arguing.

Step 1. Well-defined property of iterations. If we restrict our considerations by the segment $-h < t < h$, where $h = \rho/M$ and the quantity M is defined by formula (7), then the procedure of successive approximations is well-defined thanks to the estimates

$$|y_{k,n}(t)| \leqslant \left| \int_0^t |F_n(\mathbf{y}_{k-1}(s))|\, ds \right| \leqslant M|t|.$$

Step 2. Construction of majorant series. In view of the Lipschitz condition, we have

$$|y_{1,n}(t)| \leqslant \left| \int_0^t |F_n(\mathbf{0})|\, ds \right| \leqslant h,$$

$$\begin{aligned}|y_{k+1,n}(t) - y_{k,n}(t)| \leqslant & \left| \int_0^t |F_n(\mathbf{y}_k(s)) - F_n(\mathbf{y}_{k-1}(s))|\, ds \right| \\ \leqslant & L \left| \int_0^t |y_{k,n+1}(s) - y_{k-1,n+1}(s)| + |y_{k,n-1}(s) - y_{k-1,n-1}(s)|\, ds \right| \leqslant 2h\frac{L^k|t|^k}{k!}.\end{aligned}$$

Step 3. Proof of uniqueness. If $\hat{\mathbf{y}}$ and $\mathbf{y}$ are two solutions, then by the Lipschitz condition we obtain

$$\|\hat{\mathbf{y}}(t) - \mathbf{y}(t)\| \leqslant 2\rho\frac{L^k|t|^k}{k!}$$

for all $t \in (-h, h)$ and $k \in \mathbb{N}$. The proof is complete. □

Corollary 1. *In the case of step* (2)*, for sufficiently small time, the sum of first derivatives of the functions b_n satisfying system of equations* (1) *is constant and is equal to*

$$\sum_{n=-\infty}^{\infty} \dot{b}_n = b^2 - a^2. \tag{10}$$

2. Break of Volterra chain and conservation laws

We consider Volterra chain (3) and we complete it a break condition defined for some fixed $N > 1$ by identities

$$b_n(t)b_{n+1}(t) = 1, \quad n \geqslant N. \tag{11}$$

The break condition of the chain gives easily the relation

$$\dot{b}_1 + \dots + \dot{b}_N = 1, \quad N \geqslant 2, \tag{12}$$

which implies immediately one conservation law:

$$b_1 + b_2 + \ldots + b_N = N + t. \tag{13}$$

In particular, the latter identity implies two-sided estimates (see Section 3)

$$1 \leqslant b_1(t) < N + t, \qquad 0 < b_n(t) < N - 1 + t, \qquad 2 \leqslant n \leqslant N,$$

allowing us to specify the value h in Theorem 1 and to formulate the following statement.

Proposition 1. *There exists a unique solution of problem* (3), (11)*, which can be continued to the entire half-line $t > 0$.*

Let us show that as $N > 2$, system of equations (3) is reduced to deforming of exponential systems of series A with a positive definite quadratic form in a Lagragian.

To simplify the writing, we introduce the notation $\beta_n = b_n b_{n+1}$ and we rewrite problem (6) in terms of β_n. Employing relations (3), (11), we get

$$\begin{aligned}&(\log \beta_1)_t = \frac{\dot{b}_1}{b_1} + \frac{\dot{b}_2}{b_2} = b_3 + b_2 - b_1,\\ &(\log \beta_n)_t = \frac{\dot{b}_n}{b_n} + \frac{\dot{b}_{n+1}}{b_{n+1}} = b_{n+1} - b_{n-1} + b_{n+2} - b_n, \quad 1 < n < N,\\ &(\log \beta_n)_t = 0, \quad n \geqslant N.\end{aligned}$$

By (11) this implies a finite system of second order equations

$$(\log\beta_n)_{tt} = \begin{cases} -2\beta_n + \beta_{n+2}, & n = 1, 2, \\ \beta_{n-2} - 2\beta_n + \beta_{n+2}, & 2 < n < N-2, \\ \beta_{n-2} - 2\beta_n + 1, & n = N-2, N-1. \end{cases} \tag{14}$$

We observe that the equations involving the derivatives of the functions $\log\beta_n$ with odd indices form an independent system. This is why, introducing the notations $z_k = \log\beta_{2k-1}$ and $n = \lfloor \frac{N}{2} \rfloor$, we arrive at the following system of differential equations

$$\begin{cases} \ddot{z}_1 = -2e^{z_1} + e^{z_2}, \\ \ddot{z}_2 = e^{z_1} - 2e^{z_2} + e^{z_3}, \\ \qquad \dots \\ \ddot{z}_{n-1} = e^{z_{n-2}} - 2e^{z_{n-1}} + e^{z_n}, \\ \ddot{z}_n = e^{z_{n-1}} - 2e^{z_n} + 1. \end{cases}$$

Rewriting this system as

$$\ddot{\mathbf{z}} + A_n \exp\mathbf{z} = \mathbf{e}_n, \tag{15}$$

where A_n is a three-diagonal matrix with 2 at the main diagonal and -1 on adjoining ones, $\exp\mathbf{z} = (e^{z_1}, \dots, e^{z_n})^T$ and $\mathbf{e}_n = (0, 0, \dots, 0, 1)^T$. We obtain the following statement.

Lemma 1. *The Lagrangian of system* (15) *is written as*

$$L_n(\mathbf{z}, \dot{\mathbf{z}}) = \frac{1}{2}\sum_{i,j=1}^{n} a_{ij}\dot{z}_i\dot{z}_j + \sum_{i=1}^{n}\gamma_i z_i - \sum_{i=1}^{n} e^{z_i}, \tag{16}$$

where $a_{ij} = a_{ji}$ are the elements of the inverse to A_n matrix:

$$A_n^{-1} = \|a_{ij}\|, \quad a_{ij} = \frac{i(n-j+1)}{n+1}, \quad 1 \leqslant i \leqslant j \leqslant n;$$

and the numbers $\gamma_i = i/(n+1)$ are components of the vector $\gamma = (\gamma_1, \gamma_2, \dots, \gamma_n)^T$, satisfying the algebraic system of equations $A_n\gamma = \mathbf{e}_n$.

Example 1. *As $n = 3$, we obtain:*

$$\ddot{z}_1 = -2\beta_1 + \beta_3, \quad \ddot{z}_2 = \beta_5 - 2\beta_3 + \beta_1, \quad \ddot{z}_3 = 1 - 2\beta_5 + \beta_3. \tag{17}$$

The corresponding Lagrangian reads as

$$L_3(\mathbf{z}, \dot{\mathbf{z}}) = \frac{3\dot{z}_1^2 + 4\dot{z}_2^2 + 3\dot{z}_3^2}{8} + \frac{2\dot{z}_1\dot{z}_2 + \dot{z}_1\dot{z}_3 + 2\dot{z}_2\dot{z}_3}{4} + \frac{z_1 + 2z_2 + 3z_3}{4} - \left(e^{z_1} + e^{z_2} + e^{z_3}\right).$$

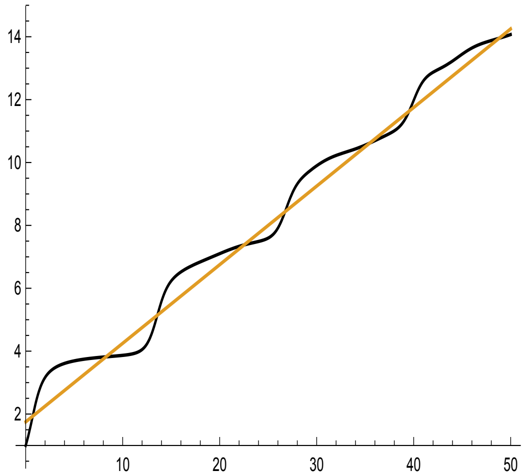
In addition to Lemma 1 we note that $\det A_n = n + 1$ and $A_n^{-1} \to A$ as $n \to \infty$,

$$A = \begin{pmatrix} 1 & 1 & 1 & 1 & 1 & \dots \\ 1 & 2 & 2 & 2 & 2 & \dots \\ 1 & 2 & 3 & 3 & 3 & \dots \\ & \ddots & \ddots & \ddots & \ddots & \end{pmatrix}, \quad \mathrm{diag}A = (1, 2, 3, 4, 5, \ \dots)$$

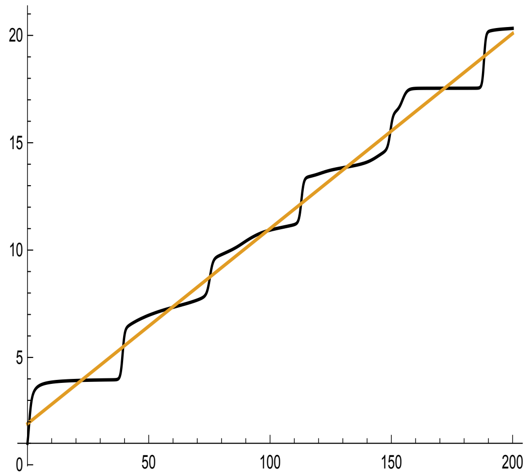
We also observe that system (3) broken by condition (11) not only preserves the Lagrangian structure but also in a good accordance with infinite chain (3) if we solve both by the iteration method. Namely, we reduce both differential system (3) and its break to systems of integral equations in terms of the variables $y_n = \log b_n$ and then we solve them by the iteration method. Then in a kth approximation $\mathbf{y}_k = \{y_{k,1}, y_{k,2}, \dots, y_{k,n}, \dots\}$ for both system the identities $y_{k,n} = 0$ as $n > k$. Such “triangle” property of two systems yields that there first $N - 1$ iterations $\mathbf{y}_k$ coincide.

3. Graphs

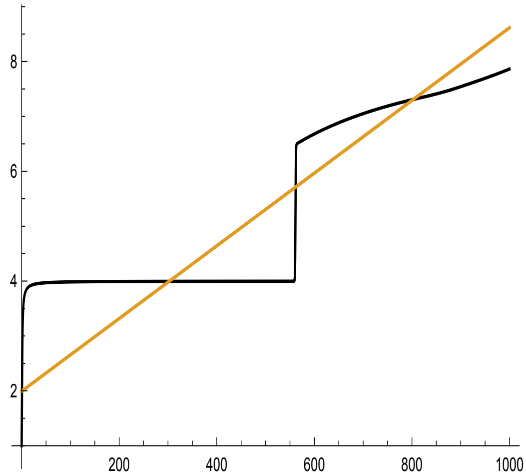
Below we provide the graphs of the function b_1 for a closed chain corresponding to the values $N = 5, 7, 101$. In all cases the graph of the function b_1 is compared with the line $f(t) = (t + N) \big/ \left\lceil \frac{N}{2} \right\rceil$.



$N = 7.$



$N = 21.$



$N = 301.$

As the provided graphs show, as N increases, the first step of the graph of the solution b_1 to a closed chain widens unboundedly approaching the value 4. Together with the remark in the end of Section 2, this allows us to conjecture that the solution b_1 of infinite chain (3) tends asymptotically to 4 as $t \to +\infty$.

4. Conclusion

There is still an open question on a possibility of continuing the solution of Cauchy problem (1), (2) on the entire real line. The issue on solvability of system (1) with unbounded initial data is also of interest. Trying various initial conditions, we can construct examples, when the corresponding Taylor series have the zero convergence radius. We hope that the studying of these issues will be fruitful.

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