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REGULARIZED ASYMPTOTICS OF SOLUTIONS TO INTEGRO-DIFFERENTIAL PARTIAL DIFFERENTIAL EQUATIONS WITH RAPIDLY VARYING KERNELS

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Abstract. We generalize the Lomov's regularization method for partial differential equations with integral operators, whose kernel contains a rapidly varying exponential factor. We study the case when the upper limit of the integral operator coincides with the differentiation variable. For such problems we develop an algorithm for constructing regularized asymptotics. In contrast to the work by Imanaliev M.I., where for analogous problems with slowly varying kernel only the passage to the limit studied as the small parameter tended to zero, here we construct an asymptotic solution of any order (with respect to the parameter). We note that the Lomov's regularization method was used mainly for ordinary singularly perturbed integro-differential equations (see detailed bibliography at the end of the article). In one of the authors' papers the case of a partial differential equation with slowly varying kernel was considered. The development of this method for partial differential equations with rapidly changing kernel was not made before. The type of the upper limit of an integral operator in such equations generates two fundamentally different situations. The most difficult situation is when the upper limit of the integration operator does not coincide with the differentiation variable. As studies have shown, in this case, the integral operator can have characteristic values, and for the construction of the asymptotics, more strict conditions on the initial data of the problem are required. It is clear that these difficulties also arise in the study of an integro-differential system with a rapidly changing kernels, therefore in this paper the case of the dependence of the upper limit of an integral operator on the variable x is deliberately avoided. In addition, it is assumed that the same regularity is observed in a rapidly decreasing kernel exponent integral operator. Any deviations from these (seemingly insignificant) limitations greatly complicate the problem from the point of view of constructing its asymptotic solution. We expect that in our further works in this direction we will succeed to weak these restrictions.

Keywords: singularly perturbed, integro-differential equation, regularization of the integral.

Mathematics Subject Classification: 35R09, 45K05

Introduction

In the present work we consider the integral-differential system

$$\begin{aligned} &\varepsilon\frac{\partial y(t,x,\varepsilon)}{\partial t}=A(t)\,y(t,x,\varepsilon)+\int_0^t e^{\frac{1}{\varepsilon}\int_s^t\mu(\theta)\,d\theta}K(t,x,s)\,y(s,x,\varepsilon)\,ds+h(t,x)\,,\\ &y(0,x,\varepsilon)=y^0(x)\ \ ((t,x)\in[0,T]\times[0,X])\,, \end{aligned}\tag{1}$$

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with a fast varying kernel. The pose the problem on constructing a regularized in the sense of S.A. Lomov [1] asymptotic solution to problem (1). Earlier, there were mostly considered systems of ordinary differential equations with slowly varying kernels ($\mu(t) \equiv 0$; see the detailed references in [2], [3]. In work [4], for the case $\mu(t) \equiv 0$, there was studied only the passage to the limit as $\varepsilon \to +0$ in an integral-differential partial differential system, while in work [5] a regularized asymptotics of arbitrary order in ε was considered for this case.

Proceeding to constructing regularized asymptotic solutions for system (1) with a fast varying kernel, we note that the dependence of the matrix A on the variable x makes no essential influence. There arise only technical difficulties but the main lines of the procedure remain unchanged. This is why from the very beginning we suppose that the matrix A is independent of x. Moreover, without loss of generality, we can assume that $T = X = 1$.

1. Regularization of problem (1)

We assume the following conditions:

1) the matrix $A(r)$ belongs to $C^\infty([0,1], \mathbb{C}^{n\times n})$, the function $h(t,x)$ belongs to $C^\infty([0,1]\times[0,1], \mathbb{C}^n)$, the function $\mu(t)$ belongs to $C^\infty([0,1], \mathbb{C}^1)$, the kernel $K(t,x,s)$ belongs to the space $C^\infty(\{0 \leqslant x \leqslant 1, 0 \leqslant s \leqslant t \leqslant 1\}, \mathbb{C}^{n\times n})$;

2) the spectrum $\{\lambda_j(t)\}$ of the matrix $A(t)$ and the spectral value $\mu(t)$ of the kernel of the integral operator satisfy the conditions:

a) $\lambda_i(t) \neq \lambda_j(t)$, $i \neq j$, $\mu(t) \neq \lambda_i(t)$, $i,j = \overline{1,n}$, $\forall t \in [0,1]$;

b) $\operatorname{Re}\lambda_j(t) \leqslant 0$, $\lambda_j(t) \neq 0$, $\operatorname{Re}\mu(t) < 0$, $j = \overline{1,n}$, $\forall t \in [0,1]$.

We denote $\mu(t) \equiv \lambda_{n+1}(t)$ and following [1], we introduce the regularized variables:

$$\tau_j = \frac{1}{\varepsilon}\int_0^t \lambda_j(s)\,ds = \frac{\psi_j(t)}{\varepsilon}, \qquad j = \overline{1,n+1}. \tag{2}$$

For the function $\tilde{y}(t,x,\tau,\varepsilon)$ we pose the following problem:

$$\begin{gathered} \varepsilon\frac{\partial \tilde{y}}{\partial t} + \sum_{j=1}^{n+1}\lambda_j(t)\frac{\partial \tilde{y}}{\partial \tau_j} - A(t)\tilde{y} - \int_0^t e^{\frac{1}{\varepsilon}\int_s^t \mu(\theta)\,d\theta} K(t,x,s)\,\tilde{y}\left(s,x,\frac{\psi(s)}{\varepsilon},\varepsilon\right)ds = h(t,x), \\ \tilde{y}(t,x,\tau,\varepsilon)\,|_{t=0,\tau=0} = y^0(x), \qquad (\tau = (\tau_1,\ldots,\tau_{n+1})), \quad \psi = (\psi_1,\ldots,\psi_{n+1})). \end{gathered} \tag{3}$$

The relation of problem (3) with original problem (1) is as follows: if $\tilde{y} = \tilde{y}(t,x,\tau,\varepsilon)$ is a solution to problem (3), then its restriction $y(t,x,\varepsilon) \equiv \tilde{y}\left(t,x,\frac{\psi(t)}{\varepsilon},\varepsilon\right)$ on regularizing functions (2) is obviously an exact solution to original problem (1). However, problem (3) can not be regarded as a completely regularized since the integral operator

$$J\tilde{y} = \int_0^t e^{\frac{1}{\varepsilon}\int_s^t \lambda_{n+1}(\theta)\,d\theta} K(t,x,s)\,\tilde{y}\left(s,x,\frac{\psi(s)}{\varepsilon},\varepsilon\right)ds$$

has not been regularized. As it is known, in order to regularize this operator [1], one needs to introduce the space M_ε asymptotically invariant w.r.t. the operator J. This is done as follows. We introduce the class U of solutions to iteration problems (see below):

$$U = \left\{\hat{y}(t,x,\tau) : \hat{y} = \sum_{j=1}^{n+1} y_j(t,x)\,e^{\tau_j} + y_0(t,x), \quad y_j(t,x) \in C^\infty([0,1]\times[0,1]),\ j = \overline{0,n+1}\right\},$$

and then we take the restriction of this class as $\tau = \psi(t)/\varepsilon$. This is exactly the space M_ε. To justify this fact, we need to show that the image $J\hat{y}(t,x,\tau)$ of the integral operator J on an element of the space U is represented as the power series

$$\sum_{k=0}^{\infty}\varepsilon^k\left(\sum_{j=1}^{n+1}y_j^{(k)}(t,x)\,e^{\frac{\psi_j(t)}{\varepsilon}}+y_0^{(k)}(t,x)\right)$$

converging asymptotically as $\varepsilon\to+0$ uniformly in $(t,x)\in[0,1]\times[0,1]$. Let us study this issue.

On an arbitrary element $\hat{y}(t,x,\tau)$ of the space U, the image of the integral operator J is of the form:

$$\begin{aligned}J\hat{y}(t,x,\tau)\equiv\sum_{j=1}^{n+1}\int_0^t e^{\frac{1}{\varepsilon}\int_s^t\lambda_{n+1}(\theta)\,d\theta}K(t,x,s)\,y_j(s,x)\,e^{\frac{\psi_j(s)}{\varepsilon}}ds\\ +\int_0^t e^{\frac{1}{\varepsilon}\int_s^t\lambda_{n+1}(\theta)\,d\theta}K(t,x,s)\,y_0(s,x)\,ds.\end{aligned}\tag{4}$$

We integrate by parts in each term of this sum. As $j=\overline{1,n}$, we have

$$\begin{aligned}&\int_0^t e^{\frac{1}{\varepsilon}\int_s^t\lambda_{n+1}(\theta)\,d\theta}K(t,x,s)\,y_j(s,x)\,e^{\frac{\psi_j(s)}{\varepsilon}}ds\\
=&e^{\frac{1}{\varepsilon}\int_0^t\lambda_{n+1}(\theta)\,d\theta}\int_0^t K(t,x,s)\,y_j(s,x)\,e^{\frac{1}{\varepsilon}\int_0^s(\lambda_j(\theta)-\lambda_{n+1}(\theta))d\theta}ds\\
=&\varepsilon e^{\frac{1}{\varepsilon}\int_0^t\lambda_{n+1}(\theta)d\theta}\int_0^t\frac{K(t,x,s)\,y_j(s,x)}{\lambda_j(s)-\lambda_{n+1}(s)}d\left(e^{\frac{1}{\varepsilon}\int_0^s(\lambda_j(\theta)-\lambda_{n+1}(\theta))d\theta}\right)\\
=&\varepsilon e^{\frac{1}{\varepsilon}\int_0^t\lambda_{n+1}(\theta)d\theta}\Big(\frac{K(t,x,s)\,y_j(s,x)}{\lambda_j(s)-\lambda_{n+1}(s)}e^{\frac{1}{\varepsilon}\int_0^s(\lambda_j(\theta)-\lambda_{n+1}(\theta))d\theta}\Big|_0^t\\
&-\int_0^t\left(\frac{\partial}{\partial s}\frac{K(t,x,s)\,y_j(s,x)}{\lambda_j(s)-\lambda_{n+1}(s)}\right)e^{\frac{1}{\varepsilon}\int_0^s(\lambda_j(\theta)-\lambda_{n+1}(\theta))d\theta}ds\Big)\\
=&\varepsilon\left[\frac{K(t,x,t)\,y_j(t,x)}{\lambda_j(t)-\lambda_{n+1}(t)}e^{\frac{1}{\varepsilon}\int_0^t\lambda_j(\theta)d\theta}-\frac{K(t,x,0)\,y_j(0,x)}{\lambda_j(0)-\lambda_{n+1}(0)}e^{\frac{1}{\varepsilon}\int_0^t\lambda_{n+1}(\theta)d\theta}\right]\\
&-\varepsilon e^{\frac{1}{\varepsilon}\int_0^t\lambda_{n+1}(\theta)d\theta}\int_0^t\left(\frac{\partial}{\partial s}\frac{K(t,x,s)\,y_j(s,x)}{\lambda_j(s)-\lambda_{n+1}(s)}\right)e^{\frac{1}{\varepsilon}\int_0^s(\lambda_j(\theta)-\lambda_{n+1}(\theta))d\theta}ds.\end{aligned}$$

We introduce the notation:

$$I_j^0(K(t,x,s)\,y_j(s,x))\equiv\frac{K(t,x,s)\,y_j(s,x)}{\lambda_j(s)-\lambda_{n+1}(s)},\qquad j=\overline{1,n}.$$

Then the result of the above transformations can be written as

$$\begin{aligned}&\int_0^t e^{\frac{1}{\varepsilon}\int_s^t\lambda_{n+1}(\theta)\,d\theta}K(t,x,s)\,y_j(s,x)\,e^{\frac{\psi_j(s)}{\varepsilon}}ds\\
=&\varepsilon\left[I_j^0(K(t,x,s)\,y_j(s,x))_{s=t}\,e^{\tau_j}-I_j^0(K(t,x,s)\,y_j(s,x))_{s=0}\,e^{\tau_{n+1}}\right]\\
&-\varepsilon e^{\frac{1}{\varepsilon}\int_0^t\lambda_{n+1}(\theta)\,d\theta}\int_0^t\frac{\partial}{\partial s}\left(I_j^0(K(t,x,s)\,y_j(s,x))\right)e^{\frac{1}{\varepsilon}\int_0^s(\lambda_j(\theta)-\lambda_{n+1}(\theta))d\theta}ds,\qquad j=\overline{1,n},\end{aligned}$$

where $\tau_j=\frac{\psi_j(t)}{\varepsilon}$, $j=\overline{1,n}$. Proceeding with this process, we obtain the series

$$\begin{aligned}&\int_0^t e^{\frac{1}{\varepsilon}\int_s^t\lambda_{n+1}(\theta)\,d\theta}K(t,x,s)\,y_j(s,x)\,e^{\frac{\psi_j(s)}{\varepsilon}}ds\\
=&\sum_{k=0}^{\infty}(-1)^k\varepsilon^{k+1}\Big[\left(I_j^k(K(t,x,s)\,y_1(s,x))\right)_{s=t}\cdot e^{\tau_j}-\left(I_1^k(K(t,x,s)\,y_1(s,x))\right)_{s=0}e^{\tau_{n+1}}\Big],\end{aligned}\tag{5}$$

where $\tau_j = \frac{\psi_j(t)}{\varepsilon}$, $j = \overline{1,n}$, and the operators I_j^k are of the form:

$$\begin{aligned}
&I_j^0\left(K\left(t,x,s\right)y_j\left(s,x\right)\right) \equiv \frac{K\left(t,x,s\right)y_j\left(s,x\right)}{\lambda_j\left(s\right)-\lambda_{n+1}\left(s\right)},\\
&I_j^1\left(K\left(t,x,s\right)y_j\left(s,x\right)\right) = \frac{1}{\lambda_j\left(s\right)-\lambda_{n+1}\left(s\right)}\frac{\partial}{\partial s}I_1^0\left(K\left(t,x,s\right)y_j\left(s,x\right)\right),\\
&\dots\\
&I_1^m\left(K\left(t,x,s\right)y_j\left(s,x\right)\right) = \frac{1}{\lambda_j\left(s\right)-\lambda_{n+1}\left(s\right)}I_1^{m-1}\left(K\left(t,x,s\right)y_j\left(s,x\right)\right), \quad m\geqslant 1, \quad j=\overline{1,n}.
\end{aligned} \tag{6}$$

We transform the term for $j = n+1$ in (4) as follows:

$$\begin{aligned}
&\int_0^t e^{\frac{1}{\varepsilon}\int_s^t \lambda_{n+1}(\theta)\,d\theta}K\left(t,x,s\right)y_{n+1}\left(s,x\right)e^{\frac{\psi_{n+1}(s)}{\varepsilon}}ds\\
&\equiv \int_0^t e^{\frac{1}{\varepsilon}\int_s^t \lambda_{n+1}(\theta)\,d\theta}K\left(t,x,s\right)y_{n+1}\left(s,x\right)e^{\frac{1}{\varepsilon}\int_0^s \lambda_{n+1}(\theta)\,d\theta}ds\\
&=e^{\frac{1}{\varepsilon}\int_0^t \lambda_{n+1}(\theta)\,d\theta}\int_0^t K\left(t,x,s\right)y_{n+1}\left(s,x\right)ds\\
&\equiv\left(\int_0^t K\left(t,x,s\right)y_{n+1}\left(s,x\right)ds\right)e^{\tau_{n+1}}, \qquad \tau_{n+1}=\frac{\psi_{n+1}\left(t\right)}{\varepsilon}.
\end{aligned} \tag{7}$$

And finally, for the last term in (4) we have:

$$\begin{aligned}
&\int_0^t e^{\frac{1}{\varepsilon}\int_s^t \lambda_{n+1}(\theta)\,d\theta}K\left(t,x,s\right)y_0\left(s,x\right)ds = \varepsilon\int_0^t \frac{K\left(t,x,s\right)y_0\left(s,x\right)}{-\lambda_{n+1}\left(s\right)}d_s e^{\frac{1}{\varepsilon}\int_s^t \lambda_{n+1}(\theta)\,d\theta}\\
&=\varepsilon\frac{K\left(t,x,s\right)y_0\left(s,x\right)}{-\lambda_{n+1}\left(s\right)}e^{\frac{1}{\varepsilon}\int_s^t \lambda_{n+1}(\theta)\,d\theta}\Big|_{s=0}^{s=t} - \varepsilon\int_0^t e^{\frac{1}{\varepsilon}\int_s^t \lambda_{n+1}(\theta)\,d\theta}\left(\frac{\partial}{\partial s}\frac{K\left(t,x,s\right)y_0\left(s,x\right)}{-\lambda_{n+1}\left(s\right)}\right)ds\\
&=\varepsilon\left[\frac{K\left(t,x,t\right)y_0\left(t,x\right)}{-\lambda_{n+1}\left(t\right)} - \frac{K\left(t,x,0\right)y_0\left(0,x\right)}{-\lambda_{n+1}\left(0\right)}e^{\tau_{n+1}}\right]\\
&\quad+\varepsilon\int_0^t e^{\frac{1}{\varepsilon}\int_s^t \lambda_{n+1}(\theta)\,d\theta}\left(\frac{\partial}{\partial s}\frac{K\left(t,x,s\right)y_0\left(s,x\right)}{\lambda_{n+1}\left(s\right)}\right)ds\\
&=\sum_{k=0}^{\infty}\varepsilon^{k+1}\left[\left(I_{n+1}^k\left(K\left(t,x,s\right)y_0\left(s,x\right)\right)_{s=0}\right)e^{\tau_{n+1}} - I_{n+1}^k\left(K\left(t,x,s\right)y_0\left(s,x\right)\right)_{s=t}\right],
\end{aligned} \tag{8}$$

where we have introduced the operators

$$\begin{aligned}
&I_{n+1}^0\left(K\left(t,x,s\right)y_0\left(s,x\right)\right) = \frac{K\left(t,x,s\right)y_0\left(s,x\right)}{\lambda_{n+1}\left(s\right)},\\
&I_{n+1}^m\left(K\left(t,x,s\right)y_0\left(s,x\right)\right) = \frac{1}{\lambda_{n+1}\left(s\right)}\frac{\partial}{\partial s}I_{n+1}^{m-1}\left(K\left(t,x,s\right)y_0\left(s,x\right)\right), \qquad m\geqslant 1.
\end{aligned} \tag{9}$$

The asymptotic convergence of series (5) and (8) can be proved in the same way as a similar statement in [2, Ch. 8]. Let $\tilde{y}\left(x,t,\tau,\varepsilon\right)$ be an arbitrary function continuous in $(t,x,\tau) \in [0,1]\times[0,1]\times \times\{\operatorname{Re}\tau_j \leqslant 0,\ j=\overline{1,n+1}\}$ and possessing the asymptotic expansion

$$\tilde{y}\left(t,x,\tau,\varepsilon\right) = \sum_{k=0}^{\infty}\varepsilon^k y_k\left(t,x,\tau\right), \qquad y_k\left(t,x,\tau\right)\in U, \tag{10}$$

converging as $\varepsilon \to +0$ uniformly $(t,x,\tau) \in [0,1]\times[0,1]\times\{\operatorname{Re}\tau_j \leqslant 0,\ j=\overline{1,n+1}\}$. We introduce the operators $R_m : U \to U$ acting on each element $\hat{y}\left(t,x,\tau\right)$ in the space U by the rule:

$$R_0\hat{y}(t,x,\tau) \equiv R_0\left(\sum_{j=1}^{n+1} y_j(t,x)\,e^{\tau_j} + y_0(t,x)\right) = e^{\tau_{n+1}}\int_0^t K(t,x,s)\,y_{n+1}(s,x)\,ds,$$

$$\begin{aligned} R_{k+1}\hat{y}(x,t,\tau) = (-1)^k \Big[&\sum_{j=1}^{n} \left(I_j^k\left(K(t,x,s)\,y_j(s,x)\right)\right)_{s=t}\cdot e^{\tau_j} \\ &- \left(I_j^k\left(K(t,x,s)\,y_j(s,x)\right)\right)_{s=0} e^{\tau_{n+1}}\Big] \\ &+ \left(I_{n+1}^k\left(K(t,x,s)\,y_0(s,x)\right)\right)_{s=0} e^{\tau_{n+1}} - \left(I_{n+1}^k\left(K(t,x,s)\,y_0(s,x)\right)\right)_{s=t}, \end{aligned} \qquad (11)$$

where the operators I_j^k, $k \geqslant 0$, are of form (6), while the operators I_{n+1}^k, $k \geqslant 0$, are of the form (9). The operators R_m are called *order operators* in ε, since being applied to a function $\hat{y}(t,x,\tau)$, they select the terms of order ε^m. It is natural to define the extended operator for the integral operator J as follows.

Definition 1. *A formal extenstion of the operator J is the operator $\tilde{J}$ acting on each function $\tilde{y}(t,x,\tau,\varepsilon)$ of form (10) by the rule*[1]

$$\tilde{J}\tilde{y} \equiv \tilde{J}\left(\sum_{k=0}^{\infty}\varepsilon^k y_k(t,x,\tau)\right) \triangleq \sum_{r=0}^{\infty}\varepsilon^r\left(\sum_{k=0}^{r} R_{r-k}y_k(t,x,\tau)\right). \qquad (12)$$

Now we can write a problem completely regularized with respect to original one (1):

$$\mathscr{L}_\varepsilon\tilde{y}(t,x,\tau,\varepsilon) \equiv \varepsilon\frac{\partial\tilde{y}}{\partial t} + \sum_{j=1}^{n+1}\lambda_j(t)\frac{\partial\tilde{y}}{\partial\tau_j} - A(t)\,\tilde{y} - \tilde{J}\tilde{y} = h(t,x)\,,\ \tilde{y}(0,x,0,\varepsilon) = y^0(x)\,, \qquad (12)$$

where $\tilde{y}(t,x,\tau,\varepsilon)$ is series (10).

2. Solvability of iteration problems

Substituting series (10) into (13) and equating the coefficients at the like powers of ε, we obtain the following iteration problems:

$$\mathscr{L}\ y_0(t,x,\tau) \equiv \sum_{j=1}^{n+1}\lambda_j(t)\frac{\partial y_0}{\partial\tau_j} - A(t)\,y_0 - R_0y_0 = h(t,x)\,,\ \ y_0(0,x,0) = y^0(x)\,; \qquad (14_0)$$

$$\mathscr{L}\ y_1(t,x,\tau) = -\tfrac{\partial y_0}{\partial t} + R_1y_0,\quad y_1(0,x,0) = 0; \qquad (14_1)$$

....

$$\mathscr{L}\ y_k(t,x,\tau) = -\frac{\partial y_{k-1}}{\partial t} + R_1y_{k-1} + \ldots + R_ky_0,\quad y_k(0,x,0) = 0,\ \ k \geqslant 1. \qquad (14_k)$$

Each of iteration problems (14_k) is of the form

$$\begin{aligned} &\mathscr{L}\ \hat{y}(t,x,\tau) \equiv \sum_{j=1}^{n+1}\lambda_j(t)\frac{\partial y}{\partial\tau_j} + \mu(t)\frac{\partial y}{\partial\tau_2} - A(t)\,y - R_0y = H(t,x,\tau)\,, \\ &\hat{y}(0,x,0) = y_*(x)\,, \end{aligned} \qquad (15)$$

where

$$H(t,x,\tau) = \sum_{j=1}^{n+1} H_j(t,x)\,e^{\tau_j} + H_0(t,x) \in U,$$

[1] The symbol $\triangleq$ means "is equal by the definition".

$y_*(x) \in C^\infty[0,1]$ are known functions and $R_0 y$ stands for the operator

$$R_0\hat{y}(x,t,\tau) \equiv R_0\left(\sum_{j=1}^{n+1} y_j(t,x)e^{\tau_j} + y_0(t,x)\right) = e^{\tau_{n+1}}\int_0^t K(t,x,s)\,y_{n+1}(s,x)\,ds.$$

Let us try to solve problem (15). Substituting the element

$$\hat{y}(t,x,\tau) = \sum_{j=1}^{n+1} y_j(t,x)e^{\tau_j} + y_0(t,x)$$

of the space U into (15), we get

$$\sum_{j=1}^{n+1}\lambda_j(t)\,y_j(t,x)e^{\tau_j} - \sum_{j=1}^{n+1} A(t)\,y_j(t,x)e^{\tau_j}$$
$$- A(t)\,y_0(t,x) - e^{\tau_{n+1}}\int_0^t K(t,x,s)\,y_{n+1}(s,x)\,ds = \sum_{j=1}^{n+1} H_j(t,x)e^{\tau_j} + H_0(t,x).$$

Equalling here the free terms and the coefficients at the like exponents, we obtain the equations

$$\begin{aligned} &- A(t)\,y_0(t,x) = H_0(t,x), \\ &(\lambda_j(t)I - A(t))\,y_j(t,x) = H_j(t,x), \qquad j = \overline{1,n}, \\ &(\lambda_{n+1}(t)I - A(t))\,y_{n+1}(t,x) - \int_0^t K(t,x,s)\,y_{n+1}(s,x)\,ds = H_{n+1}(t,x). \end{aligned} \tag{13}$$

We denote by $\varphi_j(t)$ an eigenvector associated with the eigenvalue $\lambda_j(t)$ of the matrix $A(t)$, while $\chi_j(t)$ denotes the eigenvector associated with the eigenvalue $\bar{\lambda}_j(t)$ of the matrix $A^*(t)$. The systems of vectors $\{\varphi_j(t)\}$ are $\{\chi_k(t)\}$ are chosen biorthogonal:

$$A(t)\,\varphi_j(t) \equiv \lambda_j(t)\,\varphi_j(t), \qquad A^*(t)\,\chi_k(t) \equiv \bar{\lambda}_k(t)\,\chi_k(t), \qquad (\varphi_j(t), \chi_k(t)) = \delta_{jk},$$

where δ_{jk} is the Kronecker delta and $j,k = \overline{1,n}$.

We proceed to systems (16). The first equation in (16) has the unique solution $y_0(t,x) = -A^{-1}(t)\,H_0(t,x)$. The second system in (16) is solvable in the space $C^\infty([0,1]\times[0,1],\mathbb{C}^n)$ for a fixed $j \in \{1,\ldots,n\}$ if and only if the conditions hold:

$$(H_j(t,x), \chi_j(t)) \equiv 0 \qquad \text{for all} \quad (x,t) \in [0,1]\times[0,1].$$

The last equation in (16) is the second kind Volterra equation with the smooth kernel $G(t,x,s) = (\lambda_{n+1}(t) - A(t))^{-1} K(t,x,s)$. Here the variable x is regarded as a parameter and therefore, this equation has the unique solution in the space $C^\infty([0,1]\times[0,1])$. If for each $(t,x) \in [0,1]\times[0,1]$, we introduce the scalar product

$$\langle \hat{y}(t,x,\tau), z(t,x,\tau)\rangle \equiv \langle\sum_{j=1}^{n+1} y_j(t,x)e^{\tau_j} + y_0(t,x), \sum_{j=1}^{n+1} z_j(t,x)e^{\tau_j} + z_0(t,x)\rangle$$
$$\triangleq \sum_{j=0}^{n+1}(y_j(t,x), z_j(t,x))$$

in the space U, where $(\,,)$ denotes the usual scalar product in the complex space $\mathbb{C}^n$, then the above arguing can be summarized as the following theorem.

Theorem 1. *Assume that the right hand side of equation (15) is of form*

$$H(t,x,\tau) \equiv \sum_{j=1}^{n+1} H_j(t,x)e^{\tau_j} + H_0(t,x) \in U$$

and Conditions 1) and 2) are satisfied. Then equation (15) in space U is solvable if and only if

$$\langle H(t,x,\tau),\, \chi_j(t)\, e^{\tau_j}\rangle \equiv 0 \qquad j=\overline{1,n}, \quad (t,x)\in[0,1]\times[0,1]. \tag{14}$$

Under condition (17), equation (15) has the following solution in the space U:

$$\begin{aligned}\hat{y}(t,x,\tau) = &\sum_{j=1}^{n}\alpha_j(t,x)\,\varphi_j(t)\,e^{\tau_j} + \Big(\int_0^t R(t,x,s)\left(\lambda_{n+1}(s)I - A(s)\right)^{-1} H_{n+1}(s,x)\,ds \\ &+ \left(\lambda_{n+1}(t)I - A(t)\right)^{-1} H_{n+1}(t,x)\Big)e^{\tau_{n+1}} - A^{-1}(t)\,H_0(t,x),\end{aligned} \tag{15}$$

where $\mathscr{R}(t,x,s)$ is the resolvent of the kernel $G(t,x,s) = \left(\lambda_{n+1}(t) - A(t)\right)^{-1} K(t,x,s)$ and $\alpha_j(t,x)\in C^\infty\left([0,1]\times[0,1],\mathbb{C}^1\right)$, $j=\overline{1,n}$, are arbitrary functions.

We impose the initial condition $y(0,x,0) = y_*(x)$ for solution (18). This implies:

$$\begin{aligned}&\sum_{j=1}^{n}\alpha_j(0,x)\,\varphi_j(0) - A^{-1}(0)\,H_0(0,x) = y_*(x) \\ &\Leftrightarrow\ \alpha_j(0,x) = \left(y_*(x) + A^{-1}(0)\,H_0(0,x)\,,\chi_j(0)\right), \quad j=\overline{1,n}.\end{aligned} \tag{16}$$

However, the functions $\alpha_j(t,x)$ are not found completely. We need an additional restriction for a solution to problem (15). Such restriction comes from iteration problems (14_k). We see that a natural additional restriction is the condition

$$\langle -\frac{\partial\hat{y}}{\partial t} + R_1\hat{y} + P(t,x,\tau)\,,\, \chi_j(t)\,e^{\tau_j}\rangle \equiv 0 \qquad j=\overline{1,n}, \quad (t,x)\in[0,1]\times[0,1], \tag{17}$$

where

$$P(t,x,\tau) \equiv \sum_{j=1}^{n+1} P_j(t,x)\,e^{\tau_j} + P_0(t,x) \in U$$

is a known vector function. Let us show that under condition (20), problem (15) has the unique solution in space U.

Theorem 2. *Assume that Conditions 1), 2) are satisfied and the right hand*

$$H(t,x,\tau) \equiv \sum_{j=1}^{n+1} H_j(t,x)\,e^{\tau_j} + H_0(t,x) \in U$$

satisfies orthogonality condition (17). The problem (15) is uniquely solvable in space U under additional condition (20).

Proof. In order to employ condition (20), we calculate the expression

$$-\frac{\partial\hat{y}}{\partial t} + R_1\hat{y}.$$

Since

$$\begin{aligned}R_1\hat{y}(x,t,\tau) = &-\Big[\sum_{j=1}^{n}\left(I_j^0\left(K(t,x,s)\,y_j(s,x)\right)\right)_{s=t}\cdot e^{\tau_j} - \left(I_1^0\left(K(t,x,s)\,y_j(s,x)\right)\right)_{s=0} e^{\tau_{n+1}}\Big] \\ &- \Big[\left(I_{n+1}^0\left(K(t,x,s)\,y_0(s,x)\right)_{s=t} - I_{n+1}^0\left(K(t,x,s)\,y_0(s,x)\right)_{s=0}\right)e^{\tau_{n+1}}\Big],\\ y_j(s,x) = &\ \alpha_j(s,x)\,\varphi_j(s), \qquad y_0(s,x) = -A^{-1}(s)\,H_0(s,x),\\ I_j^0&\left(K(t,x,s)\,y_j(s,x)\right) \equiv \frac{K(t,x,s)\,y_j(s,x)}{\lambda_j(s) - \lambda_{n+1}(s)},\end{aligned}$$

then

$$-\frac{\partial \hat{y}}{\partial t}+R_1\hat{y}+P\left(t,x,\tau\right)=-\sum_{j=1}^{n}\frac{\partial\left(\alpha_j\left(t,x\right)\varphi_j\left(t\right)\right)}{\partial t}e^{\tau_j}$$
$$-\frac{\partial}{\partial t}\left[\int_0^t R\left(t,x,s\right)\left(\lambda_{n+1}\left(s\right)I-A\left(s\right)\right)^{-1}H_{n+1}\left(s,x\right)ds\right.$$
$$\left.+\left(\lambda_{n+1}\left(t\right)I-A\left(t\right)\right)^{-1}H_{n+1}\left(t,x\right)\right]e^{\tau_{n+1}}+\frac{\partial}{\partial t}A^{-1}\left(t\right)H_0\left(t,x\right)$$
$$-\sum_{j=1}^{n}\left[\left(I_j^0\left(K\left(t,x,s\right)\alpha_j\left(s,x\right)\varphi_j\left(s\right)\right)\right)_{s=t}\cdot e^{\tau_j}-\left(I_j^0\left(K\left(t,x,s\right)\alpha_j\left(s,x\right)\varphi_j\left(s\right)\right)\right)_{s=0}e^{\tau_{n+1}}\right]$$
$$-\left[\left(I_{n+1}^0\left(K\left(t,x,s\right)y_0\left(s,x\right)\right)_{s=t}-I_{n+1}^0\left(K\left(t,x,s\right)y_0\left(s,x\right)\right)_{s=0}\right)e^{\tau_{n+1}}\right]$$
$$+\sum_{j=1}^{n+1}P_j\left(t,x\right)e^{\tau_j}+P_0\left(t,x\right),$$

and this is why condition (20) becomes

$$-\frac{\partial\left(\alpha_j\left(t,x\right)\right)}{\partial t}+\left(\frac{K\left(t,x,t\right)\varphi_j\left(t\right)}{\lambda_j\left(t\right)-\lambda_{n+1}\left(t\right)}-\dot{\varphi}_j\left(t\right),\chi_j\left(t\right)\right)\alpha_j\left(t,x\right)$$
$$+\left(P_j\left(t,x\right),\chi_j\left(t\right)\right)\equiv 0,\qquad j=\overline{1,n}.$$

In view of initial condition (19), this equation has the unique solution:

$$\alpha_j\left(t,x\right)=e^{q_j\left(t,x\right)}\left[\alpha_j\left(0,x\right)+\int_0^t\left(P_j\left(s,x\right),\chi_j\left(s\right)\right)e^{-q\left(s,x\right)}ds\right],\qquad(18)$$

where

$$q_j\left(t,x\right)=\int_0^t\left(\frac{K\left(s,x,s\right)\varphi_j\left(s\right)}{\lambda_j\left(s\right)-\lambda_{n+1}\left(s\right)}-\dot{\varphi}_j\left(s\right),\chi_j\left(s\right)\right)ds,\qquad j=\overline{1,n}.$$

Hence, under the assumptions of the theorem, there exists the unique solution in space U satisfying (14). □

Applying Theorems 1 and 2 to iteration problems (14_k), we construct series (10) with coefficients in class U. Let

$$y_{\varepsilon N}\left(t,x\right)=\sum_{k=0}^{N}\varepsilon^k y_k\left(t,x,\frac{\psi\left(t\right)}{\varepsilon}\right)$$

be the restriction of Nth partial sum of this series as $\tau=\frac{\psi(t)}{\varepsilon}$. Then as in [2, Ch. 8], we can easily prove the following result.

Theorem 3. *Assume that Conditions 1) and 2) are satisfied. Then as* $\varepsilon\in\left(0,\varepsilon_0\right]$, *where* $\varepsilon_0>0$ *is sufficiently small, problem (1) has the unique solution* $y\left(t,x,\varepsilon\right)\in C^1\left(\left[0,1\right]\times\left[0,1\right]\right)$ *and the estimate holds:*

$$\left\|y\left(t,x,\varepsilon\right)-y_{\varepsilon N}\left(t,x\right)\right\|_{C([0,1]\times[0,1])}\leqslant C_N\varepsilon^{N+1},\qquad N=0,1,2,\ldots,$$

where the constant $C_N>0$ *is independent of* $\varepsilon\in\left(0,\varepsilon_0\right]$.

3. Solution to first iteration problem. Study of initialization problem

Since the vector function $H(t,x,\tau) \equiv h(t,x)$ in system (14_0) is independent of τ, it satisfies condition (17). Hence, system (14_0) has a solution in the space U and it can be written as (see (18))

$$y_0(t,x,\tau) = \sum_{j=1}^{n} \alpha_j^{(0)}(t,x)\,\varphi_j(t)\,e^{\tau_j} - A^{-1}(t)\,h(t,x), \tag{19}$$

where[1] $\alpha_j^{(0)}(t,x) \in C^\infty([0,1]\times[0,1], \mathbb{C}^1)$ are arbitrary functions. In order to find these functions, we find first their values at the point $t=0$. Since $y_0(0,x,0) = y^0(x)$, then

$$\begin{aligned} &\sum_{j=1}^{n} \alpha_j^{(0)}(0,x)\,\varphi_j(0) - A^{-1}(0)\,h(0,x) = y^0(x) \\ &\Leftrightarrow\ \alpha_j^{(0)}(0,x) = \left(A^{-1}(0)\,h(0,x) + y^0(x), \chi_j(0)\right), \qquad j=\overline{1,n}. \end{aligned} \tag{20}$$

To find completely the functions $\alpha_j^{(0)}(t,x)$, we should proceed to next problem (14_1) and impose orthogonality condition (17) for its right hand side. As a result, we obtain the equations

$$-\frac{\partial\left(\alpha_j^{(0)}(t,x)\right)}{\partial t} + \left(\frac{K(t,x,t)\,\varphi_j(t)}{\lambda_j(t)-\lambda_{n+1}(t)} - \dot{\varphi}_j(t), \chi_j(t)\right)\alpha_j^{(0)}(t,x) \equiv 0, \qquad j=\overline{1,n},$$

and by identity (23) we find that

$$\alpha_j^{(0)}(t,x) = e^{q_j(t,x)}\left(A^{-1}(0)\,h(0,x) + y^0(x), \chi_j(0)\right), \qquad j=\overline{1,n}, \tag{24}$$

where

$$q_j(t,x) \equiv \int_0^t \left(\frac{K(s,x,s)\,\varphi_j(s)}{\lambda_j(s)-\lambda_{n+1}(s)} - \dot{\varphi}_j(s), \chi_j(s)\right) ds, \qquad j=\overline{1,n}.$$

Thus, we find uniquely solution (22) to first iteration problem (14_0).

We proceed to studying the initialization problem. Let $\operatorname{Re}\lambda_j(t) < 0$, $t\in[0,1]$, $j=\overline{1,n}$. Then by Theorem 3 we have

$$\begin{aligned} &\|y(t,x,\varepsilon) - y_{\varepsilon 0}(t,x)\|_{C([0,1]\times[0,1])} \leqslant c_0\varepsilon \Leftrightarrow \\ &\Leftrightarrow \|y(t,\varepsilon) - \left(\sum_{j=1}^{n} \alpha_j^{(0)}(t,x)\,\varphi_j(t)\,e^{\tau_j} - A^{-1}(t)\,h(t,x)\right)\|_{C([0,1]\times[0,1])} \leqslant c_0\varepsilon. \end{aligned}$$

Hence, by each $\delta\in(0,1]$ we get

$$\begin{aligned} c_0\varepsilon \geqslant &\|y(t,x,\varepsilon) - \left(\sum_{j=1}^{n} \alpha_j^{(0)}(t,x)\,\varphi_j(t)\,e^{\frac{\psi_j(t)}{\varepsilon}} - A^{-1}(t)\,h(t,x)\right)\|_{C([\delta,1]\times[0,1])} \\ \geqslant &\|y(t,x,\varepsilon) - \sum_{j=1}^{n} \alpha_j^{(0)}(t,x)\,\varphi_j(t)\,e^{\frac{\psi_j(t)}{\varepsilon}} + A^{-1}(t)\,h(t,x)\,\|_{C([\delta,1]\times[0,1])} \\ \geqslant &\|y(t,x,\varepsilon) + A^{-1}(t)\,h(t,x)\,\|_{C([\delta,1]\times[0,1])} - \|\sum_{j=1}^{n} \alpha_j^{(0)}(t,x)\,\varphi_j(t)\,e^{\frac{\psi_j(t)}{\varepsilon}}\|_{C([\delta,1]\times[0,1])}, \end{aligned}$$

[1] In the expression $y_j^{(k)}$ the superscript (k) indicate the iteration number; it should not be mixed up with the kth derivative.

and this implies that

$$\|y(t,x,\varepsilon)+A^{-1}(t)\,h\,(t,x)\,\|_{C([\delta,1]\times[0,1])}\leqslant c_0\varepsilon+\|\sum_{j=1}^{n}\alpha_j^{(0)}\,(t,x)\,\varphi_j\,(t)\,e^{\frac{\psi_j(t)}{\varepsilon}}\|_{C([\delta,1]\times[0,1])}$$
$$\leqslant c_0\varepsilon+\left\|\sum_{j=1}^{n}\alpha_j^{(0)}\,(t,x)\,\varphi_j\,(t)\right\|_{C([\delta,1]\times[0,1])}e^{-\frac{\varkappa\delta}{\varepsilon}},$$

where $\varkappa=\min\limits_{i=\overline{1,n},t\in[0,1]}(-\mathrm{Re}\;\lambda_i\,(t))>0$. Therefore,

$$(t)\,h\,(t,x)\,\|_{C([\delta,1]\times[0,1])}\to 0\,(\varepsilon\to+0). \tag{21}$$

We have obtained the following result.

Theorem 4. *If Conditions 1) and 2) hold and* $\mathrm{Re}\,\lambda_j\,(t)<0$, $t\in[0,1]$, $j=\overline{1,n}$, *then passage to the limit (25) holds, where* $y=y(t,x,\varepsilon)$ *is the exact solution to problem (1), and the function* $\bar{\bar{y}}\,(t,x)=-A^{-1}\,(t)\,h\,(t,x)$ *solve the degenerate w.r.t. (1) equation*

$$A\,(t)\,\bar{\bar{y}}\,(t,x)+h\,(t,x)=0.$$

However, in our case there can be pure imaginary eigenvalues $\lambda_j\,(t)$. For instance, let

$$\lambda_j\,(t)=\pm i\omega_j\,(t)\,,\quad \omega_j\,(t)>0,\quad j=\overline{1,m},\quad \mathrm{Re}\,\lambda_k\,(t)<0,\quad t\in[0,1]\,,\quad k=\overline{2m+1,n}. \tag{22}$$

In this case the passage to limit (25) in the metrics of the space $C\,([0,1]\times[0,1])$ becomes impossible. Because of this, the *initialization problem* arises: what the initial data in problem (1) should be to ensure the uniform passage to the limit $y\,(t,x,\varepsilon)\to\bar{\bar{y}}\,(t,x)$ as $\varepsilon\to+0$ on the set $[0,1]\times[0,1]$ including the boundary layer in t? The initial data in problem (1) obeying this condition are called initialization class Σ. Since

$$y\,(t,x,\varepsilon)=\sum_{j=1}^{m}\alpha_j^{(0)}(t,x)\varphi_j\,(t)\,e^{\frac{-i}{\varepsilon}\int_0^t\omega_j(\theta)d\theta}+\sum_{j=1}^{m}\beta_j^{(0)}(t,x)\varphi_{m+j}\,(t)\,e^{\frac{+i}{\varepsilon}\int_0^t\omega_j(\theta)d\theta}$$
$$+\sum_{k=2m+1}^{n}\alpha_k^{(0)}(t,x)\varphi_k\,(t)\,e^{\frac{1}{\varepsilon}\int_0^t\lambda_k(\theta)d\theta}-A^{-1}\,(t)\,h\,(t,x)+O(\varepsilon),$$

then the first $2m$ terms fast oscillate and prevent the existence of the passage to the limit $y\,(t,x,\varepsilon)\to y_0^{(0)}\,(t,x)$ on the set $[0,1]\times[0,1]$. This is why we need to remove them, that is, we need to let

$$\alpha_j^{(0)}\,(t,x)\equiv 0,\quad \beta_j^{(0)}\,(t,x)\equiv 0,\quad (t,x)\in[0,1]\times[0,1]\,,\quad j=\overline{1,m}.$$

It follows from formula (25) that this holds if and only if

$$\left(y^0\,(x)+A^{-1}\,(0)\,h\,(0,x)\,,\chi_j\,(0)\right)=0,\quad j=\overline{1,2m},\quad x\in[0,1]\,. \tag{$*$}$$

We have proved the following result.

Theorem 5. *Assume that problem (1) satisfies Conditions 1), 2) and (26). Then the passage to the limit*

$$\|y(t,x,\varepsilon)+A^{-1}\,(t)\,h\,(t,x)\,\|_{C([\delta,1]\times[0,1])}\to 0,\quad \varepsilon\to+0,$$

holds if and only if condition (*) holds.

However, condition (*) does not describe the initialization class since the exponents $\exp\left\{\frac{1}{\varepsilon}\int_0^t\lambda_k\,(\theta)\,d\theta\right\}$, $k=\overline{2m+1,n}$ do not tend to zero uniformly in the vicinity of the point $t=0$ and this is why they should be removed in the description of the class Σ. Finally, we obtain the following result.

Theorem 6. *Assume that problem (1) satisfies Conditions 1), 2) and (26). Then the passage to the limit*

$$\|y(t,x,\varepsilon)+A^{-1}(t)\,h\,(t,x)\,\|_{C([0,1]\times[0,1])}\to 0,\quad \varepsilon\to +0,$$

holds if and only if the condition

$$\left(y^{0}\,(x)+A^{-1}\,(0)\,h\,(0,x)\,,\chi_{j}\,(0)\right)=0,\quad j=\overline{1,n},$$

holds.

Thus, the initialization class is independent of the kernel and is described as follows:

$$\Sigma=\left\{\left(y^{0},h,K,A\right):\ \left(y^{0}\,(x)+A^{-1}\,(0)\,h\,(0,x)\,,\chi_{j}\,(0)\right)=0,\ j=\overline{1,n},\right\}.$$

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