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# ON SPECTRAL GAPS OF A LAPLACIAN IN A STRIP WITH A BOUNDED PERIODIC PERTURBATION

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**Abstract.** In the work we consider the Laplacian subject to the Dirichlet condition in an infinite planar strip perturbed by a periodic operator. The perturbation is introduced as an arbitrary bounded periodic operator in $L_2$ on the periodicity cell; then this operator is extended periodically on the entire strip.

We study the band spectrum of such operator. The main obtained result is the absence of the spectral gaps in the lower part of the spectrum for a sufficiently small potential. The upper bound for the period ensuring such result is written explicitly as a certain number. It also involves a certain characteristics of the perturbing operator, which can be nonrigorously described as "the maximal oscillation of the perturbation". We also explicitly write out the length of the part of the spectrum, in which the absence of the gaps is guaranteed. Such result can be regarded as a partial proof of the strong Bethe-Sommerfeld conjecture on absence of internal gaps in the band spectra of periodic operators for sufficiently small periods.

**Keywords:** periodic operator, Schrödinger operator, strip, Bethe-Sommerfeld conjecture.

**Mathematics Subject Classification:** 35P05; 35B10

## 1. Introduction

One of the classical conjecture in the theory of periodic differential operators is the Bethe-Sommerfeld conjecture. It conjectures that at least for a wide class of multi-dimensional periodic operators, their spectra can have only finitely many gaps. This conjecture was proved for a series of operators in multi-dimensional spaces. For the Schrödinger operators with a periodic potential this conjecture was proved for various dimensions and under various assumptions for the potential in works [1]–[6]. For the magnetic Schrödinger operator this conjecture was established in papers [7], [8]. In works [9]–[11], a polyharmonic operator was considered with various perturbations. The most general perturbation has a pseudo-differential operator of a lower order. Under certain conditions for the perturbation, the Bethe-Sommerfeld conjecture was proved. We also note that the cited list of papers does not pretend to be complete; further works can be found in the references of the cited papers.

The Bethe-Sommerfeld conjecture can be interpreted as the absence of gaps in an upper part of the spectrum of the considered operator. In other words, to the right to (upper than) some point the spectrum is a half-line and hence, it contains no gaps. An independent interest is the issue on the absence of the gaps in a lower part of the spectrum. A similar result is provided in Chapter 15 in book [6]. Here, the Laplacian was considered in a multidimensional space of dimension at least three. This operator was perturbed by a self-adjoint operator symmetric w.r.t. some rational lattice. It was proved in Theorem 15.2 that for a sufficiently small norm of the perturbing operator, the spectrum of the considered operator contains no gaps. In

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particular, this means the absence of the gaps in a lower part of the spectrum, that is, to the left to some point. In the case when the perturbation is a multiplication by a periodic potential, in the dimension three, the condition of the rationality can be replaced by the condition of the inclusion of cubic sublattice, see [6, Ch. IV, Sect. 16, Thm. 16.2].

In the present paper we consider the Laplacian in a strip perturbed by a bounded periodic operator. Such choice of the domain distinguish our works from the above cited ones. We consider the case when the period of the perturbing operator is small enough. The main obtained result is the absence of the gaps in a lower part of the spectrum for a sufficiently small period. An advantage of the obtained result is the fact that the upper bounds for the period and the length of the range of the perturbing operator ensuring the absence of the gaps are written explicitly in terms of particular numbers. We also find explicitly the length of the lower part of the spectrum, in which the absence of the gaps is guaranteed. The main property of this part is that its length grows in a power law as the period becomes smaller.

An interest to the case of a small period is partially motivated by a series of works on homogenization of operators with fast oscillating coefficients and with various perturbations from the theory of boundary homogenization in domains like strips and infinite cylinders [12]–[18]. In all these works in the case of a pure periodic perturbations the perturbed operators turns out to be periodic with a small period. In these works there was proved the norm resolvent convergence of the perturbed operators to certain homogenized operators and this implies that their spectra converge to the spectra of homogenized operators. At the same time, the general convergence results does not imply the absence of the gaps but the only fact that if it exists, each such gap escape to infinity as the small parameter is lessening. By the escaping we mean the situation when the distance from the bottom of the essential spectrum to this gap grows as the small parameter is lessening. The issue on the growth rate for such distance was considered in [16]–[18]. Here there were constructed two-parameter asymptotics for the first band functions and these asymptotics implied that the distance from the bottom of the spectrum to the first gap is of order $O(\varepsilon^{-2})$, where $\varepsilon$ is the small parameter. In the present work a similar result is significantly better, here the distance is at least $O(\varepsilon^{-6})$, the lower bound for this distance is written explicitly with not undetermined constants.

After this paper was submitted, the author learnt about one more work, PhD thesis [19], in which the Bethe-Sommerfeld conjecture was proved for an operator with constant coefficients in a strip with a bounded symmetric perturbation. More precisely, in a strip of width $\pi r$, $r > 0$, the operator

$$- a^2\frac{\partial^2}{\partial x_1^2} - \frac{\partial^2}{\partial x_2^2} + \mathcal{B} \tag{1.1}$$

was considered subject to the Dirichlet condition, where $\mathcal{B}$ is a periodic bounded symmetric operator in the space $L_2$. The main result of [19] on spectral gaps states the following: under the condition

$$ar > 16,$$

there are finitely many gaps in the spectrum of the considered operator. The issue on the total absence of internal spectral gaps was not discussed in [19]. We also stress that by the key estimates, our method differs qualitatively from the approach in work [19], see the discussion in Section 6.

## 2. Formulation of problem and main results

Let $x = (x_1, x_2)$ be Cartesian coordinates in $\mathbb{R}^2$, $\Pi := \{x : 0 < x_2 < \pi\}$ be a horizontal strip of width $\pi$, $\varepsilon$ be a sufficiently small positive number. We denote $\square_\varepsilon := \{x : |x_1| < \varepsilon\pi, 0 < x_2 < \pi\}$, $\mathcal{L}_\varepsilon$ is is a symmetric operator in $L_2(\square_\varepsilon)$ bounded for each considered value $\varepsilon$. This operator generates an operator in $L_2(\Pi)$ a follows. Since the restriction of a function in $L_2(\Pi)$ on $\square_\varepsilon$ is an element in $L_2(\square_\varepsilon)$, in the sense of such restriction, the operator $\mathcal{L}_\varepsilon$ can be applied to the functions in $L_2(\Pi)$. We extend the result of the action of the operator $\mathcal{L}_\varepsilon$ by zero outside $\square_\varepsilon$. After such extension, the operator $\mathcal{L}_\varepsilon$ acts in $L_2(\Pi)$. Let $\mathcal{S}(n)$ be a translation operator in $L_2(\Pi)$ acting by the rule $(\mathcal{S}(n)u)(x) = u(x_1 - 2\varepsilon\pi n, x_2)$. By $\mathcal{V}_\varepsilon$ we denote the following operator in $L_2(\Pi)$:

$$\mathcal{V}_\varepsilon := \sum_{n\in\mathbb{Z}} \mathcal{S}(-n)\mathcal{L}_\varepsilon\mathcal{S}(n).$$

The operator $\mathcal{V}_\varepsilon$ is symmetric, bounded and periodic. The latter property is understood in the sense of the identity

$$\mathcal{V}_\varepsilon\mathcal{S}(p) = \mathcal{S}(p)\mathcal{V}_\varepsilon \quad \text{for all} \quad p \in \mathbb{Z}.$$

In the present work we consider the periodic operator in the strip $\Pi$

$$\mathcal{H}_\varepsilon := -\Delta + \mathcal{V}_\varepsilon \tag{2.1}$$

subject to the Dirichlet condition. This operator is regarded as an unbounded one in $L_2(\Pi)$ on the domain $\mathring{W}_2^2(\Pi)$. Here $\mathring{W}_2^j(\Omega) := \mathring{W}_2^j(\Omega, \partial\Omega)$, and $\mathring{W}_2^j(\Omega, S)$ is the space of functions in $W_2^j(\Omega)$ defined on some domain $\Omega$ and vanishing on a curve $S$.

Since the operator $\mathcal{V}_\varepsilon$ is symmetric and bounded, by Kato-Rellich theorem the operator $\mathcal{H}_\varepsilon$ is self-adjoint.

The operator $\mathcal{H}_\varepsilon$ has a band spectrum, which is introduced as the union of the images of the band functions. The band functions $E^k_{\mathcal{L}_\varepsilon}(\tau)$, $k \geqslant 1$, are the eigenvalues of the corresponding operators on the periodicity cell $\square_\varepsilon$ depending on the rescaled quasi-momentum $\tau \in \big[-\frac{1}{2}, \frac{1}{2}\big)$. The eigenvalues are taken in the ascending order counting multiplicities. For the operator $\mathcal{H}_\varepsilon$, the corresponding operator on the cell is

$$\mathcal{H}_\varepsilon(\tau) := \left(\mathrm{i}\frac{\partial}{\partial x_1} + \frac{\tau}{\varepsilon}\right)^2 - \frac{\partial^2}{\partial x_2^2} + \mathcal{L}_\varepsilon(\tau), \quad \mathcal{L}_\varepsilon(\tau)u := e^{\frac{\mathrm{i}\tau}{\varepsilon}x_1}\mathcal{L}_\varepsilon e^{-\frac{\mathrm{i}\tau}{\varepsilon}\cdot} u,$$

on $\square_\varepsilon$ subject to the Dirichlet condition on upper and lower boundaries of the cell $\square_\varepsilon$ and to the periodic condition on the lateral sides $l_\pm$ of the cell $\square_\varepsilon$, $l_\pm := \{x : x_1 = \pm\varepsilon\pi,\ x_2 \in (0,\pi)\}$. The operator $\mathcal{H}_\varepsilon(\tau)$ is considered as an operator in $L_2(\square)$ on the domain $\mathring{W}^2_{2,per}(\square_\varepsilon, \partial\square_\varepsilon \cap \partial\Pi)$, which is the space of functions in $\mathring{W}_2^j(\square_\varepsilon, \partial\square_\varepsilon \cap \partial\Pi)$ obeying periodic boundary condition on the lateral boundaries $l_\pm$.

For the operator $\mathcal{L}_\varepsilon$ we denote:

$$\begin{aligned} \omega_{\mathcal{L}_\varepsilon} &:= \sup_{\substack{u\in L_2(\square_\varepsilon)\\ \|u\|_{L_2(\square_\varepsilon)}=1}} (\mathcal{L}_\varepsilon u, u)_{L_2(\square_\varepsilon)} - \inf_{\substack{u\in L_2(\square_\varepsilon)\\ \|u\|_{L_2(\square_\varepsilon)}=1}} (\mathcal{L}_\varepsilon u, u)_{L_2(\square_\varepsilon)}, \\ \lambda_\varepsilon &:= \inf_{\substack{u\in L_2(\square_\varepsilon)\\ \|u\|_{L_2(\square_\varepsilon)}=1}} (\mathcal{L}_\varepsilon u, u)_{L_2(\square_\varepsilon)}. \end{aligned}$$

By $\sigma(\cdot)$ we denote the spectrum of an operator and $[\cdot]$ stands for the integer part of a number.

The main aim of the work is to prove the absence of the gaps in a certain part of the band spectrum of the operator $\mathcal{H}_\varepsilon$. The main result is formulated in the following theorem.

**Theorem 2.1.** *Let $\varepsilon \leqslant \varepsilon_0$, $\varepsilon^2\omega_{\mathcal{L}_\varepsilon} \leqslant b_0$, where*

$$5\varepsilon_0 + \frac{\pi b_0}{4} \leqslant 2A_0, \quad A_0 := \frac{3\sqrt{2}}{128} - \frac{5\sqrt{7}}{896}. \tag{2.2}$$

*We denote*

$$K_\varepsilon := \frac{A_0 + \sqrt{A_0^2 - \frac{\pi}{4}b_0\varepsilon - \varepsilon^2}}{2\varepsilon}. \tag{2.3}$$

*Then the part of the spectrum*

$$\left(-\infty, \frac{([K_\varepsilon^2]+1)^2}{\varepsilon^2} + \lambda_\varepsilon\right] \cap \sigma(\mathcal{H}_\varepsilon) \tag{2.4}$$

*of the operator $\mathcal{H}_\varepsilon$ has no gaps.*

Let us discuss the main result. We observe right now that by (2.2), the radicand in definition (2.3) of quantity $K_\varepsilon$ satisfies the estimate:

$$A_0^2-\frac{\pi b_0\varepsilon}{4}-\varepsilon^2\geqslant\left(\frac{5\varepsilon_0}{2}+\frac{\pi b_0}{8}\right)^2-\frac{\pi b_0\varepsilon_0}{4}-\varepsilon_0^2=\frac{21\varepsilon_0^2}{4}+\frac{3\pi b_0\varepsilon_0}{8}+\frac{\pi b_0^2}{64}\geqslant 0,$$

and this is why the quantity $K_\varepsilon$ is well-defined.

In fact, Theorem 2.1 states that for small periods, the operator $\mathcal{H}_\varepsilon$ has no spectral gaps in a lower part of the spectrum. Moreover, there is an estimate for the length of the part of the spectrum without gaps, see (2.4). For small $\varepsilon$, the quantity $K_\varepsilon$ behaves as $O(\varepsilon^{-1})$. This is why by (2.4), the lower part of the spectrum free of the gaps is of the length at least $O(\varepsilon^{-6})$. This is an essentially stronger result in comparison with the results of works [17], [18] for the model with frequent alternation of boundary conditions, where the absence of the gaps was stated for a part of the spectrum of a length not exceeding $C\varepsilon^{-2}$ with an unknown constant $C$. We also note that the length of the discussed part of the spectrum grows as $\varepsilon$ becomes smaller.

Explicit numerical constants in the statement of Theorem 2.1 are non-optimal and can be improved. At the same time, this requires using additional bulky technical details, which would complicate seriously the proof of Theorem 2.1 give below. This is why we have decided on non-optimal constants.

It should be stressed that we chose the Dirichlet condition on the boundary of the strip $\Pi$ just for being definite. If on the boundaries of the strip we impose the Neumann condition or a combination of Dirichlet and Neumann conditions on the upper and lower boundaries, the technique of our work is applicable in this case, too and it leads us to a result like in Theorem 2.1.

We also pay an attention to the fact that condition (2.2) is imposed to the maximal admissible values of the period and the size of the range of the operator $\mathcal{L}_\varepsilon$. In particular, it follows from the definition of $b_0$ that for each operator $\mathcal{L}_\varepsilon$ bounded uniformly in $\varepsilon$, by an appropriate choice of a sufficiently small $\varepsilon_0$, we can always achieve condition (2.2) and as a result, the absence of the gaps in the lower part of the spectrum.

In conclusion we compare our main result with that of thesis [19]. For simplicity we assume that $\mathcal{B}$ in (1.1) is the operator of multiplication by a bounded measurable real potential $V$, which is $2\pi$-periodic in $x_1$. By means of the change

$$x_1\mapsto\frac{x_1}{ar},\quad x_2\mapsto\frac{x_2}{r},$$

the operator (1.1) is reduced to the operator

$$-\Delta+r^2V(arx_1,x_2)$$

in the strip $\Pi$. By denoting $\varepsilon:=\frac{1}{ar}$, the latter operator becomes operator (2.1) with

$$\mathcal{L}_\varepsilon=\frac{1}{a^2\varepsilon^2}V\left(\frac{x_1}{\varepsilon},x_2\right). \tag{2.5}$$

For such operator the result of [19] stares only the finiteness of the number of the gaps in the spectrum. At that, the case of total absence of the gaps or the location of the first gap were not studied.

In order to apply our result, we first note that for operator (2.5), the quantity $\omega_{\mathcal{L}_\varepsilon}$ is of the form:

$$\omega_{\mathcal{L}_\varepsilon}=\frac{\omega_*}{a^2\varepsilon^2},\quad \omega_*:=\sup_{[0,2\pi]\times[0,\pi]}V(x)-\inf_{[0,2\pi]\times[0,\pi]}V(x)$$

and condition (2.2) is rewritten as

$$6\varepsilon_0+\frac{\pi\omega_*}{4a^2}\leqslant 2A_0.$$

By this inequality, it follows from Theorem 2.1 that for not very large oscillations of the potential $V$, lower part of the spectrum (2.4) has no internal gaps. For the spectrum of the original operator (1.1) this means that the part of its spectrum to the point $([K_\varepsilon^2]+1)^2\varepsilon^{-2}+\lambda_\varepsilon$ is free of internal gaps. We also note that the latter quantity is $O(\varepsilon^{-6})$ as $\varepsilon\to+0$.

## 3. Examples

In the present section we provide two main examples of the operator $\mathcal{L}$.

**3.1. Potential.** The first example is the operator of multiplication by the potential $V_\varepsilon(x)=V\left(\frac{x_1}{\varepsilon},x_2,\varepsilon\right)$. Here $V=V(\xi,x_2,\varepsilon)\in L_\infty(\square_1)$ is a bounded measurable real function for each $\varepsilon$. The potential $V_\varepsilon$ belongs to $L_\infty(\square_\varepsilon)$. The corresponding operator $\mathcal{V}_\varepsilon$ is the multiplication by the potential obtained by the $2\pi\varepsilon$-periodic continuation of $V_\varepsilon$ w.r.t. $x_1$ to the entire strip $\Pi$. Such operator satisfies all needed conditions and the quantity $\omega_{\mathcal{L}_\varepsilon}$ becomes the oscillation of the potential:

$$\omega_{\mathcal{L}_\varepsilon}=\omega_V:=\operatorname*{ess\,sup}_{[0,2\pi]\times[0,\pi]}V-\operatorname*{ess\,inf}_{[0,2\pi]\times[0,\pi]}V.$$

Moreover, Theorem 2.1 works also for potentials $V$ singularly depending on the parameter. For instance, let

$$V(\xi,x_2,\varepsilon)=\varepsilon^{-a(\varepsilon)}W(\xi,x_2),$$

where $W\in L_\infty(\square_1)$ is a bounded measurable function, $a(\varepsilon)\leqslant q\leqslant 2$. Then $\omega_V=\varepsilon^{-a}\omega_W$ and condition (2.2) becomes

$$5\varepsilon_0+\frac{\pi\varepsilon_0^{2-q}\omega_W}{4}\leqslant 2A_0.$$

If $q<2$, the latter inequality is satisfied for each $\omega_W$ for sufficiently small $\varepsilon_0$. If $q=2$, the needed inequality is true for not very large $\omega_W$.

**3.2. Integral operator.** Our second example is an integral operator of form

$$\big(\mathcal{L}_\varepsilon u\big)(x)=\int\limits_{\square_\varepsilon\times\square_\varepsilon}P\left(\frac{x_1}{\varepsilon},x_2,\frac{y_1}{\varepsilon},y_2,\varepsilon\right)u(y)\,dy,$$

where the kernel $P(\xi,x_2,\zeta,y_2,\varepsilon)$ is a function in $L_2(\square_1^2)$ for each $\varepsilon$ obeying the condition

$$P(\zeta,y_2,\xi,x_2,\varepsilon)=\overline{P(\xi,x_2,\zeta,y_2,\varepsilon)}.$$

Such operator satisfies all needed conditions, too. We do not suceed to calculate explicitly $\omega_{\mathcal{L}_\varepsilon}$, but we can estimate it from above:

$$\omega_{\mathcal{L}_\varepsilon}\leqslant 2\left(\int\limits_{\square_\varepsilon^2}\left|K\left(\frac{x_1}{\varepsilon},x_2,\frac{y_1}{\varepsilon},y_2,\varepsilon\right)\right|^2dx\,dy\right)^{\frac{1}{2}}=2\varepsilon\|P(\cdot,\varepsilon)\|_{L_2(\square_1^2)}. \tag{3.1}$$

The kernel $K$ can also depend singularly on $\varepsilon$:

$$P(x,y,\varepsilon)=\varepsilon^{-a(\varepsilon)}Q(x,y),$$

where $a(\varepsilon)\leqslant q\leqslant 3$. In this case it follows from estimate (3.1) that

$$\omega_{\mathcal{L}_\varepsilon}\leqslant 2\varepsilon^{3-q}\|Q\|_{L_2(\square_1^2)}$$

and condition (2.2) casts into the form

$$5\varepsilon_0+\frac{\pi\varepsilon_0^{3-q}}{2}\|Q\|_{L_2(\square_1^2)}\leqslant 2A_0.$$

We also observe that as an operator $\mathcal{L}_\varepsilon$, we can take the sum of the potential and the integral operator from the above examples.

## 4. Counting functions

In the present section we introduce a series of auxiliary notions and we discuss preliminary statements, which will be used later in the proof of Theorem 2.1.

For an arbitrary $L>0$, by $N_{\mathcal{L}_\varepsilon}(L,\tau)$ we denote the counting function of the operator $\mathcal{H}_\varepsilon(\tau)$, which is the number of the eigenvalues of this operator counting the multiplicities not exceeding $\frac{L^2}{\varepsilon^2}$:

$$N_{\mathcal{L}_\varepsilon}(L,\tau) := \max\left\{k:\ E^k_{\mathcal{L}_\varepsilon}(\tau)\leqslant \frac{L^2}{\varepsilon^2}\right\}. \tag{4.1}$$

Since the band functions $E^k_{\mathcal{L}_\varepsilon}(\tau)$ are taken in the ascending order

$$E^1_{\mathcal{L}_\varepsilon}(\tau)\leqslant E^2_{\mathcal{L}_\varepsilon}(\tau)\leqslant\dots\leqslant E^k_{\mathcal{L}_\varepsilon}(\tau)\leqslant\dots,$$

for a fixed $L$ the quantity $\sup\limits_{\tau\in[-\frac12,\frac12)} N_{\mathcal{L}_\varepsilon}(L,\tau)$ is the number of band functions whose minima do not exceed $\frac{L^2}{\varepsilon^2}$, and the quantity $\inf\limits_{\tau\in[-\frac12,\frac12)} N_{\mathcal{L}_\varepsilon}(L,\tau)$ is the number of band functions, whose maxima do not exceed $\frac{L^2}{\varepsilon^2}$. Hereafter, to simplify the notations, instead of $\sup\limits_{\tau\in[-\frac12,\frac12)}$ and $\inf\limits_{\tau\in[-\frac12,\frac12)}$ we shall write shortly $\sup\limits_\tau$ and $\inf\limits_\tau$.

Assume that two neighbouring bands of the spectrum overlap, that is,

$$\big[\min_\tau E^k_{\mathcal{L}_\varepsilon}(\tau),\max_\tau E^k_{\mathcal{L}_\varepsilon}(\tau)\big]\cap\big[\min_\tau E^{k+1}_{\mathcal{L}_\varepsilon}(\tau),\max_\tau E^{k+1}_{\mathcal{L}_\varepsilon}(\tau)\big]\neq\emptyset$$

or

$$\max_\tau E^k_{\mathcal{L}_\varepsilon}(\tau)\geqslant\min_\tau E^{k+1}_{\mathcal{L}_\varepsilon}(\tau).$$

It is equivalent to the fact that as

$$\min_\tau E^k_{\mathcal{L}_\varepsilon}(\tau)\leqslant\frac{L^2}{\varepsilon^2}\leqslant\max_\tau E^k_{\mathcal{L}_\varepsilon}(\tau)$$

the inequality

$$\sup_\tau N_{\mathcal{L}_\varepsilon}(L,\tau)-\inf_\tau N_{\mathcal{L}_\varepsilon}(L,\tau)\geqslant 1 \tag{4.2}$$

holds. Thus, a part of the band spectrum between the points $\lambda_-$ and $\lambda_+$ does not possess spectral gaps if for all $\frac{L^2}{\varepsilon^2}\in[\lambda_-,\lambda_+]$, inequality (4.2) holds.

It is a very complicated problem to check directly inequality (4.2). This is why our next step is to estimate the left hand side of inequality (4.2) by a similar difference but for counting functions of simpler operators.

In view of the possibility to shift the spectral parameter, without loss of generality we can assume that

$$\inf_{\substack{u\in L_2(\square_\varepsilon)\\ \|u\|_{L_2(\square_\varepsilon)}=1}}(\mathcal{L}_\varepsilon u,u)_{L_2(\square_\varepsilon)}=0.$$

Then

$$\omega_{\mathcal{L}_\varepsilon}=\sup_{\substack{u\in L_2(\square_\varepsilon)\\ \|u\|_{L_2(\square_\varepsilon)}=1}}(\mathcal{L}_\varepsilon u,u)_{L_2(\square_\varepsilon)} \tag{4.3}$$

The quadratic form associated with the operator $\mathcal{H}_\varepsilon(\tau)$ is

$$\mathfrak{h}^\tau_{\mathcal{L}_\varepsilon}[u]:=\left\|\left(\mathrm{i}\frac{\partial}{\partial x_1}+\tau\right)u\right\|^2_{L_2(\square_\varepsilon)}+\left\|\frac{\partial u}{\partial x_2}\right\|^2_{L_2(\square_\varepsilon)}+\big(\mathcal{L}_\varepsilon e^{-\frac{\mathrm{i}\tau}{\varepsilon}x_1}u,e^{-\frac{\mathrm{i}\tau}{\varepsilon}x_1}u\big)_{L_2(\square_\varepsilon)}$$

on $\mathring{W}^1_{2,per}(\square_\varepsilon,\partial\square_\varepsilon\cap\partial\Pi)$. Taking into consideration the estimate

$$0\leqslant(\mathcal{L}_\varepsilon u,u)_{L_2(\square_\varepsilon)}\leqslant\omega_{\mathcal{L}_\varepsilon}\|u\|^2_{L_2(\square_\varepsilon)},$$

by the minimax principle we immediately see that the band functions $E^K_{\mathcal{L}_\varepsilon}$ satisfy the estimates:

$$E^k_0(\tau)\leqslant E^k_{\mathcal{L}_\varepsilon}(\tau)\leqslant E^k_{\omega_{\mathcal{L}_\varepsilon}}(\tau). \tag{4.4}$$

Therefore,

$$N_0(\sqrt{L^2-\varepsilon^2\omega_{\mathcal{L}_\varepsilon}},\tau)=N_{\omega_{\mathcal{L}_\varepsilon}}(L,\tau)\leqslant N_{\mathcal{L}_\varepsilon}(L,\tau)\leqslant N_0(L,\tau).$$

Then

$$\sup_\tau N_{\mathcal{L}_\varepsilon}(L,\tau)-\inf_\tau N_{\mathcal{L}_\varepsilon}(L,\tau)\geqslant \sup_\tau N_0(\sqrt{L^2-\varepsilon^2\omega_{\mathcal{L}_\varepsilon}},\tau)-\inf_\tau N_0(L,\tau),$$

and in order to check inequality (4.2), it is sufficient to prove that

$$\sup_\tau N_0(\sqrt{L^2-\varepsilon^2\omega_{\mathcal{L}_\varepsilon}},\tau)-\inf_\tau N_0(L,\tau)\geqslant 1.$$

And since the function $N_0$ is integer-valued, to check the latter inequality, it is sufficient to confirm that

$$\sup_\tau N_0(\sqrt{L^2-\varepsilon^2\omega_{\mathcal{L}_\varepsilon}},\tau)-\inf_\tau N_0(L,\tau)> 0. \tag{4.5}$$

Exactly the latter inequality will be checked in the proof of Theorem 2.1.

Let us write out the explicit formula for the function $N_0$. The eigenvalues of the operator $\mathcal{H}_0(\tau)$ and the associated eigenfunctions can be easily found by the separation of the variables:

$$\Lambda^0_{n,m}(\tau)=\frac{(n+\tau)^2}{\varepsilon^2}+m^2,\quad \Psi^0_{n,m}(x)=e^{\frac{inx_1}{\varepsilon}}\sin mx_2,\quad n\in\mathbb{Z},\quad m\in\mathbb{N}. \tag{4.6}$$

This is why the band functions $E^k_0(\tau)$ are numbers $\Lambda^0_{n,m}(\tau)$ taken in the ascending order counting the multiplicities. Returning back to definition (4.1) of counting functions, we see that $N_0(L,\tau)$ is the number of integer points $(n,m)$ in the plane satisfying the inequality $(n+\tau)^2+\varepsilon^2m^2\leqslant L^2$, that is,

$$\begin{aligned}N_0(L,\tau)&=\#\big\{(n,m):\,(n+\tau)^2+\varepsilon^2m^2\leqslant L^2,\ n\in\mathbb{Z},\ m\in\mathbb{N}\big\}\\ &=\sum_{n:|n+\tau|\leqslant L}\left[\frac{\sqrt{L^2-(n+\tau)^2}}{\varepsilon}\right]=\sum_{n=-[L+\tau]}^{[L-\tau]}\left[\frac{\sqrt{L^2-(n+\tau)^2}}{\varepsilon}\right].\end{aligned} \tag{4.7}$$

## 5. Proof of main result

In the present section we prove Theorem 2.1. It is convenient to split the proof into several stages.

**5.1. Intersection of first two bands.** We first observe that condition (2.2) implies immediately the apriori estimates for $\varepsilon_0$ and $b_0$:

$$\varepsilon_0\leqslant\frac{2A_0}{5},\quad b_0\leqslant\frac{8A_0}{\pi},\quad \frac{\pi\varepsilon\omega_{\mathcal{L}_\varepsilon}}{4}\leqslant\frac{2A_0}{\varepsilon}. \tag{5.1}$$

The main idea is to find the interval of values $L$ obeying inequality (4.5). The absence of the gaps should be checked for positive $L$ obeying the estimate

$$L^2\geqslant\varepsilon^2\inf\sigma(\mathcal{H}_\varepsilon).$$

At the same time, by the non-negativeness of the operator $\mathcal{L}$ and identity (4.3) for $\omega_{\mathcal{L}_\varepsilon}$, the bottom of the essential spectrum of the operator $\mathcal{H}_\varepsilon$ satisfies the estimate:

$$1\leqslant\inf\sigma(\mathcal{H}_\varepsilon)\leqslant 1+\omega_{\mathcal{L}_\varepsilon}$$

and as

$$\varepsilon^2\inf\sigma(\mathcal{H}_\varepsilon)\leqslant L^2<\varepsilon^2\omega_{\mathcal{L}_\varepsilon},$$

inequality (4.5) loses the sense since in the first term the square root becomes pure imaginary. This is why the absence of the gaps in the beginning of the spectrum will be proved on the base of analysing the location of first bands in the spectrum.

It follows from formula (4.6) and the estimate for $\varepsilon_0$ in (5.1) that the first two band functions $E_0^1(\tau)$ and $E_0^2(\tau)$ are even in $\tau$ functions of the form:

$$E_0^1(\tau) = \frac{\tau^2}{\varepsilon^2} + 1, \quad E_0^2(\tau) = \min\left\{\frac{(1-\tau)^2}{\varepsilon^2} + 1, \frac{\tau^2}{\varepsilon^2} + 4\right\}, \quad \tau \in \big[0, \tfrac{1}{2}\big].$$

This implies that

$$\max_\tau E_0^1(\tau) = \frac{1}{4\varepsilon^2} + 1, \quad \min_\tau E_0^2(\tau) = 4, \quad \max_\tau E_0^2(\tau) = \frac{(1-3\varepsilon^2)^2}{4\varepsilon^2} + 4,$$

where the right hand side in the latter identity arises as the value of the functions $\tau \mapsto \frac{(1-\tau)^2}{\varepsilon^2} + 1$, $\tau \mapsto \frac{\tau^2}{\varepsilon^2} + 4$ at the intersection point of the graphs. By estimate (4.4) and (5.1) this implies that

$$\max_\tau E_{\mathcal{L}_\varepsilon}^1 \geqslant \frac{1}{4\varepsilon^2} + 1, \quad \min_\tau E_{\mathcal{L}_\varepsilon}^2 \leqslant 4 + \omega_{\mathcal{L}_\varepsilon}, \quad \max_\tau E_{\mathcal{L}_\varepsilon}^2 \geqslant \frac{(1-3\varepsilon^2)^2}{4\varepsilon^2} + 4.$$

Since by (5.1) and the positiveness $\omega_{\mathcal{L}_\varepsilon}$

$$\frac{1}{4\varepsilon^2} - \omega_{\mathcal{L}_\varepsilon} \geqslant \frac{1}{\varepsilon^2}\left(\frac{1}{4} - b_0\right) > 3 \quad \Rightarrow \quad 4 + \omega_{\mathcal{L}_\varepsilon} \leqslant \frac{1}{4\varepsilon^2} + 1,$$

the first two bands of the spectrum intersect and the interval

$$\left[\inf \sigma(\mathcal{H}_\varepsilon), \frac{(1-3\varepsilon^2)^2}{4\varepsilon^2} + 4\right]$$

contains no spectral gaps. Hence, it is sufficient to check inequality (4.5) for

$$L^2 \geqslant \frac{(1-3\varepsilon^2)^2}{4} + 4\varepsilon^2 = \frac{1 + 10\varepsilon^2 + 9\varepsilon^4}{4} > \frac{1}{4}.$$

In what follows the latter inequality for $L$ is assumed to hold.

**5.2. Case $\frac{1}{2} \leqslant L < 1$.** Here we assume that

$$\frac{1}{2} < \frac{\sqrt{1 + 10\varepsilon^2 + 9\varepsilon^4}}{2} \leqslant L < 1.$$

By (5.1), for such values of $L$ we have

$$\frac{\sqrt{3}}{4} < \sqrt{L^2 - \varepsilon^2\omega_{\mathcal{L}_\varepsilon}} < 1.$$

Therefore,

$$N_0(\sqrt{L^2 - \varepsilon^2\omega_{\mathcal{L}_\varepsilon}}, 0) = \left[\frac{\sqrt{L^2 - \varepsilon^2\omega_{\mathcal{L}_\varepsilon}}}{\varepsilon}\right],$$

$$N_0(L, 1-L) = \left[\frac{\sqrt{2L-1}}{\varepsilon}\right], \tag{5.2}$$

and under the additional condition

$$L^2 \geqslant \frac{1}{4} + \varepsilon^2\omega_{\mathcal{L}_\varepsilon} \tag{5.3}$$

we have

$$N_0\left(\sqrt{L^2 - \varepsilon^2\omega_{\mathcal{L}_\varepsilon}}, \frac{1}{2}\right) = 2\left[\frac{\sqrt{L^2 - \varepsilon^2\omega_{\mathcal{L}_\varepsilon} - \frac{1}{4}}}{\varepsilon}\right]. \tag{5.4}$$

Let number $L_* \in \left(\frac{1}{2}, 1\right)$ be such that

$$2\frac{\sqrt{L_*^2 - \varepsilon^2\omega_{\mathcal{L}_\varepsilon} - \frac{1}{4}}}{\varepsilon} - 1 = \frac{\sqrt{L_*^2 - \varepsilon^2\omega_{\mathcal{L}_\varepsilon}}}{\varepsilon},$$
$$L_* = \sqrt{\frac{(\sqrt{3+4\varepsilon^2}+\varepsilon)^2}{9} + \varepsilon^2\omega_{\mathcal{L}_\varepsilon}}.$$

Then as

$$\frac{1}{2} \leqslant L \leqslant L_*$$

by (5.1) we get

$$\begin{aligned}\sup_\tau N_0(\sqrt{L^2-\varepsilon^2\omega_{\mathcal{L}_\varepsilon}},\tau) - \inf_\tau N_0(L,\tau) &\geqslant N_0(\sqrt{L^2-\varepsilon^2\omega_{\mathcal{L}_\varepsilon}},0) - N_0(L,1-L)\\ &\geqslant \frac{\sqrt{L^2-\varepsilon^2\omega_{\mathcal{L}_\varepsilon}} - \sqrt{2L-1}}{\varepsilon} - 1\\ &= \frac{(L-1)^2 - \varepsilon^2\omega_{\mathcal{L}_\varepsilon}}{\varepsilon(\sqrt{L^2-\varepsilon^2\omega_{\mathcal{L}_\varepsilon}} + \sqrt{2L-1})} - 1\\ &\geqslant \left.\frac{(L_*-1)^2 - \varepsilon^2\omega_{\mathcal{L}_\varepsilon}}{\varepsilon_0(\sqrt{L_*^2-\varepsilon^2\omega_{\mathcal{L}_\varepsilon}} + \sqrt{2L_*-1})}\right|_{\substack{\varepsilon=\frac{2A_0}{5}\\ \varepsilon^2\omega_{\mathcal{L}_\varepsilon}=\frac{8A_0}{\pi}}} - 1 > 0.\end{aligned}$$

As

$$L_* \leqslant L < 1,$$

condition (5.3) is satisfied and in view of (5.2), (5.4) we obtain

$$\begin{aligned}&\sup_\tau N_0(\sqrt{L^2-\varepsilon^2\omega_{\mathcal{L}_\varepsilon}},\tau) - \inf_\tau N_0(L,\tau)\\ &\qquad\geqslant N_0\left(\sqrt{L^2-\varepsilon^2\omega_{\mathcal{L}_\varepsilon}}, \frac{1}{2}\right) - N_0(L,1-L)\\ &\qquad\geqslant \left.\frac{\sqrt{L^2-\varepsilon^2\omega_{\mathcal{L}_\varepsilon}-\frac{1}{4}} - \sqrt{2L-1}}{\varepsilon}\right|_{\substack{\varepsilon=\frac{2A_0}{5}\\ \varepsilon^2\omega_{\mathcal{L}_\varepsilon}=\frac{8A_0}{\pi}}} - 2 > 0.\end{aligned} \tag{5.5}$$

**5.3. General case: auxiliary estimates.** Hereafter we assume that $L \geqslant 1$. We denote $K := [L]$, $\alpha := \{L\}$, where $\{\cdot\}$ is the fractional part of a number. We begin with the obvious relations

$$0 \leqslant L - \sqrt{L^2-\varepsilon^2\omega_{\mathcal{L}_\varepsilon}} = \frac{\varepsilon^2\omega_{\mathcal{L}_\varepsilon}}{L+\sqrt{L^2-\varepsilon^2\omega_{\mathcal{L}_\varepsilon}}} \leqslant \frac{\varepsilon^2\omega_{\mathcal{L}_\varepsilon}}{L} \leqslant \frac{8A_0}{\pi} \tag{5.6}$$

valid thanks to estimates (5.1) and $L \geqslant 1$.

We denote

$$F_1(L,X,B) := \sqrt{L^2-B-X^2}, \quad F_2(L,X,A) := \sqrt{L^2-(X+A)^2},$$
$$F_0(L,X,A,B) := 2F_1(L,X,B) - F_2(L,X,A) - F_2(L,X,-A).$$

By straightforward calculations we check that

$$\begin{aligned}F_0(L,X,A,B) &= (A^2-B)\left(\frac{1}{F_1(L,X,B)+F_2(L,X,A)} + \frac{1}{F_1(L,X,B)+F_2(L,X,-A)}\right)\\ &+ \frac{8A^2X^2}{F_1(L,X,B)+F_2(L,X,A)}\frac{1}{F_1(L,X,B)+F_2(L,X,-A)}\frac{1}{F_2(L,X,A)+F_2(L,X,-A)}.\end{aligned}$$

This representation yields that for positive radicands, the function $F_0(L,X,A,B)$ is positive for $B \leqslant A^2$, $A>0$ and increases monotonically as $X \geqslant A \geqslant 0$. Moreover, for such $X$ the estimate holds:

$$F_0(L,X,A,0) \geqslant A^2\left(\frac{1}{F_2(L,X,-A)}+\frac{X^2}{F_2^3(L,X,-A)}\right) \geqslant A^2\frac{L^2+2A(X-A)}{\big(L^2-(X-A)^2\big)^{\frac{3}{2}}}. \tag{5.7}$$

We observe one more obvious inequality for $F_0(L,X,A,B)$:

$$\begin{aligned}
0 \geqslant & \int\limits_0^{\sqrt{L^2-B}} \big(F_0(L,X,A,B)-F_0(L,X,A,0)\big)\,dx \\
= & -\int\limits_0^{\sqrt{L^2-B}} \frac{B\,dx}{F_1(L,X,B)+F_1(L,X,0)} \geqslant -\int\limits_0^{\sqrt{L^2-B}} \frac{B\,dx}{2F_1(L,X,B)} \\
= & -\frac{B}{2}\arcsin\frac{X}{\sqrt{L^2-B}}\bigg|_0^{\sqrt{L^2-B}} = -\frac{\pi B}{4}.
\end{aligned} \tag{5.8}$$

**5.4. General case: $\alpha \leqslant \frac{1}{4}$.** In view of (4.7) we have

$$\begin{aligned}
&\sup_\tau N_0(\sqrt{L^2-\varepsilon^2\omega_{\mathcal{L}_\varepsilon}},\tau)-\inf_\tau N_0(L,\tau) \geqslant N_0\left(\sqrt{L^2-\varepsilon^2\omega_{\mathcal{L}_\varepsilon}},\frac{1}{2}\right)-N_0(L,\alpha) \\
&= \sum_{n=-\left[\sqrt{L^2-\varepsilon^2\omega_{\mathcal{L}_\varepsilon}}+\frac{1}{2}\right]}^{\left[\sqrt{L^2-\varepsilon^2\omega_{\mathcal{L}_\varepsilon}}-\frac{1}{2}\right]} \left[\frac{\sqrt{L^2-\varepsilon^2\omega_{\mathcal{L}_\varepsilon}-\big(n+\frac{1}{2}\big)^2}}{\varepsilon}\right] - \sum_{n=-[L+\alpha]}^{[L-\alpha]}\left[\frac{\sqrt{L^2-(n+\alpha)^2}}{\varepsilon}\right] \\
&= \sum_{n=0}^{\left[\sqrt{L^2-\varepsilon^2\omega_{\mathcal{L}_\varepsilon}}-\frac{1}{2}\right]} 2\left[\frac{\sqrt{L^2-\varepsilon^2\omega_{\mathcal{L}_\varepsilon}-\big(n+\frac{1}{2}\big)^2}}{\varepsilon}\right] - \sum_{n=-K}^{K-1}\left[\frac{\sqrt{L^2-(n+\alpha)^2}}{\varepsilon}\right].
\end{aligned} \tag{5.9}$$

By (5.6) this implies:

$$\begin{aligned}
\sup_\tau\, & N_0(\sqrt{L^2-\varepsilon^2\omega_{\mathcal{L}_\varepsilon}},\tau)-\inf_\tau N_0(L,\tau) \geqslant \sum_{n=0}^{K-1} 2\left[\frac{\sqrt{L^2-\varepsilon^2\omega_{\mathcal{L}_\varepsilon}-\big(n+\frac{1}{2}\big)^2}}{\varepsilon}\right] \\
& -\sum_{n=0}^{K-1}\left(\left[\frac{\sqrt{L^2-(n+\alpha)^2}}{\varepsilon}\right]+\left[\frac{\sqrt{L^2-(n+1-\alpha)^2}}{\varepsilon}\right]\right) \geqslant \frac{S_1(L)}{\varepsilon}-2K,
\end{aligned} \tag{5.10}$$

where for brevity we have denoted:

$$S_1(L) := \sum_{n=0}^{K-1} f_1(n,L), \qquad f_1(x,L) := F_0\left(L, x+\frac{1}{2},\frac{1}{2}-\alpha,\varepsilon^2\omega_{\mathcal{L}_\varepsilon}\right).$$

The above described properties of the function $F_0$ implies the monotonous increasing in $x \geqslant 0$ of the function $f_1(x,L)$ and the positivity of $f_1(0,L)$:

$$S_1(L) \geqslant f_1(0,L)+\int\limits_0^{K-1} f_1(x,L)\,dx, \quad f_1(0,L)>0. \tag{5.11}$$

Although the integral in the left hand side of this inequality can be explicitly calculated, it is more convenient to estimate it before integration.

As $K=1$, the integral in (5.11) vanishes and by (5.1),

$$f_1(0,1+\alpha)\geqslant\left(2\sqrt{(1+\alpha)^2-\frac{1}{4}-b_0}-\sqrt{4\alpha}-\sqrt{1+2\alpha}\right)\bigg|_{\alpha=\frac{1}{4}}>2\varepsilon_0. \tag{5.12}$$

We proceed to the case $K\geqslant 2$. By (5.8) we immediately obtain

$$\begin{aligned}&\int\limits_0^{K-1}\left(f_1(x,L)-F_0\left(L,x+\frac{1}{2},\frac{1}{2}-\alpha,0\right)\right)dx\\ &\geqslant\int\limits_0^{\sqrt{L^2-\varepsilon^2\omega_{\mathcal{L}_\varepsilon}}}\left(F_0\left(L,x,\frac{1}{2}-\alpha,\varepsilon^2\omega_{\mathcal{L}_\varepsilon}\right)-F_0\left(L,x,\frac{1}{2}-\alpha,0\right)\right)dx\geqslant-\frac{\pi\varepsilon^2\omega_{\mathcal{L}_\varepsilon}}{4}.\end{aligned} \tag{5.13}$$

By (5.7) we have

$$\begin{aligned}\int\limits_0^{K-1}F_0\left(L,x,\frac{1}{2}-\alpha,0\right)dx&\geqslant\int\limits_0^{K-1}\frac{L^2+(1-2\alpha)(x+\alpha)}{16\big((K+\alpha)^2-(x+\alpha)^2\big)^{\frac{3}{2}}}\,dx\\ &=\int\limits_\alpha^{K-1+\alpha}\frac{(K+\alpha)^2+(1-2\alpha)x}{16\big((K+\alpha)^2-x^2\big)^{\frac{3}{2}}}\,dx\\ &=\frac{x+(1-2\alpha)}{16\sqrt{(K+a)^2-x^2}}\bigg|_\alpha^{K-1+\alpha}=\frac{F_3(\alpha)}{16},\\ F_3(\alpha):=\frac{K-\alpha}{\sqrt{2(K+\alpha)-1}}-\frac{1-\alpha}{\sqrt{K^2+2\alpha K}}.&\end{aligned} \tag{5.14}$$

The function $f_*(\alpha)$ decreases monotonically in $\alpha\in\left[0,\frac{1}{4}\right]$ since

$$\begin{aligned}F_3'(\alpha)=&-\frac{1}{\big(2(K+a)-1\big)^{\frac{1}{2}}}-\frac{K-\alpha}{\big(2(K+\alpha)-1\big)^{\frac{3}{2}}}+\frac{1}{(K^2+2\alpha K)^{1/2}}+\frac{(1-\alpha)K}{(K^2+2\alpha K)^{\frac{3}{2}}}\\ \leqslant&-\frac{1}{\big(2K-\frac{1}{2}\big)^{\frac{1}{2}}}-\frac{K-\frac{1}{4}}{\big(2K-\frac{1}{2}\big)^{\frac{3}{2}}}+\frac{1}{K}+\frac{1}{K^2}\\ =&-\frac{3}{2^{\frac{3}{2}}\left(K-\frac{1}{4}\right)^{\frac{1}{2}}}+\frac{1}{K}+\frac{1}{K^2}\leqslant-\frac{3}{\sqrt{14}}+\frac{3}{4}<0.\end{aligned}$$

This is why by (5.14)

$$\begin{aligned}\int\limits_0^{K-1}F_0\left(L,x,\frac{1}{2}-\alpha,0\right)dx&\geqslant\frac{1}{16}F_3\left(\frac{1}{4}\right)=\frac{\sqrt{K-\frac{1}{4}}}{16\sqrt{2}}-\frac{3}{64\sqrt{K^2+\frac{1}{2}K}}\\ &\geqslant\left(\frac{\sqrt{K-\frac{1}{4}}}{16\sqrt{2}\sqrt{K}}-\frac{3}{64\sqrt{K^2+\frac{1}{2}K}\sqrt{K}}\right)\bigg|_{K=2}\sqrt{K}\geqslant\frac{13}{500}\sqrt{K}.\end{aligned} \tag{5.15}$$

This and (5.10), (5.12), (5.13), (5.14) finally imply

$$\sup_\tau N_0(\sqrt{L^2-\varepsilon^2\omega_{\mathcal{L}_\varepsilon}},\tau)-\inf_\tau N_0(L,\tau)\geqslant\frac{2\varepsilon_0}{\varepsilon}-2>0$$

as $L=1+\alpha$ and

$$\sup_{\tau} N_0(\sqrt{L^2-\varepsilon^2\omega_{\mathcal{L}_\varepsilon}},\tau)-\inf_{\tau} N_0(L,\tau)\geqslant \frac{13}{500}\frac{\sqrt{K}}{\varepsilon}-\frac{\pi\varepsilon\omega_{\mathcal{L}_\varepsilon}}{4}-2K \tag{5.16}$$

as $L=K+\alpha$, $K\geqslant 2$.

Other cases $\frac{1}{4}<\alpha<\frac{1}{2}$, $\frac{1}{2}\leqslant\alpha<\frac{3}{4}$, $\frac{3}{4}\leqslant\alpha<1$ are considered in a similar way. This is why we describe these cases rather briefly dwelling only on main formulae.

**5.5. General case: $\frac{1}{4}<\alpha<\frac{1}{2}$.** Similar to (5.9), (5.10) we have

$$\begin{aligned}
&\sup_{\tau} N_0(\sqrt{L^2-\varepsilon^2\omega_{\mathcal{L}_\varepsilon}},\tau)-\inf_{\tau} N_0(L,\tau)\geqslant N_0\left(\sqrt{L^2-\varepsilon^2\omega_{\mathcal{L}_\varepsilon}},0\right)-N_0(L,\alpha)\\
&\quad=\sum_{n=1}^{K}2\left[\frac{\sqrt{L^2-\varepsilon^2\omega_{\mathcal{L}_\varepsilon}-n^2}}{\varepsilon}\right]\\
&\quad-\sum_{n=1}^{K}\left(\left[\frac{\sqrt{L^2-(n+\alpha)^2}}{\varepsilon}\right]+\left[\frac{\sqrt{L^2-(n-\alpha)^2}}{\varepsilon}\right]\right)\\
&\quad+\left[\frac{\sqrt{L^2-\varepsilon^2\omega_{\mathcal{L}_\varepsilon}}}{\varepsilon}\right]-\left[\frac{\sqrt{L^2-\alpha^2}}{\varepsilon}\right]\geqslant\frac{S_2(L)}{\varepsilon}-2K-1,
\end{aligned} \tag{5.17}$$

where

$$\begin{aligned}
&S_2(L):=\sum_{n=0}^{K-1}f_2(n,L)+F_1(L,0,\varepsilon^2\omega_{\mathcal{L}_\varepsilon})-F_2(L,0,\alpha),\\
&f_2(x,L):=F_0(L,x+1,\alpha,\varepsilon^2\omega_{\mathcal{L}_\varepsilon}).
\end{aligned}$$

Similarly to (5.11) we obtain

$$\begin{aligned}
&S_2(L)\geqslant f_2(0,L)+F_1(L,0,\varepsilon^2\omega_{\mathcal{L}_\varepsilon})-F_2(L,0,\alpha)+\int_0^{K-1}f_2(x,L)\,dx,\\
&f_2(0,L)+F_1(L,0,\varepsilon^2\omega_{\mathcal{L}_\varepsilon})-F_2(L,0,\alpha)>0.
\end{aligned} \tag{5.18}$$

As $K=1$, the integral in the right hand side of the latter inequality vanishes and

$$S_2(1+\alpha)\geqslant\left(2\sqrt{(1+\alpha)^2-\frac{1}{4}-b_0}+\sqrt{(1+\alpha)^2-b_0}-\sqrt{1+2\alpha}-\sqrt{4\alpha}\right)\Bigg|_{\alpha=\frac{1}{4}}>3\varepsilon_0. \tag{5.19}$$

As $K\geqslant 2$, by (5.8) and completely similarly to (5.13),

$$\int_0^{K-1}\big(f_2(x,L)-F_0(L,x+1,\alpha,0)\big)\,dx\geqslant-\frac{\pi\varepsilon^2\omega_{\mathcal{L}_\varepsilon}}{4}. \tag{5.20}$$

Similarly to (5.14), (5.15) we obtain

$$\begin{aligned}\int\limits_0^{K-1} F_0(L,x+1,\alpha,0)\,dx \geqslant \int\limits_{1-\alpha}^{K-\alpha} \frac{(K+\alpha)^2+2\alpha x}{16\big((K+\alpha)^2-x^2\big)^{\frac{3}{2}}}\,dx = \frac{x+2\alpha}{16\sqrt{(K+a)^2-x^2}}\bigg|_{1-\alpha}^{K-\alpha}\\ \geqslant \frac{1}{16}\left(\frac{K+\alpha}{2\sqrt{\alpha K}} - \frac{1+\alpha}{\sqrt{K^2+2\alpha K+2\alpha-1}}\right)\bigg|_{\alpha=\frac{1}{2}}\\ = \frac{1}{16}\left(\frac{K+\frac{1}{2}}{\sqrt{2K}} - \frac{3}{2\sqrt{K^2+K}}\right) \geqslant \frac{9\sqrt{K}}{320}.\end{aligned}$$

This and (5.17), (5.18), (5.19), (5.20) yield

$$\sup_\tau N_0(\sqrt{L^2-\varepsilon^2\omega_{\mathcal{L}_\varepsilon}},\tau) - \inf_\tau N_0(L,\tau) \geqslant \frac{3\varepsilon_0}{\varepsilon} - 3 > 0$$

as $K=1$ and

$$\sup_\tau N_0(\sqrt{L^2-\varepsilon^2\omega_{\mathcal{L}_\varepsilon}},\tau) - \inf_\tau N_0(L,\tau) \geqslant \frac{9\sqrt{K}}{320\varepsilon} - 2K - \frac{\pi\varepsilon\omega_{\mathcal{L}_\varepsilon}}{4} - 1 \tag{5.21}$$

as $K \geqslant 2$.

### 5.6. General case: $\frac{1}{2} \leqslant \alpha < \frac{3}{4}$.

Here

$$\begin{aligned}&\sup_\tau N_0(\sqrt{L^2-\varepsilon^2\omega_{\mathcal{L}_\varepsilon}},\tau) - \inf_\tau N_0(L,\tau) \geqslant N_0\left(\sqrt{L^2-\varepsilon^2\omega_{\mathcal{L}_\varepsilon}},0\right) - N_0(L,1-\alpha)\\ &= \sum_{n=1}^{K} 2\left[\frac{\sqrt{L^2-\varepsilon^2\omega_{\mathcal{L}_\varepsilon}-n^2}}{\varepsilon}\right]\\ &\quad - \sum_{n=1}^{K}\left(\left[\frac{\sqrt{L^2-(n+1-\alpha)^2}}{\varepsilon}\right] + \left[\frac{\sqrt{L^2-(n-1+\alpha)^2}}{\varepsilon}\right]\right)\\ &\quad + \left[\frac{\sqrt{L^2-\varepsilon^2\omega_{\mathcal{L}_\varepsilon}}}{\varepsilon}\right] - \left[\frac{\sqrt{L^2-(1-\alpha)^2}}{\varepsilon}\right] \geqslant \frac{S_3(L)}{\varepsilon} - 2K - 1,\end{aligned} \tag{5.22}$$

where

$$\begin{aligned}&S_3(L) := \sum_{n=0}^{K-1} f_3(n,L) + F_1(L,0,\varepsilon^2\omega_{\mathcal{L}_\varepsilon}) - F_2(L,0,1-\alpha),\\ &f_3(x,L) := F_0(L,x+1,1-\alpha,\varepsilon^2\omega_{\mathcal{L}_\varepsilon})\end{aligned}$$

As $K=1$, similarly to (5.19) we have

$$\begin{aligned}S_3(1+\alpha) \geqslant &\Big(2\sqrt{(1+\alpha)^2-1-b_0} + \sqrt{(1+\alpha)^2-b_0}\\ &- \sqrt{1+2\alpha} - \sqrt{4\alpha} - \sqrt{6\alpha-3}\Big)\bigg|_{\alpha=\frac{3}{4}} > 3\varepsilon_0.\end{aligned}$$

As $K \geqslant 2$, similarly to (5.13), (5.14) we obtain

$$\int\limits_0^{K-1}\big(f_3(x,L) - F_0(L,x+1,1-\alpha,0)\big)\,dx \geqslant -\frac{\pi\varepsilon^2\omega_{\mathcal{L}_\varepsilon}}{4}$$

and

$$\int\limits_0^{K-1} F_0(L,x+1,1-\alpha,0)\,dx \geqslant \frac{1}{16}\left(\frac{K+1-\alpha}{\sqrt{2(K+\alpha)-1}} - \frac{2-\alpha}{\sqrt{K^2+2\alpha K}}\right)$$
$$\geqslant \frac{\sqrt{K+\frac{1}{4}}}{16\sqrt{2}} - \frac{5}{32\sqrt{4K^2+6K}} \geqslant \sqrt{2}A_0\sqrt{K}.$$

Therefore,

$$\sup_\tau N_0(\sqrt{L^2-\varepsilon^2\omega_{\mathcal{L}_\varepsilon}},\tau) - \inf_\tau N_0(L,\tau) \geqslant \frac{3\varepsilon_0}{\varepsilon} - 3 > 0$$

as $K=1$ and

$$\sup_\tau N_0(\sqrt{L^2-\varepsilon^2\omega_{\mathcal{L}_\varepsilon}},\tau) - \inf_\tau N_0(L,\tau) \geqslant \frac{2A_0\sqrt{K}}{\varepsilon} - 2K - \frac{\pi\varepsilon\omega_{\mathcal{L}_\varepsilon}}{4} - 1 \tag{5.23}$$

as $K \geqslant 2$.

**5.7. General case: $\frac{3}{4}<\alpha<1$.** In this case the first estimate is similar to (5.9), (5.10):

$$\begin{aligned}
&\sup_\tau N_0(\sqrt{L^2-\varepsilon^2\omega_{\mathcal{L}_\varepsilon}},\tau) - \inf_\tau N_0(L,\tau) \geqslant N_0\left(\sqrt{L^2-\varepsilon^2\omega_{\mathcal{L}_\varepsilon}},\frac{1}{2}\right) - N_0(L,1-\alpha)\\
&\quad=\sum_{n=0}^{K} 2\left[\frac{\sqrt{L^2-\varepsilon^2\omega_{\mathcal{L}_\varepsilon}-\left(n+\frac{1}{2}\right)^2}}{\varepsilon}\right]\\
&\quad-\sum_{n=0}^{K}\left(\left[\frac{\sqrt{L^2-(n+\alpha)^2}}{\varepsilon}\right]+\left[\frac{\sqrt{L^2-(n+1-\alpha)^2}}{\varepsilon}\right]\right)\\
&\quad\geqslant \frac{S_4(L)}{\varepsilon} - 2K - 2,
\end{aligned} \tag{5.24}$$

where

$$S_4(L) := \sum_{n=0}^{K} f_4(n,L), \quad f_4(x,L) := F_0\left(L, x+\frac{1}{2}, \alpha-\frac{1}{2}, \varepsilon^2\omega_{\mathcal{L}_\varepsilon}\right).$$

Here the analogue of inequality (5.11) reads as

$$S_4(L) \geqslant f_4(0,L) + \int\limits_0^K f_4(x,L)\,dx \tag{5.25}$$

and this is why there is no need to consider independently the case $K=1$. As in (5.13), we estimate

$$\int\limits_0^K \left(f_4(x,L) - F_0\left(L,x+\frac{1}{2},\alpha-\frac{1}{2},0\right)\right)dx \geqslant -\frac{\pi\varepsilon^2\omega_{\mathcal{L}_\varepsilon}}{4}$$

and similarly to (5.14) we integrate:

$$\int\limits_0^K F_0\left(L,x+\frac{1}{2},\frac{1}{2}-\alpha,0\right)dx \geqslant \int\limits_{1-\alpha}^{K+1-\alpha} \frac{L^2+(2\alpha-1)x}{16\big((K+\alpha)^2-x\big)^{\frac{3}{2}}}\,dx = \frac{x+(2\alpha-1)}{16\sqrt{(K+a)^2-x^2}}\Bigg|_{1-\alpha}^{K+1-\alpha}$$
$$\geqslant \frac{1}{16}\left(\frac{K+\alpha}{\sqrt{2\alpha-1}\sqrt{2K+1}} - \frac{\alpha}{\sqrt{K^2+2\alpha K+2\alpha-1}}\right)\Bigg|_{\alpha=1} > \frac{11\sqrt{K}}{250}.$$

Hence, by (5.24), (5.25),

$$\sup_\tau N_0(\sqrt{L^2-\varepsilon^2\omega_{\mathcal{L}_\varepsilon}},\tau)-\inf_\tau N_0(L,\tau)\geqslant\frac{\sqrt{K}}{25\varepsilon}-2K-\frac{\pi\varepsilon\omega_{\mathcal{L}_\varepsilon}}{4}-2 \tag{5.26}$$

for $K\geqslant 1$. As $K=1$, the latter inequality obviously holds thanks to condition (2.2).

**5.8. End of proof.** We compare the right hand sides of inequalities (5.16), (5.21), (5.23), (5.26) as $K\geqslant 2$. Since by (5.1)

$$\left(\frac{11}{250}-2A_0\right)\frac{\sqrt{K}}{\varepsilon}>1,$$

the minimal of the compared right hand sides is that in inequality (5.23). This is why the right hand sides of inequalities (5.16), (5.21), (5.23), (5.26) are positive as $K\geqslant 2$ provided

$$\frac{2A_0\sqrt{K}}{\varepsilon}-2K-\frac{\pi\varepsilon\omega_{\mathcal{L}_\varepsilon}}{4}-1\geqslant 0.$$

As $K=2$, this inequality holds thanks to condition (2.2). Solving then it w.r.t. $K$, we obtain that it holds as $\sqrt{K}\leqslant K_\varepsilon$. Thus, for such values $K$ inequality (4.5) holds and since the spectrum is a closed set, this completes the proof of the theorem.

## 6. Some features of proof

In the present section we discuss certain features of the proof of main result given above.

At the first stage, in estimates (5.9), (5.17), (5.22), (5.24), the difference

$$\sup_\tau N_0(\sqrt{L^2-\varepsilon^2\omega_{\mathcal{L}_\varepsilon}},\tau)-\inf_\tau N_0(L,\tau)$$

is estimated by the values of the function

$$N_0(\sqrt{L^2-\varepsilon^2\omega_{\mathcal{L}_\varepsilon}},\tau_1)-N_0(L,\tau_2)$$

at certain points $\tau_1$, $\tau_2$. It is clear that the choice of these points is arbitrary. To get the best result, the point $\tau_1$ for the function $N_0(\sqrt{L^2-\varepsilon^2\omega_{\mathcal{L}_\varepsilon}},\tau)$ should be chosen so that the value of this function at this point is as close the maximal one as possible. The choice of the point $\tau_2$ for the function $N_0(L,\tau)$ should be made in order to minimize its value. The issue on finding exactly the maximum and the minimum point for the function $N_0$ is very complicated. At the same time, many preliminary numerical calculations suggested that the function $N_0(L,\tau)$ is quite close to the maximal value as $\tau=0$ or $\tau=\frac{1}{2}$ and the particular choice is subject to the fractional part $\alpha$ of the number $L$. If the fractional part satisfies either $\alpha<\alpha_0\approx\frac{1}{10}$ or $\alpha>\alpha_0+\frac{1}{2}$, we should choose $\tau=\frac{1}{2}$, while as $\alpha_0<\alpha<\alpha_0+\frac{1}{2}$ we should choose $\tau=0$. The obtained values of the function $N_0(L,\tau)$ differ a little from the maximal ones. In the work we worsen this observation replacing $\alpha_0$ by $\frac{1}{4}$. Similar approximation for the minimum of the function $N_0(L,\tau)$ are again obtained under an appropriate choice of the number $\tau_2$ subject to the fractional part of the number $L$. Namely, here $\tau_2$ should be equal to the distance from $\alpha$ to the nearest integer: $\tau_2=\min\{\alpha,1-\alpha\}$. At such points the function $N_0(L,\tau)$ differs a little from the minimal values. In all previous works on Bethe-Sommerfeld conjecture we know, the similar difference

$$\sup_\tau N_0(\sqrt{L^2-\varepsilon^2\omega_{\mathcal{L}_\varepsilon}},\tau)-\inf_\tau N_0(L,\tau)$$

was estimated by some other methods allowing to avoid the issue on location of extrema of the counting functions. This is why the proof provided in the present work suggest the way for approximate finding the extremal points for the counting functions. At the same time, we stress that the proposed choice of the numbers $\tau_1$, $\tau_2$ does not necessary provides the exact values of extrema for the function $N_0$. In particular, as $L = 39.623$, $\varepsilon = 0.0035$, we have

$$N_0(L, 0) = 704646, \quad N_0(L, 0.499088) = 704816, \quad N_0(L, 0.5) = 704808,$$

and as $L = 39.635$, $\varepsilon = 0.0035$ we have

$$N_0(L, 0.365) = 704704, \quad N_0(L, 0.3636) = 704701.$$

We also note that in (5.9), (5.17), (5.22), (5.24), we estimate the integer parts of various quantities as follows: $[z] \geqslant z - 1$. Of course, this is too rough and the quantity $-2K$ in inequalities (5.9), (5.17), (5.22), (5.24) arises exactly due to this rough estimate. The attempts to apply appropriate methods of the number theory from [20] did not lead us to more gentle estimates, which could have changed essentially $K_\varepsilon$.

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