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“QUANTIZATIONS” OF ISOMONODROMIC HAMILTON SYSTEM $H^{\frac{7}{2}+1}$

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Abstract. We consider two compatible linear evolution equations with times s_1 and s_2 depending on two spatial variables. These evolution equations are the analogues of the non-stationary Schrödinger equations determined by the two Hamiltonians $H^{\frac{7}{2}+1}_{s_k}(s_1, s_2, q_1, q_2, p_1, p_2)$ $(k = 1, 2)$ of the Hamilton system $H^{\frac{7}{2}+1}$ formed by a pair of compatible Hamiltonian systems of equations admitting the application of isomonodromic deformations method. These analogues arise from canonical non-stationary Schrödinger equations determined by the Hamiltonians $H^{\frac{7}{2}+1}_{s_k}$. They arise by the formal replacement of the Planck constant by the imaginary unit. We construct explicit solutions of these analogues of Schrödinger equations in terms of the solutions of the corresponding linear systems of ordinary differential equations in the isomonodromic deformations method, whose compatibility condition is the Hamiltonian system $H^{\frac{7}{2}+1}$. The key role in the construction of these explicit solutions is played by the change, which was used earlier in constructing the solutions of non-stationary Schrödinger equation determined by the Hamiltonians of isomonodromic Hamiltonian Garnier system with two degrees of freedom as well as of two isomonodromic degenerations of the latter. We discuss the applicability of this change for constructing the solutions to analogues of non-stationary Schrödinger equations determined by the Hamiltonians of the entire hierarchy of isomonodromic Hamiltonian systems with two degrees of freedom being the degenerations of this Garnier system. We mention also a relation of solutions to Hamilton systems $H^{\frac{7}{2}+1}$ with some problems of modern nonlinear mathematical physics. In particular, we show that the solutions of these Hamiltonian systems are determined explicitly by the simultaneous solutions to the Korteweg-de Vries equation $u_t + u_{xxx} + uu_x = 0$ and a non-autonomous fifth order ordinary differential equations, which are used in universal description of the influence of a small dispersion on the transformation of weak hydrodynamical discontinuities into the strong ones.

Keywords: Hamilton systems, quantization, Shrödinger equation, Painlevé equations, isomonodromic deformations method.

Mathematics Subject Classification: 34M56, 35Q41

1. Introduction

In wave quantum mechanics, the classical Hamilton systems of ordinary differential equations (ODEs) with n degrees of freedom

$$(\lambda_i)'_\tau = H'_{\mu_i}, \quad (\mu_i)'_\tau = -H'_{\lambda_i} \quad (i = 1, \dots, n), \tag{1}$$

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defined by the Hamiltonians $H(\tau,\lambda_1,\ldots,\lambda_n,\mu_1,\ldots,\mu_n)$ correspond to the evolution Schrödinger equation

$$\varepsilon\Psi'_\tau = H(\tau,\zeta_1,\ldots,\zeta_n,-\varepsilon\frac{\partial}{\partial\zeta_1},\ldots,-\varepsilon\frac{\partial}{\partial\zeta_n})\Psi, \tag{2}$$

where the dependence on the Planck constant is taken into consideration by means of the parameter $\varepsilon = i\hbar$.

Around 30 years ago, the second author of this paper found out that equations of form (2) with $n = 1$ and $\varepsilon = 1$ arise in the context of the theory of Painlevé equation. It turned out [39], [40], [42] that such evolution equations are related with the representations of each of sixth canonical Painlevé ODEs $\lambda''_{tt} = f_j(t,\lambda,\lambda'_t)$ $(j = 1,\ldots,6)$ both via the coordinate of the Hamilton system with one degree of freedom of form (1) and via the compatibility condition of linear differential equations in the isomonodromic deformations method (IDM):

$$V''_{\zeta\zeta} = P(\zeta,\tau,\lambda,\lambda'_\tau)V, \quad V'_\tau = B(\zeta,\tau,\lambda,\lambda'_\tau)V'_\zeta - \frac{B_\zeta(\zeta,\tau,\lambda,\lambda'_\tau)}{2}V \tag{3}$$

written out in the classical work by R. Garnier [6].

It was shown in [39], [40] that by explicit changes of form $\Psi = V\exp(S(\tau,\zeta))$ the simultaneous solutions V of equations (3) are transformed to solutions of the evolution equations

$$\varepsilon\frac{\partial\Psi}{\partial\tau} = H(\tau,\zeta,\varepsilon\frac{\partial}{\partial\zeta})\Psi \qquad (\varepsilon = 1). \tag{4}$$

The right hand sides of equations (4) are independent of $\lambda(\tau)$ and $\mu(\tau)$. For a particular choice of order of acting of the differentiation in the variable ζ and the multiplication by this variable, these right hand sides are defined by the Hamiltonians $H = H_j(\tau,\lambda,\mu)$ $(j = \overline{1,6})$ of Hamilton systems (1). By excluding the momenta $\mu(\tau)$ we get a second order ODE for the coordinate $\lambda(\tau)$ coinciding with the corresponding Painlevé equation. Another choice of order admits a symbolic writing of six linear evolution equations in form of (2) [42]. Following the terminology of paper [41], in what follows we call evolution equations (2) with constants $\varepsilon \neq i\hbar$ "quantizations" of corresponding Hamilton systems.

During the last decade, there were written quite a lot of works on relations of the equations of IDM for Painlevé type ODEs with evolution linear equations of quantum mechanics and, starting from work [36], of quantum field theory [1]–[3], [7], [8], [14]–[17], [19]–[24], [26], [27], [32], [35]–[38], [41], [43]–[45].

In particular, in works [38], [42], in terms of the corresponding solutions to the linear equations in IDM there were constructed the solutions to "quantizations" (2) for three compatibly isomonodromic Hamilton pair of systesm of ODEs with *two* degrees of freedom. At that, in [38] there was considered the so-called Garnier system heading an entire hierarchy of isomonodromic Hamilton systems with two degrees of freedom; these systems can be obtained from the Garnier system by the procedure of successive degeneration [11], [13]. In [42] there were constructed "quantizations" (2) of the pairs of Hamilton systems of ODEs being two lowest representatives in this hierarchy. In Remark 2 in [38] there was formulated a conjecture that by means of this procedure, constructions [38] can be extended to the entire hierarchy of the degenerations of the Garnier system. This conjecture is likely true. But in order to realize such extension, the procedure of successive degeneration described in [11], [13] should be also generalized for quantum operators corresponding not only to classical coordinates but also to classical momenta. As opposite to the known procedure of successive degeneration of the hierarchy of six classical Painlevé equations, for a part of successive degenerations given in [11], [13], the combinations of coordinates and momenta are employed. It is not so easy to make such generalization since the Hamiltonians in the hierarchy of Hamilton systems considered in [11], [13], are *quadratic* *only w.r.t. the momenta* but not w.r.t. the coordinates. Because of this reason, at present, the issue on constructing solutions to "quantizations" (2) via the solutions of the corresponding linear equations of IDM for all systems in the hierarchy of the degenerations of the Garnier system with two degrees of freedom is still open.

This paper is devoted to solving this issue for one of the system in this hierarchy, for so-called system $H^{\frac{7}{2}+1}$ [12]. This system is represented by a pair of compatible isomonodromic Hamilton systems (1) with the Hamiltonians

$$\begin{aligned}H^{\frac{7}{2}+1}_{s_1}(s_1,s_2,q_1,q_2,p_1,p_2) = &-6s_1p_1^2+4q_2p_1p_2+2(q_1+s_1)q_2p_2^2+4\gamma p_1\\&+4\gamma(q_1+s_1)p_2+\frac{3}{2}s_1q_1^3+\frac{1}{2}q_1^2q_2-2s_1q_1q_2\\&-\frac{1}{2}q_2^2-\frac{3}{2}s_1(3s_1^2-2s_2)q_1-\frac{1}{2}(5s_1^2-2s_2)q_2,\end{aligned} \tag{5}$$

$$\begin{aligned}H^{\frac{7}{2}+1}_{s_2}(s_1,s_2,q_1,q_2,p_1,p_2) =&2p_1^2-2q_2p_2^2-4\gamma p_2-\frac{1}{2}q_1^3\\&+(q_1+s_1)q_2+\frac{(3s_1^2-2s_2)q_1}{2},\end{aligned} \tag{6}$$

where γ is a constant and the times are respectively $\tau = s_1$, $\tau = s_2$.

Before proceeding to the main part of the paper, we observe that in the initial list of isomonodromic Hamilton degenerations of the Garnier system with two degree of freedom provided in paper [13], the pair of systems defined by Hamiltonians (5), (6) was not written out. This gap in the list in [13] was covered by H. Kawamuko [12]. Meanwhile, similar to classical Painlevé equations, the Hamilton system $H^{\frac{7}{2}+1}$ has relations with various issues in nonlinear mathematical physics:

1) In Section 2 of the present work we show that the solutions to the pair of Hamilton systems defined by Hamiltonians (5), (6), can be expressed via the simultaneous solutions to the Korteweg-de Vires equation (KdV)

$$u_t + u_{xxx} + uu_x = 0 \tag{7}$$

and the stationary part of its symmetry

$$\frac{5\beta}{16}\left(u_{xxxx}+\frac{5uu_{xx}}{3}+\frac{5(u_x)^2}{6}+\frac{5u^3}{18}\right)'_x+2u+xu_x-3t(u_{xxx}+uu_x)=0, \tag{8}$$

β is an arbitrary constant. In the general situation, by means of these solutions, the influence of a small dispersion on the transformation of weak hydrodynamical discontinuities into the strong one is described [28]–[30].

2) D.P. Novikov attracted the attention of the authors to the fact that the same Hamilton system determines the solutions to a non-autonomous Hénon-Heilis Hamilton system with the Hamiltonian

$$H_{HH} = \frac{1}{2}(p_1^2+p_2^2)+q_1^3+\frac{1}{2}q_1q_2^2-\frac{(\alpha+1/2)^2}{2q_2^2}-\frac{1}{2}\tau q_1, \tag{9}$$

where α is a constant. This system was introduced in work [9]. In terms of the solutions to this Hamilton system, the solutions to the fourth order ODE

$$w_{\tau\tau\tau\tau}-10(w^2w_{\tau\tau}+ww_\tau^2)+6w^5-\tau w-\alpha=0 \tag{10}$$

can be represented. After [9], such solutions were considered from various points of view in many papers, see, for instance, [4], [5], [31]. Special solutions to ODEs (10) with $\alpha = 0$ were studied earlier in a widely cited work by V. Periwal and D. Shevitz [18] devoted to some integrable models in the string theory.

2. Special isomonodormic solution to KdV equation and Hamilton system $H^{\frac{7}{2}+1}$

2.1. KdV equation (7) is the compatibility condition for the systems of linear ODEs in the problem of inverse scattering method (PISM)

$$V_x = L(\lambda,t,x)V, \quad V_t = Q(\lambda,t,x)V, \tag{11}$$

where

$$L(\lambda,t,x) = \begin{pmatrix} -i\lambda & \frac{u}{6} \\ -1 & i\lambda \end{pmatrix},$$

$$Q(\lambda,t,x) = (-4i\lambda^3 + \frac{i\lambda u}{3} - \frac{u_x}{6})\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} + 4\lambda^2 \begin{pmatrix} 0 & \frac{u}{6} \\ -1 & 0 \end{pmatrix} + \frac{i\lambda u_x}{3}\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} + \begin{pmatrix} 0 & -\frac{u_{xx}}{6} - \frac{u^2}{18} \\ \frac{u}{3} & 0 \end{pmatrix}.$$

Its solutions compatible with ODE (8) are isomonodromical [33]: it turns out that for such solutions to KdV equation there exist fundamental solutions to linear systems (11) satisfying one more system of linear ODEs

$$V_\lambda = \lambda^4 A(\lambda,t,x)V, \tag{12}$$

where $A(\lambda,t,x)$ is a polynomial in λ^{-1} matrix:

$$A(\lambda,t,x) = A_0(t,x) + \frac{A_1(t,x)}{\lambda} + \frac{A_2(t,x)}{\lambda^2} + \frac{A_3(t,x)}{\lambda^3} + \frac{A_4(t,x)}{\lambda^4} + \frac{A_5(t,x)}{\lambda^5}, \tag{13}$$

where β is a constant, with the coefficients

$$A_0(t,x) = -5i\beta\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad A_1(t,x) = 5\beta\begin{pmatrix} 0 & \frac{u}{6} \\ -1 & 0 \end{pmatrix},$$

$$A_2(t,x) = i\begin{pmatrix} \frac{5\beta u}{12} - 12t & \frac{5\beta u_x}{12} \\ 0 & -\frac{5\beta u}{12} + 12t \end{pmatrix}, \quad A_3(t,x) = \begin{pmatrix} -\frac{5\beta u_x}{24} & -\frac{5\beta u_{xx}}{24} - \frac{5\beta u^2}{72} + 2tu \\ \frac{5\beta u}{12} - 12t & \frac{5\beta u_x}{24} \end{pmatrix},$$

$$A_4(t,x) = i\begin{pmatrix} -\frac{5\beta u_{xx}}{48} - \frac{5\beta u^2}{96} + tu - x & -\frac{5\beta u_{xxx}}{48} - \frac{5\beta u u_x}{48} + tu_x \\ 0 & \frac{5\beta u_{xx}}{48} + \frac{5\beta u^2}{96} - tu + x \end{pmatrix},$$

$$A_5(t,x) = \begin{pmatrix} \frac{5\beta u_{xxx}}{96} + \frac{5\beta u u_x}{96} - \frac{tu_x}{2} + \frac{1}{2} & \frac{5\beta u_{xxxx}}{96} + \frac{5\beta u_x^2}{96} + \frac{5\beta u u_{xx}}{72} + \frac{5\beta u^3}{576} + \frac{xu}{6} - \frac{tu^2}{6} - \frac{tu_{xx}}{2} \\ -\frac{5\beta u_{xx}}{48} - \frac{5\beta u^2}{96} + tu - x & -\frac{5\beta u_{xxx}}{96} - \frac{5\beta u u_x}{96} + \frac{tu_x}{2} - \frac{1}{2} \end{pmatrix}.$$

The compatibility condition of the first system in pair (11) with system (12) is exactly ODE (8). At that, the identity $\det A_5 = -(\theta^0)^2 = const$ holds. This identity is a fourth order ODE satisfied by all simultaneous solutions $u(t,x)$ of KdV equation (7) and ODE (8).

Remark 1. *We found the precise form of system of linear equations* (12) *by means of the Fourier integral*

$$J = \int_{-\infty}^{\infty} \lambda \exp((x\lambda - t\lambda^3 + \beta\lambda^5/16))d\lambda. \tag{14}$$

As it was mentioned in [28], *this integral satisfies linear parts of KdV equation* (7) *and ODE* (8). *The coefficients of this system were defined in view of the condition that the pair of equations PISM* (11) *corresponding to simultaneous solutions of* (7) *and ODE* (8) *should possesses a fundamental solution $V(\lambda,t,x)$ with the only essential singular point $\lambda = \infty$ having the asymptotics*

$$V(\lambda,t,x) \approx \exp\left\{-i(\lambda x + 4t\lambda^3 + \beta\lambda^5 + const \ln\lambda)\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\right\}$$

as $\lambda \to \infty$ in some sector of the complex λ-plane. The leading term in this asymptotics is similar to the integrand in (14). *Thus, the similarities between this Fourier integral and a* *simultaneous solution of the pair of equations* (7), (8) *are extended to the corresponding IDM equations according the general theory of such isomonodromic analogues of Fourier integrals of special form and the practice of applying them, see* [34], [46] *and the references in the latter work.*

By means of the transformation

$$V(\lambda,t,x)=T(\lambda,t,x)\Phi(\zeta,t,x)=\begin{pmatrix} i\lambda & -1\\ 1 & 0\end{pmatrix}\Phi(\zeta,t,x),\quad \zeta=-\lambda^2$$

the simultaneous solutions to systems (11), (12) are transformed to solutions of three slightly more nestled systems of linear ODEs

$$\begin{cases}\Phi_x=L_1(\zeta,t,x)\Phi,\\ \Phi_t=Q_1(\zeta,t,x)\Phi,\\ \Phi_\zeta=B(\zeta,t,x)\Phi\end{cases}\tag{15}$$

with matrix coefficients

$$L_1(\zeta,t,x)=T^{-1}LT=\begin{pmatrix}0 & 1\\ \zeta-\frac{u}{6} & 0\end{pmatrix},$$
$$Q_1(\zeta,t,x)=T^{-1}QT=\begin{pmatrix}\frac{u_x}{6} & -4\zeta-\frac{u}{3}\\ \frac{u_{xx}}{6}+\frac{u^2}{18}+\frac{u}{3}\zeta-4\zeta^2 & -\frac{u_x}{6}\end{pmatrix},$$
$$B(\zeta,t,x)=-\frac{1}{2\lambda}(\lambda^4T^{-1}AT-T^{-1}T'_\lambda)=\begin{pmatrix}B_{11}(\zeta,t,x) & B_{12}(\zeta,t,x)\\ B_{21}(\zeta,t,x) & B_{22}(\zeta,t,x)\end{pmatrix},$$

where

$$B_{11}(\zeta,t,x)=-B_{22}(\zeta,t,x)=-\frac{5\beta u_x}{48}-\frac{5\beta u_{xxx}}{192\zeta}-\frac{5\beta uu_x}{192\zeta}+\frac{tu_x}{4\zeta}-\frac{1}{4\zeta},$$
$$B_{12}(\zeta,t,x)=\frac{5\beta\zeta}{2}+\frac{5\beta u}{24}-6t+\frac{5\beta u_{xx}}{96\zeta}+\frac{5\beta u^2}{192\zeta}-\frac{tu}{2\zeta}+\frac{x}{2\zeta},$$
$$B_{21}(\zeta,t,x)=\frac{5\beta\zeta^2}{2}-\frac{5\beta u\zeta}{24}-6t\zeta-\frac{5\beta u^2}{576}-\frac{5\beta u_{xx}}{96}+\frac{tu}{2}+\frac{x}{2}$$
$$-\frac{5\beta u_{xxxx}}{192\zeta}-\frac{5\beta u_x^2}{192\zeta}-\frac{5\beta uu_{xx}}{144\zeta}-\frac{5\beta u^3}{1152\zeta}-\frac{xu}{12\zeta}+\frac{tu^2}{12\zeta}+\frac{tu_{xx}}{4\zeta}.$$

In what follows, for convenience and without loss of generality we assume that $\beta=0.4$.

2.2. The Hamilton system $H^{\frac{7}{2}+1}$ defined by the pair of Hamiltonians (5), (6) is two compatible systems of ODEs

$$\begin{cases}\dfrac{\partial q_1}{\partial s_1}=4p_2q_2-12s_1p_1+4\gamma,\\ \dfrac{\partial q_2}{\partial s_1}=4p_1q_2+4(q_1+s_1)(p_2q_2+\gamma),\\ \dfrac{\partial p_1}{\partial s_1}=-4\gamma p_2-2p_2^2q_2-\dfrac{9}{2}s_1(q_1^2-s_1^2)-q_1q_2+2s_1q_2-3s_1s_2,\\ \dfrac{\partial p_2}{\partial s_1}=-4p_1p_2-2(q_1+s_1)p_2^2-\dfrac{1}{2}(q_1^2-5s_1^2+2s_2)+2s_1q_1+q_2,\end{cases}\tag{16}$$

$$\begin{cases} \dfrac{\partial q_1}{\partial s_2} = 4p_1, \\ \dfrac{\partial q_2}{\partial s_2} = -4p_2q_2 - 4\gamma, \\ \dfrac{\partial p_1}{\partial s_2}q_1 = \dfrac{3}{2}q_1^2 - q_2 + s_2 - \dfrac{3s_1^2}{2}, \\ \dfrac{\partial p_2}{\partial s_2} = -q_1 - s_1 + 2p_2^2. \end{cases} \tag{17}$$

By the changes

$$\begin{aligned} &\mu_1 = \sqrt[5]{4}p_1, \quad \lambda_1 = \frac{1}{\sqrt[5]{4}}q_1, \quad \mu_2 = \frac{1}{\sqrt[5]{16}}q_2, \quad \lambda_2 = -\sqrt[5]{16}p_2; \\ &t_1 = \sqrt[5]{2}\left(s_2 - \frac{3s_1^2}{2}\right), \quad t_2 = -\sqrt[5]{8}s_1; \quad \theta^0 = 2\gamma \end{aligned} \tag{18}$$

these two Hamilton systems of ODEs are reduced to compatible Hamilton systems

$$\begin{cases} \dfrac{\partial \lambda_1}{\partial t_1} = \dfrac{\partial K_1}{\partial \mu_1} = 2\mu_1, \\ \dfrac{\partial \lambda_2}{\partial t_1} = \dfrac{\partial K_1}{\partial \mu_2} = 2\lambda_1 - \lambda_2^2 - t_2, \\ \dfrac{\partial \mu_1}{\partial t_1} = -\dfrac{\partial K_1}{\partial \lambda_1} = -2\mu_2 + 3\lambda_1^2 + t_1, \\ \dfrac{\partial \mu_2}{\partial t_1} = -\dfrac{\partial K_1}{\partial \lambda_2} = 2\mu_2\lambda_2 - \theta^0, \end{cases} \tag{19}$$

$$\begin{cases} \dfrac{\partial \lambda_1}{\partial t_2} = \dfrac{\partial K_2}{\partial \mu_1} = 2\mu_2\lambda_2 - \theta^0, \\ \dfrac{\partial \lambda_2}{\partial t_2} = \dfrac{\partial K_2}{\partial \mu_2} = 2\mu_2 + 2\mu_1\lambda_2 - t_2\lambda_2^2 - t_2^2 - \lambda_1\lambda_2^2 - \lambda_1^2 + \lambda_1 t_2 - t_1, \\ \dfrac{\partial \mu_1}{\partial t_2} = -\dfrac{\partial K_2}{\partial \lambda_1} = \mu_2(\lambda_2^2 + 2\lambda_1 - t_2) - \theta^0\lambda_2, \\ \dfrac{\partial \mu_2}{\partial t_2} = -\dfrac{\partial K_2}{\partial \lambda_2} = -2\mu_1\mu_2 + 2\mu_2\lambda_2(\lambda_1 + t_2) - \theta^0(\lambda_1 + t_2) \end{cases} \tag{20}$$

with the Hamiltonians

$$K_1 = \mu_1^2 + (2\lambda_1 - \lambda_2^2 - t_2)\mu_2 - \lambda_1^3 - t_1\lambda_1 + \theta^0\lambda_2,$$

$$K_2 = \mu_2^2 + 2\lambda_2\mu_1\mu_2 - \theta^0\mu_1 - \mu_2(\lambda_1^2 + (\lambda_1 + t_2)\lambda_2^2 - t_2\lambda_1 + t_1 + t_2^2) + \theta^0 t_2\lambda_2 + \theta^0\lambda_1\lambda_2,$$

times $\tau = t_1$, $\tau = t_2$, coordinates λ_1, λ_2 and momenta μ_1, μ_2.

2.3. In recent work by H. Kawakami, Hamilton systems (19), (20) was represented as the compatibility condition of the following three systems of linear IDM equations

$$\begin{cases} \dfrac{\partial Y_1}{\partial t_1} = (\zeta + 2\lambda_1 - t_2)Y_2, \\ \dfrac{\partial Y_2}{\partial t_1} = Y_1, \end{cases} \tag{21}$$

$$\begin{cases} \dfrac{\partial Y_1}{\partial t_2} = \mu_1 Y_1 + (-\zeta^2 + \zeta(2t_2 - \lambda_1) + 2\mu_2 - \lambda_1^2 + t_2\lambda_1 - t_1 - t_2^2)Y_2, \\ \dfrac{\partial Y_2}{\partial t_2} = (-\zeta + \lambda_1 + t_2)Y_1 - \mu_1 Y_2, \end{cases} \tag{22}$$

$$\begin{cases} \dfrac{\partial Y_1}{\partial \zeta} = \left(\dfrac{\mu_2\lambda_2}{\zeta} - \mu_1\right) Y_1 \\ \qquad + \left(\zeta^2 + \zeta(\lambda_1 - 2t_2) + \lambda_1^2 + t_1 + t_2^2 - \mu_2 - t_2\lambda_1 + \dfrac{\theta^0\lambda_2 - \mu_2\lambda_2^2}{\zeta}\right) Y_2, \\ \dfrac{\partial Y_2}{\partial \zeta} = \left(\zeta - \lambda_1 - t_2 + \dfrac{\mu_2}{\zeta}\right) Y_1 + \left(\mu_1 + \dfrac{\theta^0 - \mu_2\lambda_2}{\zeta}\right) Y_2. \end{cases} \tag{23}$$

The condition of compatibility of systems (21), (22) is the the following relations:

$$(\mu_1)_{t_1} = 3\lambda_1^2 - 2\mu_2 + t_1, \quad (\lambda_1)_{t_1} = 2\mu_1, \quad (\mu_2)_{t_1} = (\lambda_1)_{t_2}.$$

They imply that the coordinate λ_1 satisfies the evolution equation

$$(\lambda_1)_{t_2} = -\frac{1}{4}(\lambda_1)_{t_1t_1t_1} + 3\lambda_1(\lambda_1)_{t_1} + \frac{1}{2},$$

which by the changes

$$\lambda_1 = -\frac{u}{12} + \frac{t_2}{2}, \quad x = t_1 + \frac{3}{4}t_2^2, \quad 4t = t_2 \tag{24}$$

is transformed to a solution to KdV equation (7). Hence, the pair of system of equations (21), (22) is in fact an $L-A$ pair for KdV equation (7). This pair is equivalent to the traditionally used in PISM $L-A$ pair represented by first two systems of equations in (15). These two pairs are reduced one to the other by a very simple transform. The third system of equations in (15) being a system of linear ODEs with the independent variable ζ is also very similar to system (23) of linear equations of IDM for the pair of Hamilton systems (19), (20). In view of these remarks it is rather easy to conclude as follows.

As $\beta = 0.4$, the changes (18), (24) and

$$\lambda_2 = \frac{\frac{u_{xxx}}{96} + \frac{uu_x}{96} - \frac{tu_x}{4} + \frac{1}{4} + \frac{\theta^0}{2}}{\frac{u_{xx}}{48} + \frac{u^2}{96} - \frac{tu}{2} + \frac{x}{2}} = \frac{\frac{u_{xxxx}}{96} + \frac{u_x^2}{96} + \frac{uu_{xx}}{72} + \frac{u^3}{576} + \frac{xu}{12} - \frac{tu^2}{12} - \frac{tu_{xx}}{4}}{\frac{u_{xxx}}{96} + \frac{uu_x}{96} - \frac{tu_x}{4} + \frac{1}{4} - \frac{\theta^0}{2}},$$

$$\mu_1 = -\frac{u_x}{24}, \quad \mu_2 = \frac{u_{xx}}{48} + \frac{u^2}{96} - \frac{tu}{2} + \frac{x}{2}, \quad Y_1 = e^{\frac{\theta^0}{2}\ln\zeta}\Phi_2, \quad Y_2 = e^{\frac{\theta^0}{2}\ln\zeta}\Phi_1$$

make the equivalence between systems of linear equations of IDM (11), (12) and three systems of equations (21)–(23) and allow us to express the general solutions of compatible systems defined by Hamiltonians (5), (6) in terms of the simultaneous solutions $u(t,x)$ of KdV equation (7) and ODE (8).

3. Solutions to "quantizations" of system $H^{\frac{7}{2}+1}$

3.1. The 2×2 matrix

$$M = \Phi^{-1}(\eta, t, x)\Phi(\zeta, t, x) \tag{25}$$

constructed by the fundamental simultaneous solution Φ of linear systems of ODEs (15) satisfies two *scalar* spatially two-dimensional evolution equations. The first of them is

$$(\eta - \zeta)M_x = \frac{\zeta + \eta}{\zeta - \eta}(M_\eta + M_\zeta) + \eta M_{\eta\eta} - \zeta M_{\zeta\zeta} + g_1(t, x, \eta, \zeta)M \tag{26}$$

with the time x and the other is

$$\begin{aligned}(\zeta - \eta)M_t = &\frac{4(6t(\zeta + \eta) - \zeta^2 - \eta^2)}{(\zeta - \eta)}(M_\eta + M_\zeta) + 4\eta(6t - \zeta)M_{\eta\eta} \\ &+ 4\zeta(\eta - 6t)M_{\zeta\zeta} + g_2(t, x, \eta, \zeta)M\end{aligned} \tag{27}$$

with the time t. The coefficients $g_1(t,x,\eta,\zeta)$, $g_2(t,x,\eta,\zeta)$ of these equations are defined by formulae

$$g_1(t,x,\eta,\zeta) = r_1(t,x,\eta,\zeta) + r_2(t,x,\eta,\zeta),$$
$$g_2(t,x,\eta,\zeta) = 24t(r_1(t,x,\eta,\zeta) + r_2(t,x,\eta,\zeta)) + 4(r_3(t,x,\eta,\zeta) + r_4(t,x,\eta,\zeta)),$$

where $(A_n)_{ij}$ are the elements of the entries (A_n) of matrix (13), and

$$r_1(t,x,\eta,\zeta) = \zeta^4 - \eta^4 - 12t(\zeta^3 - \eta^3) + (x + 36t^2)(\zeta^2 - \eta^2) - 6xt(\zeta - \eta) - \frac{1}{4}\frac{\zeta - \eta}{\zeta\eta}(\theta^0)^2,$$
$$r_2(t,x,\eta,\zeta) = \left(\frac{u_x^2}{576} + \frac{u}{12}(A_5)_{21} - \frac{1}{2}(A_5)_{12} + \left(\frac{u^2}{72} + 2x\right)\left(\frac{u}{24} - 3t\right) + 6xt\right)(\zeta - \eta),$$
$$r_3(t,x,\eta,\zeta) = \zeta\eta(\eta^3 - \zeta^3) - 12t\zeta\eta(\eta^2 - \zeta^2) + \left((x + 36t^2)\zeta\eta - \frac{x^2}{4}\right)(\eta - \zeta) + \frac{1}{4}\frac{\zeta^2 - \eta^2}{\zeta\eta}(\theta^0)^2,$$
$$r_4(t,x,\eta,\zeta) = \left(\frac{u_x}{24}(A_5)_{11} + \left(3t - \frac{u}{24}\right)(A_5)_{12} + \left(\frac{u^2}{576} + \frac{u_{xx}}{96} - \frac{ut}{4} - \frac{x}{4}\right)(A_5)_{21} - \frac{x^2}{4}\right)(\zeta - \eta).$$

The change

$$M = (\zeta - \eta)\exp(S(t,x))W$$

defined by the function $S(t,x)$ satisfying two compatible relations

$$S_x(t,x) = f_1(t,x) = -\frac{r_2(t,x,\eta,\zeta)}{\zeta - \eta}, \quad S_t = f_2(t,x) = \frac{24tr_2(t,x,\eta,\zeta) + 4r_4(t,x,\eta,\zeta)}{\zeta - \eta},$$
$$12f_1(t,x)'_t = 12f_2(t,x)'_x = u_{xx} + \frac{u^2}{2},$$

transforms each simultaneous solution of equations (26), (27) to a simultaneous solution of the pair of linear evolution equations

$$(\zeta - \eta)W_x = \zeta W_{\zeta\zeta} - \eta W_{\eta\eta} + W_\zeta - W_\eta$$
$$- \left[\zeta^4 - \eta^4 - 12t(\zeta^3 - \eta^3) + (x + 36t^2)(\zeta^2 - \eta^2) - 6xt(\zeta - \eta) - \frac{1}{4}\frac{\zeta - \eta}{\zeta\eta}(\theta^0)^2\right]W,$$
$$(\zeta - \eta)W_t = 4\eta(6t - \zeta)W_{\eta\eta} + 4\zeta(\eta - 6t)W_{\zeta\zeta} + 24t(W_\eta - W_\zeta) - 4(\zeta - \eta)(W_\eta + W_\zeta)$$
$$+ \left[24t\left(\zeta^4 - \eta^4 - 12t(\zeta^3 - \eta^3) + (x + 36t^2)(\zeta^2 - \eta^2) - 6xt(\zeta - \eta) - \frac{1}{4}\frac{\zeta - \eta}{\zeta\eta}(\theta^0)^2\right)\right.$$
$$\left. + 4\left(\zeta\eta(\eta^3 - \zeta^3) - 12t\zeta\eta(\eta^2 - \zeta^2) + ((x + 36t^2)\zeta\eta - \frac{x^2}{4})(\eta - \zeta)) + \frac{1}{4}\frac{\zeta^2 - \eta^2}{\zeta\eta}(\theta^0)^2\right)\right]W,$$

which contain *no dependence* on $u(t,x)$ in its coefficients. In these two evolution equations we pass to the independent variables

$$s_1 = \sqrt[5]{4}(-2t), \quad s_2 = \frac{x}{\sqrt[5]{2}}, \quad \xi = \sqrt[5]{4}(\zeta + \eta - 4t), \quad \rho = \sqrt[5]{16}\zeta\eta,$$

and make the change

$$W = \exp(tx^2 - 16t^3x - \frac{384}{5}t^5)(\zeta\eta)^{\frac{\theta^0}{2}}\Psi.$$

This reduces the equations to the pair of evolution equations:

$$\Psi_{s_1} = -6s_1\Psi_{\xi\xi} + 2\rho(\xi + s_1)\Psi_{\rho\rho} + 4\rho\Psi_{\xi\rho} + 2(\xi + s_1)(1 + \theta^0)\Psi_\rho + 2(1 + \theta^0)\Psi_\xi$$
$$+ \left[\frac{3}{2}s_1\xi^3 + \frac{1}{2}\rho\xi^2 - \frac{1}{2}\rho^2 - 2s_1\xi\rho - \frac{3}{2}s_1(3s_1^2 - 2s_2)\xi - \frac{1}{2}(5s_1^2 - 2s_2)\rho\right]\Psi, \tag{28}$$

$$\Psi_{s_2} = 2\Psi_{\xi\xi} - 2\rho\Psi_{\rho\rho} - 2(1+\theta^0)\Psi_\rho + \left[-\frac{1}{2}\xi^3 + \rho\xi + \frac{1}{2}(3s_1^2 - 2s_2)\xi + s_1\rho\right]\Psi \qquad (29)$$

with the polynomial coefficients. By the operator relations

$$\frac{\partial}{\partial\xi}\xi - \xi\frac{\partial}{\partial\xi} = 1, \qquad \frac{\partial}{\partial\rho}\rho - \rho\frac{\partial}{\partial\rho} = 1 \qquad (30)$$

the latter pair of the equations can be written as "quantizations" ($\varepsilon = 1$)

$$\varepsilon\frac{\partial\Psi}{\partial s_i} = H_{s_i}^{\frac{7}{2}+1}\left(s_1, s_2, \xi, \rho, -\varepsilon\frac{\partial}{\partial\xi}, -\varepsilon\frac{\partial}{\partial\rho}\right)\Psi \qquad (i = 1, 2) \qquad (31)$$

determined by Hamiltonians (5), (6) of the Hamilton system $H^{\frac{7}{2}+1}$.

Remark 2. *Thanks to relations (30), equations (28), (29) can be written as ($\varepsilon = 1$)*

$$\varepsilon\frac{\partial\Psi}{\partial s_i} = H_{s_i}^{\frac{7}{2}+1}\left(s_1, s_2, \xi, \rho, \varepsilon\frac{\partial}{\partial\xi}, \varepsilon\frac{\partial}{\partial\rho}\right)\Psi \qquad (i = 1, 2).$$

Thus, the constructions of Sections 2 and 3 give the solutions to such "quantizations" (31). These solutions are written explicitly in terms of solutions to compatible equations IDM (21)–(23). At that, the coefficients of these equations IDM are also explicitly expressed via the set of simultaneous solutions to classical Hamilton systesm of ODEs with Hamiltonians (5), (6).

4. Conclusion

While constructing in the paper solutions to "quantizations" of the Hamilton isomonodromic system $H^{\frac{7}{2}+1}$, an essential role is played by change (25). Such change also played a key role in constructing solutions to "quantizations" of the Garnier system with two degrees of freedom in paper [38] and in constructing solutions to "quantizations" of two lowest representatives in the hierarchy of degenerations of this system in paper [42]. The results of these two papers and of the present paper suggest a conjecture that this change should help in constructing solutions to "quantizations" of the entire hierarchy. For some other purposes this change employed earlier by D.P. Novikov in [36], see also formula (2.3.36) in [25].

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