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STUDY OF DIFFERENTIAL OPERATOR WITH SUMMABLE POTENTIAL AND DISCONTINUOUS WEIGHT FUNCTION

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Abstract. In the work we propose a new approach for studying differential operators with a discontinuous weight function. We study the spectral properties of a differential operator on a finite segment with separated boundary conditions and with "matching" condition at the discontinuity point of the weight function. We assume that the potential of the operator is a summable function on the segment, on which the operator is considered. For large value of the spectral parameter we obtain an asymptotics for the fundamental system of solutions of the corresponding differential equation. By means of this asymptotics we study the "matching" conditions of the considered differential operator. Then we study the boundary conditions of the considered operator. As a result, we obtain an equation for the eigenvalues of the operator, which an entire function. We study the indicator diagram of the equation for the eigenvalues; this diagram is a regular octagon. In various sectors of the indicator diagram we find the asymptotics for the eigenvalues of the studied differential operator.

Keywords: spectral theory of differential operators, spectral parameter, summable potential, discontinuous weight function, indicator diagram, asymptotics of eigenvalues.

Mathematics Subject Classification: 34L05, 45C05

1. Formulation of problem. We consider the differential operator

$$L(y(x)) = y^{(8)}(x) + q(x)y(x) - \lambda\rho(x)y(x)$$

on the segment $[0;\pi]$, where λ is a spectral parameter, subject to separated boundary conditions. The potential $q(x)$ is a summable function $[0;\pi]$, while the weight function $\rho(x)$ is piecewise-constant with a discontinuity at x_1:

$$\rho(x) = \begin{cases} a^8, & a > 0, \quad 0 \leqslant x < x_1, \\ b^8, & b > 0, \quad x_1 < x \leqslant \pi. \end{cases}$$

In more details, we study the differential operator defined by differential equations of form

$$\begin{cases} y_1^{(8)}(x) + q_1(x)y_1(x) = \lambda a^8 y_1(x), \quad 0 \leqslant x < x_1, \quad a > 0, & (1) \\ y_2^{(8)}(x) + q_2(x)y_2(x) = \lambda b^8 y_2(x), \quad x_1 < x \leqslant \pi, \quad b > 0, & (2) \end{cases}$$

subject to "matching" condition at the discontinuity point x_1:

$$\begin{gathered} y_1(x_1 - 0) = y_2(x_1 + 0),\ b^m y_1^{(m)}(x_1 - 0) = a^m y_2^{(m)}(x_1 + 0),\ m = 1, 2, \ldots, 7, \\ \left(y_1(x_1 - 0) = \lim_{x \to x_1, x < x_1} y_1(x),\ y_2(x_1 + 0) = \lim_{x \to x_1, x > x_1} y_2(x) \right), \end{gathered} \tag{3}$$

and to separated boundary conditions

$$\begin{gathered} y_1^{(m_1)}(0) = y_1^{(m_2)}(0) = \cdots = y_1^{(m_7)}(0) = y_2^{(n_1)}(\pi) = 0, \\ m_1 < m_2 < \cdots < m_7;\ m_p, n_1 \in \{0, 1, 2, \ldots, 7\},\ p = 1, 2, \ldots, 7. \end{gathered} \tag{4}$$

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At that we assume that the functions $q_1(x)$ and $q_2(x)$ are summable on the segments $[0; x_1]$ and $[x_1; \pi]$, respectively:

$$\begin{aligned} q_1 \in L_1[0; x_1] &\Leftrightarrow \left(\int_0^x q_1(t)dt\right)'_x = q_1(x) \quad \text{a.e. in} \quad [0; x_1], \\ q_2 \in L_1[x_1; \pi] &\Leftrightarrow \left(\int_{x_1}^x q_2(t)dt\right)'_x = q_2(x) \quad \text{a.e. in} \quad [x_1; \pi]. \end{aligned} \tag{5}$$

Differential operator with a discontinuous weight function (even in the case of a continuous or a smooth potential) were not studied a lot. A classical work in this field is [1], in which for a self-adjoint second order differential operator the theorem on equiconvergence at a discontinuity point of the coefficients was obtained.

Even the case of a non-constant weight function was not studied sufficiently in detail, especially for the operators of order greater than two. In work [2], for the Sturm-Liouville operator, there was studied the issue on the maximal growth rate for normalized eigenfunctions if the weight function is continuous and positive. The issue on estimates for the normalized eigenfunctions of the Sturm-Liouville operator with a positive weight function was studied in work [3]. The author does not know whether such issues were studied for the operators of fourth, sixth and higher orders.

The need of studying differential operators with discontinuous potentials (and a discontinuous weight function) arise in many problems of mechanics, physics and mathematics, for instance, in problems on longitudinal oscillations of rods or transversal oscillations of beams formed by materials of different densities. Problems for differential operators with discontinuous coefficients also arise in forecasting earthquakes and tsunamis. Such issues were considered by the author in works [4, 5].

In work [6], there was studied the (first order) differentiation operator with a discontinuous (piecewise-constant) weight function. In work [7] there was obtained an analogue of Dirichlet theorem for expansions over eigenfunctions of differential equation with discontinuous coefficients. In work [8] there were studied spectral properties of a boundary value problem for a second order operator with a discontinuous (piecewise-constant) weight function. In work [9] there were studied first and second order operators with a sign-changing weight function. The operators of order higher than two were not considered in the above cited papers.

In works [10, 11] the smoothness of the potential was reduced and there was proposed a new topical method for studying Sturm-Liouville operator with a summable potential; at that, the weight function was equal to one.

In work [12] there was proposed a method different from that in works [10, 11] for studying the spectral properties of a fourth order differential operator with a summable potential, which confirmed the results of works [10, 11] for a second order operator. In work [13], the method of work [12] was extended for functional-differential operators, while in work [14] it was extended for a sixth order operator with a summable potential of a delayed variable.

The issue on studying the spectral properties of operators with non-smooth coefficients is still topical. In works [15, 16], as a potential for a second order operator, the δ-function served.

In work [17] the author studied a second order differential operator with a summable potential and an arbitrary smooth weight function.

In work [18, Ch. 4] there were studied differential operators with a discontinuous weight function of second order with a piecewise-constant weight function and a piecewise-smooth potential, of fourth order with a summable potential, of fourth order with a piecewise-smooth potential and a piecewise-smooth weight function. The study of differential operator (1)–(5) is the continuation of studies in work [18, Ch. 4]. The use of "matching" condition (3) is motivated by physical arguments, see [19, Ch. 1, 2].

In the terminology of work [20, Ch. 2], boundary conditions (4) are irregular, at that, we study an entire family of differential operators; in total, there are $8 \cdot 8 = 64$ various types of boundary conditions (4) under varying the numbers $m_1, m_2, \ldots, m_7$ and n_1. All operators in this family have similar spectral properties.

2. Asymptotics of solutions of differential equations (1), (2) for large values of spectral parameter λ. Let $\lambda = s^8$, $s = \sqrt[8]{\lambda}$, and for being correct, we fix the branch of the root by the requirement $\sqrt[8]{1}=+1$. By w_k we denote various eight roots of one:

$$\begin{aligned} &w_k^8=1, \quad w_k=e^{\frac{2\pi i}{8}(k-1)} \quad (k=1,2,\dots,8);\\ &w_1=1, \quad w_2=e^{\frac{2\pi i}{8}}=\cos\left(\frac{2\pi}{8}\right)+i\sin\left(\frac{2\pi}{8}\right)=\frac{\sqrt{2}}{2}+i\frac{\sqrt{2}}{2}=z\neq 0,\\ &w_3=e^{\frac{4\pi i}{8}}=z^2=i, \quad w_4=e^{\frac{6\pi i}{8}}=z^3=-\frac{\sqrt{2}}{2}+i\frac{\sqrt{2}}{2},\dots;\\ &w_k=z^{k-1}, \quad k=1,2,\dots,8. \end{aligned} \tag{6}$$

The numbers w_k, $(k=1,2,\dots,8)$ in (6) partition the unit circumference in eight equal arcs. These numbers satisfy the relations:

$$\sum_{k=1}^{8} w_k^p=0, \quad p=1,2,\dots,7; \qquad \sum_{k=1}^{8} w_k^p, \quad p=0, \quad p=8. \tag{7}$$

By the methods of works [20, Ch. 2], [21, Ch. 1], [22, Ch. 4], the following statements can be proved.

Theorem 1. *The general solution of differential equation (1) is*

$$y_1(x,s)=\sum_{k=1}^{8} C_{1k}y_{1k}(x,s); \quad y_1^{(m)}(x,s)=\sum_{k=1}^{8} C_{1k}y_{1k}^{(m)}(x,s), \ m=1,2,\dots,7, \tag{8}$$

where C_{11}, C_{12},…, C_{18} are arbitrary constants and as $|s|\to\infty$, for the fundamental system of solutions $\{y_{1k}(x,s)\}_{k=1}^{8}$ the asymptotic expansions and estimates hold:

$$y_{1k}(x,s)=e^{aw_ksx}-\frac{A_{7k}^0(x,s)}{8a^7s^7}+\underline{O}\left(\frac{e^{|\operatorname{Im}s|x}}{s^{14}}\right), \quad k=1,2,\dots,7, \tag{9}$$

$$y_{1k}^{(m)}(x,s)=(as)^m\left\{w_k^m e^{aw_ksx}-\frac{A_{7k}^m(x,s)}{8a^7s^7}+\underline{O}\left(\frac{e^{|\operatorname{Im}s|x}}{s^{14}}\right)\right\}, \quad k=1,2,\dots,8, \quad m=1,2,\dots,7, \tag{10}$$

$$\begin{aligned} A_{7k}^0(x,s)=&w_1e^{aw_1sx}\int_0^x q_1(t)e^{a(w_k-w_1)st}dt_{ak1}+w_2e^{aw_2sx}\int_0^x q_1(t)e^{a(w_k-w_2)st}dt_{ak2}+\dots\\ &+w_8e^{aw_8sx}\int_0^x q_1(t)e^{a(w_k-w_8)st}dt_{ak8}, \quad k=1,2,\dots,8, \end{aligned} \tag{11}$$

$$A_{7k}^m(x,s)=\sum_{n=1}^{8} w_nw_n^m e^{aw_nsx}\left(\int_0^x \dots\right)_{akn}, \quad k=1,2,\dots,8, \quad m=1,2,\dots,7. \tag{12}$$

While proving formulae (9)–(12), the variation of constants and relations are to be employed (7).

Theorem 2. *The general solution to differential equation (2) is*

$$y_2(x,s)=\sum_{k=1}^{8} C_{2k}y_{2k}(x,s); \quad y_2^{(m)}(x,s)=\sum_{k=1}^{8} C_{2k}y_{2k}^{(m)}(x,s), \quad m=1,2,\dots,7, \tag{13}$$

where C_{2k} $(k=1,2,\dots,8)$ are arbitrary constants,

$$y_{2k}(x,s)=e^{bw_ksx}-\frac{B_{7k}^0(x,s)}{8a^7s^7}+\underline{O}\left(\frac{e^{|\operatorname{Im}s|x}}{s^{14}}\right), \quad k=1,2,\dots,8, \tag{14}$$

$$\frac{y_{2k}^{(m)}(x,s)}{(bs)^m}=w_k^m e^{bw_ksx}-\frac{B_{7k}^m(x,s)}{8a^7s^7}+\underline{O}\left(\frac{e^{|\operatorname{Im}s|x}}{s^{14}}\right), \quad k=1,2,\dots,8, \quad m=1,2,\dots,7, \tag{15}$$

$$B_{7k}^0(x,s)=\sum_{n=1}^{8} w_ne^{bw_nsx}\int_{x_1}^x q_2(t)e^{b(w_k-w_n)st}dt_{bkn}, \quad k=1,2,\dots,8, \tag{16}$$

$$B_{7k}^m(x,s)=\sum_{n=1}^{8} w_nw_n^m e^{bw_nsx}\int_{x_1}^x q_2(t)e^{b(w_k-w_n)st}dt_{bkn}, \quad k=1,2,\dots,8, \quad m=1,2,\dots,7. \tag{17}$$

While proving formulae (8)–(17), we supposed the following initial conditions

$$\begin{aligned} &A^0_{7k}(0,s)=0; \quad A^m_{7k}(0,s)=0;\ y_{1k}(0,s)=1; \quad y^{(m)}_{1k}(0,s)=w^m_k(as)^m;\\ &B^0_{7k}(x_1,s)=0; \quad B^m_{7k}(x_1,s)=0; \quad y_{2k}(x_1,s)=e^{bw_ksx_1};\\ &y^{(m)}_{2k}(x_1,s)=(bs)^m w^m_k e^{bw_ksx_1}; \quad k=1,2,\dots,8; \quad m=1,2,\dots,7. \end{aligned} \tag{18}$$

3. Study of "matching" condition (3). Substituting formulae (8) and (13) into "matching" conditions (3), we obtain:

$$\begin{cases} y_1(x_1-0)=y_2(x_1+0)\Leftrightarrow \sum\limits_{k=1}^{8} C_{2k}y_{2k}(x_1+0)=\sum\limits_{k=1}^{8} C_{1k}y_{1k}(x_1-0);\\ (bs)^m y_1^{(m)}(x_1-0)=(as)^m y_2^{(m)}(x_1+0)\\ \Leftrightarrow \sum\limits_{k=1}^{8} C_{2k}\dfrac{y^{(m)}_{2k}(x_1+0)}{(bs)^m}=\sum\limits_{k=1}^{8} C_{1k}\dfrac{y^{(m)}_{1k}(x_1-0)}{(as)^m}, \quad m=1,2,\dots,7. \end{cases} \tag{19}$$

System (19) is the system of eight linear equations with eight unknowns $C_{21},C_{22},\dots,C_{28}$, at that, $C_{11},C_{12},\dots,C_{18}$ are regarded as parameters. The Cramer's rule implies that such system has the unique solution:

$$C_{2k}=\frac{\Delta_k}{\Delta(s)}, \quad k=1,2,\dots,8, \tag{20}$$

and the determinant $\Delta(s)$ of the system is non-zero:

$$\Delta(s)=\left.\begin{vmatrix} y_{21}(x,s) & y_{22}(x,s) & \dots & y_{27}(x,s) & y_{28}(x,s)\\ \dfrac{y'_{21}(x,s)}{bs} & \dfrac{y'_{22}(x,s)}{bs} & \dots & \dfrac{y'_{27}(x,s)}{bs} & \dfrac{y'_{28}(x,s)}{bs}\\ \dots & \dots & \dots & \dots & \dots\\ \dfrac{y^{(7)}_{21}(x,s)}{(bs)^7} & \dfrac{y^{(7)}_{22}(x,s)}{(bs)^7} & \dots & \dfrac{y^{(7)}_{27}(x,s)}{(bs)^7} & \dfrac{y^{(7)}_{28}(x,s)}{(bs)^7} \end{vmatrix}\right|_{x=x_1+0} \neq 0. \tag{21}$$

Indeed, the determinant $\Delta(s)$ in (21) is the Wronskin of eight linearly independent solutions $\{y_{2k}(x,s)\}_{k=1}^8$ of differential equation (2) and this is why it is non-zero at each point of the semi-interval $(x_1;\pi]$. The same fact can be obtained by initial conditions (18) and formulae (14), (15):

$$\begin{aligned} \Delta(s)=&\begin{vmatrix} e^{bw_1sx_1} & e^{bw_2sx_1} & \dots & e^{bw_7sx_1} & e^{bw_8sx_1}\\ w_1e^{bw_1sx_1} & w_2e^{bw_2sx_1} & \dots & w_7e^{bw_7sx_1} & w_8e^{bw_8sx_1}\\ w_1^2e^{bw_1sx_1} & w_2^2e^{bw_2sx_1} & \dots & w_7^2e^{bw_7sx_1} & w_8^2e^{bw_8sx_1}\\ \dots & \dots & \dots & \dots & \dots\\ w_1^7e^{bw_1sx_1} & w_2^7e^{bw_2sx_1} & \dots & w_7^7e^{bw_7sx_1} & w_8^7e^{bw_8sx_1} \end{vmatrix}\\ =&e^{b(w_1+w_2+\dots+w_7+w_8)sx_1}\Delta_0 \overset{(7)}{=} e^0\Delta_0=\Delta_0\neq 0, \end{aligned} \tag{22}$$

where

$$\Delta_0=\begin{vmatrix} 1 & 1 & \dots & 1 & 1\\ w_1 & w_2 & \dots & w_7 & w_8\\ w_1^2 & w_2^2 & \dots & w_7^2 & w_8^2\\ \dots & \dots & \dots & \dots & \dots\\ w_1^7 & w_2^7 & \dots & w_7^7 & w_8^7 \end{vmatrix} = \prod_{k>n;k,n=1,2,\dots,8}(w_k-w_n)\neq 0 \tag{23}$$

and Δ_0 is the Vandermonde determinant of the numbers $w_1,w_2,\dots,w_8$.

The determinants Δ_k $(k=1,2,\dots,8)$ in formula (20) are obtained from the determinant $\Delta(s)$ in (21) by replacing the kth column by the column

$$\left.\left(\sum_{k=1}^{8} C_{1k}y_{1k}(x,s); \sum_{k=1}^{8} C_{1k}\frac{y'_{1k}(x,s)}{as}; \dots; \sum_{k=1}^{8} C_{1k}\frac{y^{(7)}_{1k}(x,s)}{(as)^7}\right)^*\right|_{x=x_1-0}.$$

For instance, Δ_1 is of the form:

$$\Delta_1 = \left| \begin{matrix} [C_{11}y_{11} + C_{12}y_{12} + \dots + C_{18}y_{18}]_{x=x_1-0} & y_{22} & y_{23} & \dots & y_{28} \\ \left[C_{11}\dfrac{y'_{11}}{as} + C_{12}\dfrac{y'_{12}}{as} + \dots + C_{18}\dfrac{y'_{18}}{as}\right]_{x=x_1-0} & \dfrac{y'_{22}}{bs} & \dfrac{y'_{23}}{bs} & \dots & \dfrac{y'_{28}}{bs} \\ \dots & \dots & \dots & \dots & \dots \\ \left[C_{11}\dfrac{y^{(7)}_{11}}{(as)^7} + C_{12}\dfrac{y^{(7)}_{12}}{(as)^7} + \dots + C_{18}\dfrac{y^{(7)}_{18}}{(as)^7}\right]_{x=x_1-0} & \dfrac{y^{(7)}_{22}}{(bs)^7} & \dfrac{y^{(7)}_{23}}{(bs)^7} & \dots & \dfrac{y^{(7)}_{28}}{(bs)^7} \end{matrix} \right|_{x=x_1+0}. \tag{24}$$

Writing the Laplace expansion of the determinants Δ_k in (21)–(24) into the sum of the determinants along kth column ($k = 1, 2, \dots, 8$), we obtain:

$$\Delta_k = \sum_{n=1}^{8} C_{1n}\Delta_{kn}, \ k = 1, 2, \dots, 8, \tag{25}$$

$$\Delta_{1n} \overset{(24)}{=} \left| \begin{matrix} y_{1n} & y_{22} & y_{23} & \dots & y_{28} \\ \dfrac{y'_{1n}}{as} & \dfrac{y'_{22}}{bs} & \dfrac{y'_{23}}{bs} & \dots & \dfrac{y'_{28}}{bs} \\ \dots & \dots & \dots & \dots & \dots \\ \dfrac{y^{(7)}_{1n}}{(as)^7} & \dfrac{y^{(7)}_{22}}{(bs)^7} & \dfrac{y^{(7)}_{23}}{(bs)^7} & \dots & \dfrac{y^{(7)}_{28}}{(bs)^7} \end{matrix} \right|_{x=x_1\pm 0},$$

$$\Delta_{2n} = \left| \begin{matrix} y_{21} & y_{1n} & y_{23} & \dots & y_{28} \\ \dfrac{y'_{21}}{bs} & \dfrac{y'_{1n}}{as} & \dfrac{y'_{23}}{bs} & \dots & \dfrac{y'_{28}}{bs} \\ \dots & \dots & \dots & \dots & \dots \\ \dfrac{y^{(7)}_{21}}{(bs)^7} & \dfrac{y^{(7)}_{1n}}{(as)^7} & \dfrac{y^{(7)}_{23}}{(bs)^7} & \dots & \dfrac{y^{(7)}_{28}}{(bs)^7} \end{matrix} \right|_{x=x_1\pm 0}, \dots, \tag{26}$$

$$\Delta_{8n} = \left| \begin{matrix} y_{21} & y_{22} & \dots & y_{27} & y_{1n} \\ \dfrac{y'_{21}}{bs} & \dfrac{y'_{22}}{bs} & \dots & \dfrac{y'_{27}}{bs} & \dfrac{y'_{1n}}{as} \\ \dots & \dots & \dots & \dots & \dots \\ \dfrac{y^{(7)}_{21}}{(bs)^7} & \dfrac{y^{(7)}_{22}}{(bs)^7} & \dots & \dfrac{y^{(7)}_{27}}{(bs)^7} & \dfrac{y^{(7)}_{1n}}{(as)^7} \end{matrix} \right|_{x=x_1\pm 0}.$$

For the determinant Δ_0 in (23) we can calculate the matrix of algebraic minors for the entries of this determinant:

$$(\Delta_{0mk}) = \left| \begin{matrix} \Delta_{011} & \Delta_{012} & \dots & \Delta_{017} & \Delta_{018} \\ \Delta_{021} & \Delta_{022} & \dots & \Delta_{027} & \Delta_{028} \\ \dots & \dots & \dots & \dots & \dots \\ \Delta_{071} & \Delta_{072} & \dots & \Delta_{077} & \Delta_{078} \\ \Delta_{081} & \Delta_{082} & \dots & \Delta_{087} & \Delta_{088} \end{matrix} \right|$$
$$= \frac{\Delta_0}{8} \left| \begin{matrix} 1 & -1 & 1 & -1 & \dots & 1 & -1 \\ -w_1^{-1} & w_2^{-1} & -w_3^{-1} & w_4^{-1} & \dots & -w_7^{-1} & w_8^{-1} \\ w_1^{-2} & -w_2^{-2} & w_3^{-2} & -w_4^{-2} & \dots & w_7^{-2} & -w_8^{-2} \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ w_1^{-6} & -w_2^{-6} & w_3^{-6} & -w_4^{-6} & \dots & w_7^{-6} & -w_8^{-6} \\ -w_1^{-7} & w_2^{-7} & -w_3^{-7} & w_4^{-7} & \dots & -w_7^{-7} & w_8^{-7} \end{matrix} \right|. \tag{27}$$

The validity of formula (27) can be checked via the Laplace expansion of the determinant Δ_0 in (23) along the rows and columns.

Employing formulae (9), (10), (14), (15) and (18), we have:

$$\Delta_{1n}=\begin{vmatrix} e^{aw_nsx_1}-\dfrac{A^0_{7n}(x_1,s)}{8a^7s^7}+\underline{O}\left(\dfrac{1}{s^{14}}\right) & e^{bw_2sx_1} & \dots & e^{bw_7sx_1} & e^{bw_8sx_1}\\ w_ne^{aw_nsx_1}-\dfrac{A^1_{7n}(x_1,s)}{8a^7s^7}+\underline{O}\left(\dfrac{1}{s^{14}}\right) & w_2e^{bw_2sx_1} & \dots & w_7e^{bw_7sx_1} & w_8e^{bw_8sx_1}\\ w_n^2e^{aw_nsx_1}-\dfrac{A^2_{7n}(x_1,s)}{8a^7s^7}+\underline{O}\left(\dfrac{1}{s^{14}}\right) & w_2^2e^{bw_2sx_1} & \dots & w_7^2e^{bw_7sx_1} & w_8^2e^{bw_8sx_1}\\ \dots\dots & & & & \\ w_n^7e^{aw_nsx_1}-\dfrac{A^7_{7n}(x_1,s)}{8a^7s^7}+\underline{O}\left(\dfrac{1}{s^{14}}\right) & w_2^7e^{bw_2sx_1} & \dots & w_7^7e^{bw_7sx_1} & w_8^7e^{bw_8sx_1} \end{vmatrix} \tag{28}$$

$$=e^{aw_nsx_1}e^{bw_2sx_1}e^{bw_3sx_1}(\dots)e^{bw_7sx_1}e^{bw_8sx_1}\phi_n-\frac{e^{-bw_1sx_1}}{8a^7s^7}\phi_{n7}+\underline{O}\left(\frac{1}{s^{14}}\right),$$

$$\phi_n=\begin{vmatrix} 1 & 1 & \dots & 1 & 1\\ w_n & w_2 & \dots & w_7 & w_8\\ w_n^2 & w_2^2 & \dots & w_7^2 & w_8^2\\ \dots & & & & \\ w_n^7 & w_2^7 & \dots & w_7^7 & w_8^7 \end{vmatrix},\quad \phi_{n7}=\begin{vmatrix} A^0_{7n}(x_1,s) & 1 & \dots & 1 & 1\\ A^1_{7n}(x_1,s) & w_2 & \dots & w_7 & w_8\\ \dots & & & & \\ A^7_{7n}(x_1,s) & w_2^7 & \dots & w_7^7 & w_8^7 \end{vmatrix},\quad n=1,2,\dots,8. \tag{29}$$

Writing formulae (7), (23), (27), by the properties of the determinants and (28), (29) we find:

$$\Delta_{1n}=\psi_{1n}(x_1,s)-\frac{\psi_{1n7}(x_1,s)}{8a^7s^7}e^{-bw_1sx_1}+\underline{O}\left(\frac{1}{s^{14}}\right),\quad n=1,2,\dots,8, \tag{30}$$

$$\psi_{1n}(x_1,s)=0\quad (n=2,3,\dots,8);\quad \psi_{11}(x_1,s)=\Delta_0e^{(aw_1-bw_1)sx_1}, \tag{31}$$

$$\psi_{1n7}(x_1,s)=\frac{\Delta_0}{8}\sum_{k=1}^{8}w_1^{1-k}A^{k-1}_{7n}(x_1,s),\quad n=1,2,\dots,8. \tag{32}$$

In the same way, by (26) we get the following formulae for the determinants $\Delta_{2n},\Delta_{3n},\dots,\Delta_{8n}$:

$$\Delta_{mn}=\psi_{mn}(x_1,s)-\frac{\psi_{mn7}(x_1,s)}{8a^7s^7}e^{-bw_msx_1}+\underline{O}\left(\frac{1}{s^{14}}\right),\quad m,n=1,2,\dots,8, \tag{33}$$

$$\psi_{mn}(x_1,s)=0\quad \text{as}\quad m\neq n;\quad \psi_{mm}(x_1,s)=\Delta_0e^{(aw_m-bw_m)sx_1},\quad m=1,2,\dots,8, \tag{34}$$

$$\psi_{mn7}(x_1,s)=\frac{\Delta_0}{8}\sum_{k=1}^{8}w_m^{1-k}A^{k-1}_{7n}(x_1,s),\quad m,n=1,2,\dots,8. \tag{35}$$

Applying formulae (11), (12), in (35) we get:

$$\begin{aligned}\sum_{k=1}^{8}w_m^{1-k}A^{k-1}_{7n}(x_1,s)&=\sum_{k=1}^{8}w_m^{1-k}\left(\sum_{p=1}^{8}w_pw_p^{k-1}e^{aw_psx_1}\left(\int_0^{x_1}\dots\right)_{anp}\right)\\ &=\sum_{p=1}^{8}w_pe^{aw_psx_1}\left(\int_0^{x_1}\dots\right)_{anp}\left(\sum_{k=1}^{8}\left(\frac{w_p}{w_m}\right)^{k-1}\right)\\ &=w_me^{aw_msx_1}\left(\int_0^{x_1}\dots\right)_{anm}\cdot 8,\quad m,n=1,2,\dots,8.\end{aligned} \tag{36}$$

Employing formulae (30)–(36), we write the matrix of elements $(\frac{\Delta_{mn}}{\Delta_0})$ $(m,n=1,2,\dots,8)$:

$$\left(\frac{\Delta_{mn}}{\Delta_0}\right)=$$
$$\begin{pmatrix} G_1\left[1-\frac{w_1T_{11}}{8a^7s^7}+\dots\right] & G_1\left[0-\frac{w_1T_{21}}{8a^7s^7}+\dots\right] & \dots & G_1\left[0-\frac{w_1T_{71}}{8a^7s^7}+\dots\right] & G_1\left[0-\frac{w_1T_{81}}{8a^7s^7}+\dots\right] \\ G_2\left[0-\frac{w_2T_{12}}{8a^7s^7}+\dots\right] & G_2\left[1-\frac{w_2T_{22}}{8a^7s^7}+\dots\right] & \dots & G_2\left[0-\frac{w_2T_{72}}{8a^7s^7}+\dots\right] & G_2\left[0-\frac{w_2T_{82}}{8a^7s^7}+\dots\right] \\ \dots & \dots & \dots & \dots & \dots \\ G_7\left[0-\frac{w_7T_{17}}{8a^7s^7}+\dots\right] & G_7\left[0-\frac{w_7T_{27}}{8a^7s^7}+\dots\right] & \dots & G_7\left[1-\frac{w_7T_{77}}{8a^7s^7}+\dots\right] & G_7\left[0-\frac{w_7T_{87}}{8a^7s^7}+\dots\right] \\ G_8\left[0-\frac{w_8T_{18}}{8a^7s^7}+\dots\right] & G_8\left[0-\frac{w_8T_{28}}{8a^7s^7}+\dots\right] & \dots & G_8\left[0-\frac{w_8T_{78}}{8a^7s^7}+\dots\right] & G_8\left[1-\frac{w_8T_{88}}{8a^7s^7}+\dots\right] \end{pmatrix} \tag{37}$$

where we introduce the following notations:

$$G_m=e^{(aw_m-bw_m)sx_1};\ T_{mn}=\int_0^{x_1}q_1(t)e^{a(w_m-w_n)st}dt_{amn};\quad m,n=1,2,\dots,8;\ +\dots=+\underline{O}\left(\frac{1}{s^{14}}\right).$$

Formulae (37) complete the study of "matching" conditions (3).

4. Study of boundary conditions (4). By means of formulae (8)–(10) and (18), the first seven boundary conditions in (4) become

$$y_1^{(m_r)}(0)=0\quad(r=1,2,\dots,7)\Leftrightarrow\sum_{k=1}^{8}C_{1k}\frac{y_{1k}^{(m_r)}(0,s)}{(as)^{m_r}}=0\Leftrightarrow\sum_{k=1}^{8}C_{1k}w_k^{m_r}=0,\quad r=1,2,\dots,7. \tag{38}$$

By help of formulae (13), (20) and (25), the eighth boundary condition in (4) casts into the form:

$$\begin{gathered}\frac{y_2^{(n_1)}(\pi,s)}{(bs)^{n_1}}=0\Leftrightarrow\sum_{k=1}^{8}C_{2k}\frac{y_{2k}^{(n_1)}(\pi,s)}{(bs)^{n_1}}=0\Leftrightarrow\sum_{k=1}^{8}\frac{\Delta_k}{\Delta(s)}\frac{y_{2k}^{(n_1)}(\pi,s)}{(bs)^{n_1}}=0\\ \Leftrightarrow\sum_{k=1}^{8}\left(\sum_{n=1}^{8}C_{1n}\Delta_{kn}\right)\frac{y_{2k}^{(n_1)}(\pi,s)}{(bs)^{n_1}}=0\Leftrightarrow\sum_{k=1}^{8}C_{1k}\left(\sum_{n=1}^{8}\Delta_{nk}\frac{y_{2n}^{(n_1)}(\pi,s)}{(bs)^{n_1}}\right)=0,\\ n_1=0,1,2,\dots,7.\end{gathered} \tag{39}$$

System (38), (39) is a homogeneous system of eight linear equations for eight unknowns $C_{11},C_{12},\dots,C_{18}$. By the Cramer's rule we conclude that such system has nonzero solutions $(\sum_{k=1}^{8}C_{1k}^8\neq0)$ if and only if its determinant is zero. This is why the following theorem holds true.

Theorem 3. *Eigenvalue equation for differential operator (1)–(4) subject to condition* (5) *of the summability of the potential is represented as*

$$h(s)=\begin{vmatrix} w_1^{m_1} & w_2^{m_1} & \dots & w_7^{m_1} & w_8^{m_1}\\ w_1^{m_2} & w_2^{m_2} & \dots & w_7^{m_2} & w_8^{m_2}\\ \dots & \dots & \dots & \dots & \dots\\ w_1^{m_7} & w_2^{m_7} & \dots & w_7^{m_7} & w_8^{m_7}\\ b_{81} & b_{82} & \dots & b_{87} & b_{88}\end{vmatrix}=0, \tag{40}$$

$$b_{8p}=\sum_{k=1}^{8}\Delta_{pk}\frac{y_{2k}^{(n_1)}(\pi,s)}{(bs)^{n_1}},\quad p=1,2,\dots,8. \tag{41}$$

By the Laplace expansion of the determinant $h(s)$ in (40) along the last row we have

$$h(s)=b_{81}R_1-b_{82}R_2+b_{83}R_3-b_{84}R_4+\dots+b_{87}R_7-b_{88}R_8=0, \tag{42}$$

where R_n $(n = 1, 2, \dots, 8)$ are algebraic minors for the elements in the last row in $h(s)$. The determinants R_n can be easily calculated by formulae (6):

$$R_8 = \begin{vmatrix} w_1^{m_1} & w_2^{m_1} & \dots & w_6^{m_1} & w_7^{m_1} \\ w_1^{m_2} & w_2^{m_2} & \dots & w_6^{m_2} & w_7^{m_2} \\ \dots & \dots & \dots & \dots & \dots \\ w_1^{m_7} & w_2^{m_7} & \dots & w_6^{m_7} & w_7^{m_7} \end{vmatrix} = \begin{vmatrix} 1^{m_1} & z^{m_1} & \dots & z^{5m_1} & z^{6m_1} \\ 1^{m_2} & z^{m_2} & \dots & z^{5m_2} & z^{6m_2} \\ \dots & \dots & \dots & \dots & \dots \\ 1^{m_7} & z^{m_7} & \dots & z^{5m_7} & z^{6m_7} \end{vmatrix}$$
$$= \det \text{Wandermond's}(z^{m_1}, z^{m_2}, \dots, z^{m_7}) = \prod_{r>k;r,k=1,2,\dots,7} (z^{m_r} - z^{m_k}) = W_7 \neq 0, \tag{43}$$

$$R_1 = \begin{vmatrix} w_2^{m_1} & w_3^{m_1} & \dots & w_7^{m_1} & w_8^{m_1} \\ w_2^{m_2} & w_3^{m_2} & \dots & w_7^{m_2} & w_8^{m_2} \\ \dots & \dots & \dots & \dots & \dots \\ w_2^{m_7} & w_3^{m_7} & \dots & w_7^{m_7} & w_8^{m_7} \end{vmatrix} = \begin{vmatrix} z^{m_1} & z^{2m_1} & \dots & z^{6m_1} & z^{7m_1} \\ z^{m_2} & z^{2m_2} & \dots & z^{6m_2} & z^{7m_2} \\ \dots & \dots & \dots & \dots & \dots \\ z^{m_7} & z^{2m_7} & \dots & z^{6m_7} & z^{7m_7} \end{vmatrix}$$
$$= z^{m_1} z^{m_2} (\dots) z^{m_7} R_8 = z^{M_7} W_7, \quad M_7 = \sum_{k=1}^{7} m_k, \tag{44}$$

The numbers m_k $(k = 1, 2, \dots, 7)$ are determined by boundary conditions (4).

In the same way we obtain the following formulae:

$$R_2 = z^{2M_7} W_7, \quad R_3 = z^{3M_7} W_7, \dots; \quad R_n = z^{nM_7} W_7, \quad n = 1, 2, \dots, 8. \tag{45}$$

We substitute formulae (43)–(45) into equation (42) and divide by $z^{M_7} W_7 \neq 0$ to get the equation

$$h(s) = \sum_{k=1}^{8} (-1)^{k-1} b_{8k} z^{(k-1)M_7} = 0, \tag{46}$$

where the quantities b_{8k} $(k = 1, 2, \dots, 8)$ are determined in (41), Δ_{pk} are defined in (37) and $y_{2k}^{(n_1)}(\pi, s)$ are found in view of (14), (15).

The study of the asymptotics of the roots to equation (46) is closely related to studying the indicator diagram for this equation, see [23, Ch. 12]. The indicator diagram is the convex hull of the exponents in the exponentials involved in this equation. Applying formulae (41), (14), (15), (37), we rewrite equation (46) in a more detailed form:

$$h(s) = \sum_{k=1}^{8} \Delta_{k1} \frac{y_{2k}^{(n_1)}(\pi, s)}{(bs)^{n_1}} - z^{M_7} \sum_{k=1}^{8} \Delta_{k2} \frac{y_{2k}^{(n_1)}(\pi, s)}{(bs)^{n_1}} + \dots$$
$$+ z^{6M_7} \sum_{k=1}^{8} \Delta_{k7} \frac{y_{2k}^{(n_1)}(\pi, s)}{(bs)^{n_1}} - z^{7M_7} \sum_{k=1}^{8} \Delta_{k8} \frac{y_{2k}^{(n_1)}(\pi, s)}{(bs)^{n_1}} = 0,$$

which implies:

$$h(s) = \frac{y_{21}^{(n_1)}(\pi, s)}{(bs)^{n_1}} \sum_{k=1}^{8} (-1)^{k-1} z^{(k-1)M_7} \Delta_{1k} + \frac{y_{22}^{(n_1)}(\pi, s)}{(bs)^{n_1}} \sum_{k=1}^{8} (-1)^{k-1} z^{(k-1)M_7} \Delta_{2k} + \dots$$
$$+ \frac{y_{27}^{(n_1)}(\pi, s)}{(bs)^{n_1}} \sum_{k=1}^{8} (-1)^{k-1} z^{(k-1)M_7} \Delta_{7k} + \frac{y_{28}^{(n_1)}(\pi, s)}{(bs)^{n_1}} \sum_{k=1}^{8} (-1)^{k-1} z^{(k-1)M_7} \Delta_{8k} = 0. \tag{47}$$

This yields that the indicator diagram is as it is shown in Figure (48).

The indicator diagram shown in Figure (48) is a regular octagon. The roots of equation (47) can be located only in eight sectors of small angles. Their bisectrix are median perpendiculars to the sides of this octagon.

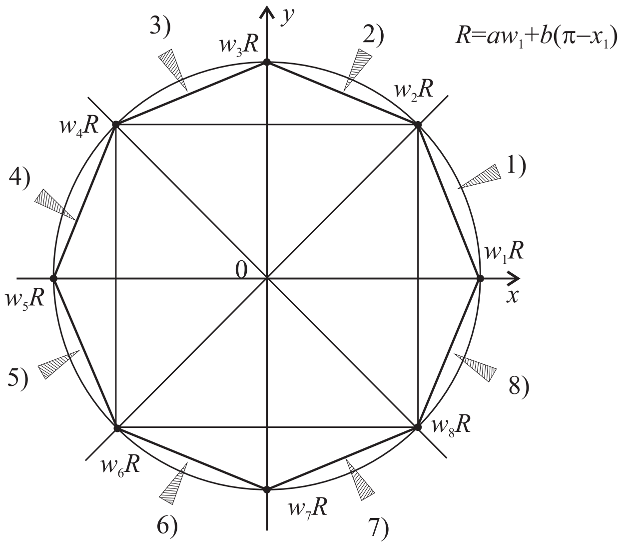

FIGURE 48.

5. Asymptotics of eigenvalues in Sector 1) of indicator diagram (48). Applying formulae (14), (15) and (37), we rewrite equation (47) as

$$\begin{aligned} h(s) =& \left[w_1^{n_1} e^{bw_1 s\pi} - \frac{B_{71}^{n_1}(\pi, s)}{8b^7 s^7} + \underline{O}\left(\frac{1}{s^{14}}\right)\right] e^{(aw_1 - bw_1)sx_1} D_1(x_1, s) \\ &+ \left[w_2^{n_1} e^{bw_2 s\pi} - \frac{B_{72}^{n_1}(\pi, s)}{8b^7 s^7} + \underline{O}\left(\frac{1}{s^{14}}\right)\right] e^{(aw_2 - bw_2)sx_1} D_2(x_1, s) + \dots \\ &+ \left[w_8^{n_1} e^{bw_8 s\pi} - \frac{B_{78}^{n_1}(\pi, s)}{8b^7 s^7} + \underline{O}\left(\frac{1}{s^{14}}\right)\right] e^{(aw_8 - bw_8)sx_1} D_8(x_1, s) = 0, \end{aligned} \tag{49}$$

$$\begin{aligned} D_m(x_1, s) =& \left[D_{m1} - \frac{w_m}{8a^7 s^7}\left(\int_0^{x_1} \dots\right)_{a1m} + \underline{O}\left(\frac{1}{s^{14}}\right)\right] \\ &- z^{M_7}\left[D_{m2} - \frac{w_m}{8a^7 s^7}\left(\int_0^{x_1} \dots\right)_{a2m} + \underline{O}\left(\frac{1}{s^{14}}\right)\right] \\ &+ z^{2m_7}\left[D_{m3} - \frac{w_m}{8a^7 s^7}\left(\int_0^{x_1} \dots\right)_{a3m} + \underline{O}\left(\frac{1}{s^{14}}\right)\right] - \dots \\ &- z^{7M_7}\left[D_{m8} - \frac{w_m}{8a^7 s^7}\left(\int_0^{x_1} \dots\right)_{a8m} + \underline{O}\left(\frac{1}{s^{14}}\right)\right], \\ & D_{mm} = 1, \quad D_{mn} = 0 \quad (m \neq n); \quad m, n = 1, 2, \dots, 8. \end{aligned} \tag{50}$$

The general theory, see [23, Ch. 12], of finding roots to quasi-polynomials of form (49), (50) states that in Sector 1) we should keep only the exponentials with exponents in the segment $[w_1 R; w_2 R]$ of indicator diagram (48). This fact implies the next theorem.

Theorem 4. *In Sector 1) of indicator diagram (48), eigenvalue equation for differential operator (1)–(4) subject to condition (5) of summability of the potential is of the form:*

$$\begin{aligned} f_1(s) =& [w_1^{n_1} e^{Rw_1 s} - z^{M_7} w_2^{n_1} e^{Rw_2 s}] - \frac{1}{8s^7}\left[\frac{w_1}{a^7} w_1^{n_1}\left(\int_0^{x_1} \dots\right)_{a11} e^{Rw_1 s}\right. \\ &- \frac{w_1}{a^7} w_1^{n_1} e^{Rw_1 s} z^{M_7}\left(\int_0^{x_1} \dots\right)_{a21} \\ &+ \frac{B_{71}^{n_1}(\pi, s)}{b^7} e^{(aw_1 - bw_1)sx_1} - \frac{w_2}{a^7} w_2^{n_1} z^{M_7} e^{Rw_2 s}\left(\int_0^{x_1} \dots\right)_{a22} \\ &\left.+ \frac{w_2}{a^7} w_2^{n_1} e^{Rw_2 s}\left(\int_0^{x_1} \dots\right)_{a12} - \frac{B_{72}^{n_1}(\pi, s)}{b^7} z^{M_7} e^{(aw_2 - bw_2)sx_1}\right] + \underline{O}\left(\frac{1}{s^{14}}\right) = 0. \end{aligned} \tag{51}$$

The exponentials of form $\exp(Rw_m s)$ $(m = 3, 4, \dots, 8)$ and the integrals of form $(\int_0^{x_1} \dots)_{a1m}$, $(\int_0^{x_1} \dots)_{a2m}, \dots$, $m = 3, 4, \dots, 8$, involved in equation (49), (50) but not involved in equation (51), are infinitesimal in Sector 1).

We divide equation (51) by $e^{Rw_2 s} w_1^{n_1} \neq 0$, observe that due to (6) we have $\frac{w_2^{n_1}}{w_1^{n_1}} = z^{n_1}$, and we rewrite equation (51) as

$$
\begin{aligned}
f_1(s) =& [e^{R(w_1-w_2)s} - z^{M_7} z^{n_1}] \\
& - \frac{1}{8s^7}\left[\frac{g_1(x_1,s)}{a^7} + \frac{g_2(x_1,s)}{a^7} + \frac{1}{b^7} e^{-Rw_2 s} w_1^{-n_1} g_3(x_1,\pi,s)\right] + \underline{O}\left(\frac{1}{s^{14}}\right) = 0; \\
g_1(x_1,s) &= w_1 \left(\int_0^{x_1} \dots\right)_{a11} e^{R(w_1-w_2)s} - w_2 z^{M_7} z^{n_1} \left(\int_0^{x_1} \dots\right)_{a22}; \\
g_2(x_1,s) &= w_2 z^{n_1} \left(\int_0^{x_1} \dots\right)_{a12} - w_1 z^{M_7} e^{R(w_1-w_2)s} \left(\int_0^{x_1} \dots\right)_{a21}; \\
g_3(x_1,\pi,s) &= B_{71}^{n_1}(\pi,s) e^{(aw_1-bw_1)sx_1} - z^{M_7} B_{72}^{n_1}(\pi,s) e^{(aw_2-bw_2)sx_1}.
\end{aligned}
\tag{52}
$$

The main approximation of equation (52) is

$$
\begin{aligned}
& e^{R(w_1-w_2)s} = z^{M_7} z^{n_1} = e^{2\pi i k} e^{\frac{2\pi i}{8} M_7} e^{\frac{2\pi i}{8} n_1} \Leftrightarrow s_{k,1,\text{bsc}} = \frac{2\pi i \tilde{k}}{R(w_1-w_2)}, \\
& R = ax_1 + b(\pi - x_1), \quad \tilde{k} = k + \frac{M_7}{8} + \frac{n_1}{8}, \quad k \in \mathbb{Z}.
\end{aligned}
\tag{53}
$$

This is why the next theorem holds.

Theorem 5. *In Sector 1) of indicator diagram (48), the asymptotics of eigenvalues of differential operator (1)–(4) subject to condition (5) of the summability of the potential are*

$$
s_{k,1} = \frac{2\pi i}{R(w_1-w_2)}\left[\tilde{k} + \frac{d_{7k,1}}{\tilde{k}^7} + \underline{O}\left(\frac{1}{\tilde{k}^{14}}\right)\right], \quad \tilde{k} = k + \frac{M_7}{8} + \frac{n_1}{8}, \quad k \in \mathbb{Z}. \tag{54}
$$

In order to prove Theorem 5, it is sufficient to show that the coefficients $d_{7k,1}$ in formula (54) are uniquely determined and to provide explicit formulae for them.

Applying Maclaurin formula, we obtain:

$$
\begin{aligned}
e^{R(w_1-w_2)s}\Big|_{s_{k,1}} &= \exp\left[R(w_1-w_2)\frac{2\pi i}{R(w_1-w_2)}\left(\tilde{k} + \frac{d_{7k,1}}{\tilde{k}^7} + \underline{O}\left(\frac{1}{\tilde{k}^{14}}\right)\right)\right] \\
&= z^{M_7} z^{n_1}\left[1 + \frac{2\pi i d_{7k,1}}{\tilde{k}^7} + \underline{O}\left(\frac{1}{\tilde{k}^{14}}\right)\right]; \\
\frac{1}{s_{k,1}^7} &= \frac{R^7(w_1-w_2)^7}{2^7\pi^7 i^7}\frac{1}{\tilde{k}^7}\left(1 + \underline{O}\left(\frac{1}{\tilde{k}^8}\right)\right).
\end{aligned}
\tag{55}
$$

Substituting formulae (53)–(55) into equation (52), we get

$$
\begin{aligned}
& \left[z^{M_7} z^{n_1} + z^{M_7} z^{n_1} 2\pi i \frac{d_{7k,1}}{\tilde{k}^7} + \underline{O}\left(\frac{1}{\tilde{k}^{14}}\right) - z^{M_7} z^{n_1}\right] \\
& - \frac{R^7(w_1-w_2)^7}{8 \cdot 2^7 \pi^7 i^3}\frac{1}{\tilde{k}^7}\left(1 + \underline{O}\left(\frac{1}{\tilde{k}^8}\right)\right)\left[\frac{g_1(x_1,s)}{a^7} + \frac{g_2(x_1,s)}{a^7} + \frac{1}{b^7} e^{-Rw_2 s} w_1^{-n_1} g_3(x_1,\pi,s)\right]\Bigg|_{sk,1} \\
& + \underline{O}\left(\frac{1}{\tilde{k}^{14}}\right) = 0
\end{aligned}
$$

and we hence find that

$$
d_{7k,1} = \frac{R^7(w_1-w_2)^7}{8 \cdot 2^7 \pi^7 i^3 2\pi i} z^{-M_7} z^{-n_1}\left[\frac{g_1(x_1,s)}{a^7} + \frac{g_2(x_1,s)}{a^7} + \frac{e^{-Rw_2 s}}{b^7} w_1^{-n_1} g_3(x_1,\pi,s)\right]\Bigg|_{s_{k,1,\text{bsc}}}, \tag{56}
$$

where $s_{k,1,\text{bsc}}$ was defined in (53).

It follows from formulae (11) that

$$\left(\int_0^{x_1}\dots\right)_{a11}=\left(\int_0^{x_1}\dots\right)_{a22}=\dots=\left(\int_0^{x_1}\dots\right)_{amm}=\int_0^{x_1}q_1(t)dt_{a11}\ (m=1,2,\dots,8).$$

Hence,

$$\begin{aligned} g_1(x_1,s)\Big|_{s_{k,1,\mathrm{bsc}}} &\overset{(52),(55)}{=} w_1\left(\int_0^{x_1}\dots\right)_{a11}z^{M_7}z^{n_1}-w_2z^{M_7}z^{n_1}\left(\int_0^{x_1}\dots\right)_{a22}\\ &= (w_1-w_2)z^{M_7}z^{n_1}\left(\int_0^{x_1}\dots\right)_{a11}; \end{aligned} \tag{57}$$

$$\begin{aligned} g_2(x_1,s)\Big|_{s_{k,1,\mathrm{bsc}}} &=\Big[w_2z^{n_1}z^{M_7}z^{-M_7}\left(\int_0^{x_1}\dots\right)_{a12}-w_1z^{M_7}z^{M_7}z^{n_1}\left(\int_0^{x_1}\dots\right)_{a21}\Big]\Big|_{s_{k,1,\mathrm{bsc}}}\\ &=z^{M_7}z^{n_1}e^{\frac{\pi i}{8}}\Big[e^{\frac{\pi i}{8}}e^{-\frac{2\pi i}{8}M_7}\int_0^{x_1}q(t)\exp\left(a(w_1-w_2)t\frac{2\pi i\tilde{k}}{R(w_1-w_2)}\right)dt_{a12}\\ &\quad -e^{-\frac{\pi i}{8}}e^{\frac{2\pi i}{8}M_7}\int_0^{x_1}q_1(t)\exp\left(-\frac{a}{R}2\pi i\tilde{k}t\right)dt_{a21}\Big]\\ &=z^{M_7}z^{n_1}e^{\frac{\pi i}{8}}2i\int_0^{x_1}q_1(t)\sin\left[2\pi\frac{a}{R}\tilde{k}t+\frac{\pi}{8}-\frac{2\pi}{8}M_7\right]dt_{q1}; \end{aligned} \tag{58}$$

$$\begin{aligned} g_3(x_1,\pi,s)\Big|_{s_{k,1,\mathrm{bsc}}} &=\Big[w_1\left(\int_{x_1}^{\pi}\dots\right)_{b11}e^{R(w_1-w_2)s}-w_2z^{n_1}z^{M_7}\left(\int_{x_1}^{\pi}\dots\right)_{b22}\Big]\Big|_{s_{k,1,\mathrm{bsc}}}\\ &\quad+\Big[w_2\left(\int_{x_1}^{\pi}\dots\right)_{b12}z^{n_1}e^{R(w_1-w_2)s}e^{-b(w_1-w_2)s\pi}\\ &\qquad -z^{M_7}w_1e^{b(w_1-w_2)s\pi}\left(\int_{x_1}^{\pi}\dots\right)_{b21}\Big]\Big|_{s_{k,1,\mathrm{bsc}}}\\ &=(w_1-w_2)z^{M_7}z^{n_1}\left(\int_{x_1}^{\pi}\dots\right)_{b11}\\ &\quad+2ie^{\frac{\pi i}{8}}z^{M_7}z^{n_1}\int_{x_1}^{\pi}q_2(t)\sin\left[\frac{2\pi b\tilde{k}}{R}t+\frac{\pi}{8}+\frac{2\pi}{8}n_1-\frac{b\pi}{R}2\pi\tilde{k}\right]dt_{q2}. \end{aligned} \tag{59}$$

Substituting formulae (57)–(59) into (56), we obtain:

$$\begin{aligned} d_{7k,1}=&\frac{R^7(w_1-w_2)^8}{8\cdot 2^8\pi^8}\Big[\frac{1}{a^7}\left(\int_0^{x_1}\dots\right)_{a11}+\frac{1}{a^7}\frac{2ie^{\frac{\pi i}{8}}}{w_1-w_2}\left(\int_0^{x_1}\dots\right)_{q1}\\ &+\frac{1}{b^7}\left(\int_{x_1}^{\pi}\dots\right)_{b11}+\frac{1}{b^7}\frac{2ie^{\frac{\pi i}{8}}}{w_1-w_2}]\left(\int_{x_1}^{\pi}\dots\right)_{q2}\Big],\ k\in\mathbb{Z}. \end{aligned} \tag{60}$$

By formulae (6) we find:

$$\frac{2ie^{\frac{\pi i}{8}}}{w_1-w_2}=\frac{2ie^{\frac{\pi i}{8}}}{1-e^{\frac{\pi i}{8}}}=\frac{2ie^{\frac{\pi i}{8}}}{e^{\frac{\pi i}{8}}(e^{-\frac{\pi i}{8}}-e^{\frac{\pi i}{8}})}=-\frac{1}{\sin(\frac{\pi}{8})}=-\frac{2}{\sqrt{2-\sqrt{2}}}.$$

Substituting this formula into (60), we get:

$$\begin{aligned} d_{7k,1} =& \frac{R^7(w_1-w_2)^8}{8\cdot 2^8\pi^8}\left\{\left[\frac{1}{a^7}\int_0^{x_1} q_1(t)dt_{a11} + \frac{1}{b^7}\int_{x_1}^{\pi} q_2(t)dt_{b11}\right]\right.\\ &-\left[\frac{1}{a^7}\frac{1}{\sin(\frac{\pi}{8})}\int_0^{x_1} q_1(t)\sin\left[\frac{2\pi a}{R}\tilde{k}t+\frac{\pi}{8}-\frac{\pi}{4}M_7\right]dt_{q1}\right.\\ &\left.\left.+\frac{1}{b^7}\frac{1}{\sin(\frac{\pi}{8})}\int_{x_1}^{\pi} q_2(t)\sin\left[\frac{2\pi b}{R}\tilde{k}t+\frac{\pi}{8}+\frac{\pi}{4}n_1-\frac{2b\pi^2}{R}\tilde{k}\right]dt_{q2}\right]\right\},\quad k\in\mathbb{Z}, \end{aligned} \tag{61}$$

$$\tilde{k} = k+\frac{M_7}{8}+\frac{n_1}{8},\quad M_7=\sum_{p=1}^{7} m_p,\quad R = ax_1+b(\pi-x_1).$$

Formula (61) shows that the coefficients $d_{7k,1}$ in (54) are determined uniquely that completes the proof of Theorem 5.

While dealing with the limits $b\to a$ or $x_1\to 0$ or $x_1\to\pi$, formula (61) becomes

$$d_{7k,1} = \frac{(w_1-w_2)^8}{8\pi 2^8}\left[\int_0^{\pi} q(t)dt_{a11} - \frac{1}{\sin(\frac{\pi}{8})}\int_0^{\pi} q(t)\sin\left(2\tilde{k}t+\frac{\pi}{8}-\frac{2\pi}{8}M_7\right)dt_{q1}\right],\ k\in\mathbb{Z}. \tag{62}$$

Formula (62) was obtained by the author earlier in work [24].

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