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DIRICHLET BOUNDARY VALUE PROBLEM IN HALF-STRIP FOR FRACTIONAL DIFFERENTIAL EQUATION WITH BESSEL OPERATOR AND RIEMANN-LIOUVILLE PARTIAL DERIVATIVE

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Abstract. In the work we study the Dirichlet boundary value problem in a half-strip for a fractional differential equations with the Bessel operator and the Riemann-Liouville partial derivatives. We formulate the unique solvability theoresm for the considered problem. We find the representations for the solutions in terms of the integral transform with the Wright function in the kernel. The proof of the existence theorem is made on the base of the mentioned integral transform and the modified Bessel function of first kind. The uniqueness of the solutions is shown in the class of the functions satisfying an analogue of Tikhonov equation. In the case, when the considered equations is the fractional order diffusion equation, we show that the obtained solutions coincides with the known solution to the Dirichlet problem for the corresponding equation. We also consider the case when the initial function is power in the spatial variable. In this case the solution to the problem is written out in terms of the Fox H-function.

Keywords: Bessel operator, Riemann–Liouville partial derivative, fractional diffusion, Wright function, integral transform with the function of Wright in the kernel, modified Bessel function of the first kind, Fox H-function, Tikhonov condition.

Mathematics Subject Classification: 35A22, 35R11, 35C15

1. Introduction

Let D^{γ}_{ay} be an integration-differentiation operator in the Riemann-Liouville sense of a fractional order γ with the origin at a point a and with the end at a point y; this operator is defined as [11], [12], [15]:

$$D^{\gamma}_{ay}\, g(y) = \frac{\operatorname{sign}(y-a)}{\Gamma(-\gamma)} \int\limits_a^y \frac{g(t)}{|y-t|^{\gamma+1}} dt, \quad \gamma<0; \quad D^{\gamma}_{ay}\, g(y) = g(y), \quad \gamma = 0;$$

$$D^{\gamma}_{ay}\, g(y) = \operatorname{sign}^n(y-a)\, \frac{d^n}{dy^n}\, D^{\gamma-n}_{ay}\, g(y), \quad n-1<\gamma\leqslant n, \quad n\in\mathbb{N}.$$

Here $\Gamma(s)$ is the Euler gamma function.

In the domain $\Omega = \{(x,y) : 0<x<\infty,\ 0<y<T\}$ we consider the equation

$$\mathbf{L}\, u(x,y) \equiv B_x\, u(x,y) - D^{\alpha}_{0y}\, u(x,y) = 0, \tag{1}$$

where $B_x = x^{-b}\frac{\partial}{\partial x}\left(x^b \frac{\partial}{\partial x}\right)$ is the Bessel operator, $b=\mathrm{const}$, $\alpha = \mathrm{const}$.

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Equation of form (1), namely,

$$D_{0t}^{\frac{2}{d_w}} P(r,t) = \frac{1}{r^{d_s - 1}} \frac{\partial}{\partial r} \left(r^{d_s - 1} \frac{\partial P(r,t)}{\partial r} \right),$$

where d_w and d_s characterize the fractal dimension of a media, $P(r,t)$ is the density of the spatial particles distribution at time t, was proposed in work by R. Metzler, W.G. Glöckle, T.F. Nonnenmacher [28] for describing transfer processes in media of a fractal dimension.

As $\alpha = 1$, equation (1) becomes the equation

$$u_{xx}(x,y) + \frac{b}{x} u_x(x,y) - u_y(x,y) = 0,$$

which is called a B-parabolic equation by I.A. Kipriyanov [5]. As $b > -1$, the latter equation was studied in work [23]; as $|b| < 1$, $x > 0$, it was considered in work [17].

A lot of works were devoted to studying equation (1) as $b = 0$, $0 < \alpha < 2$, and to studying its generalizations. Let us mention some of them.

In work [4], the method of integral transforms was applied for studying the Cauchy problem for the equation

$$D_{0t}^{\alpha} u(x,t) = \lambda^2 \Delta_x u(x,t), \quad x \in \mathbb{R}^m, \quad t > 0, \tag{2}$$

where

$$\Delta_x = \sum_{j=1}^{m} \partial^2 / \partial x_j^2, \quad n - 1 < \alpha < n, \quad n \in \mathbb{N}.$$

As $0 < \alpha \leqslant 1$ and $1 < \alpha < 2$, the solutions were written in terms of the Fox H-function. The solution to the Cauchy problem for equation (2) as $\lambda = 1$, $0 < \alpha \leqslant 1$ was written in work [1] in terms of the Wright function. The Cauchy problem for equation (2) in the case, when the Riemann-Laplace operator is replaced by Caputo operator, was studied in work [3].

For equation (2) with $m = 1$, the Dirichlet boundary value problem in the first quadrant was studied in work [2].

In work [16], there was constructed a fundamental solution and studied the Cauchy problem for a multi-dimensional diffusion-wave equation with the Dzhrbashyan-Nersesyan operator.

Works [6], [7], [26] were devoted to studying the Cauchy problem for a fractional order diffusion equation with the Caputo derivative and elliptic operator with the coefficients depending on spatial variables.

The interest to equation (1) is also motivated by its applications in studying problems in physics, astronomy and other applied sciences [18], [29], [30].

2. Formulation of problem

Let $0 < \alpha \leqslant 1$. A *regular solution* of equation (1) in the domain Ω is a function $u = u(x,y)$ satisfying equation (1) in the domain Ω such that $y^{1-\alpha} u \in C(\bar{\Omega})$, u_x, u_{xx}, $D_{0y}^{\alpha} u \in C(\Omega)$, where $\bar{\Omega}$ is the closure of the domain Ω.

Problem 1. *Find a regular in the domain Ω solution to equation* (1) *satisfying the boundary conditions*

$$\lim_{y \to 0} D_{0y}^{\alpha - 1} u(x,y) = \varphi(x), \qquad 0 < x < \infty, \tag{3}$$

$$u(0,y) = 0, \qquad 0 < y < T, \tag{4}$$

where $\varphi(x)$ is a given function.

Problem 1 with $\varphi(x) \equiv 0$, $u(0,y) = \tau(y)$, $y^{1-\alpha} \tau(y) \in C[0,T]$ was studied in work [20]. The solution was written in terms of the Fox H-function [14].

3. Auxiliary statements

Here we provide some statements from the theory of integral transforms and the theory of special functions, which will be used in further exposition of the work.

In work [15], A.V. Pskhu introduced an integral transform for a function $v(y)$ defined on the positive semi-axis:

$$A^{\alpha,\mu} v(y) = y^{\mu-1} \int_0^\infty v(t)\, \phi\left(-\alpha, \mu; -t\, y^{-\alpha}\right) dt, \quad 0 < \alpha < 1, \tag{5}$$

where $\phi(\rho, \mu; z)$ is the Wright function defined by the series [24]

$$\phi(\rho, \mu; z) = \sum_{n=0}^{\infty} \frac{z^n}{n!\,\Gamma(\rho n + \mu)}, \quad \rho > -1.$$

In the case $\mu = 0$, we denote $A^{\alpha,0} v(x,y) = A^{\alpha} v(x,y)$. If the transform $A^{\alpha,\mu}$ is applied to a function of several variables and if there is a need, by the subscript we denote the variable, w.r.t. which the transformation is made. For instance, $A_y^{\alpha,\mu} v(x,y)$.

Integral (5) converges if the function $v(y)$ is integrable on each finite segment in the positive semi-axis and the estimates

$$|v(y)| < c\, y^{\lambda}, \quad y \to 0,$$

hold true, where $\lambda > -1$ if $\mu \neq 0$ and $\lambda > -2$ if $\mu = 0$, and

$$|v(y)| < c \exp\left(k\, y^{\varepsilon}\right), \quad y \to \infty,$$

where $\varepsilon < 1/(1-\alpha)$, c and k are positive constants.

We provide some properties of the transform $A^{\alpha,\mu}$ [15].

1°. Assume that $v(y)$ is continuous at the point $y = 0$ and is differentiable as $y > 0$. Then

$$D_{0y}^{\alpha} A^{\alpha,\mu} v(y) = A^{\alpha,\mu} v'(y) + \frac{y^{\mu-1}}{\Gamma(\mu)} v(0).$$

In particular, the formula

$$D_{0y}^{\alpha} A^{\alpha} v(y) = A^{\alpha} v'(y). \tag{6}$$

holds.

2°. Let $0 \leqslant \mu \leqslant \alpha$ and $\lim\limits_{y\to 0} D_{0y}^{-\mu/\alpha} v(y) = v_0 < \infty$. Then

$$\lim_{y\to 0} D_{0y}^{\alpha-1} A^{\alpha,\mu} v(y) = v_0. \tag{7}$$

3°. If $u(y) \leqslant v(y)$ and $\mu \geqslant 0$, then

$$A^{\alpha,\mu} u(y) \leqslant A^{\alpha,\mu} v(y). \tag{8}$$

Transformation (5) for a power function and the Wright function are calculated by the formulae [15]

$$A^{\alpha,\mu} y^{\delta-1} = y^{\alpha\delta+\mu-1} \frac{\Gamma(\delta)}{\Gamma(\alpha\delta+\mu)}, \quad \delta > 0, \quad \mu \neq 0; \quad \delta > -1, \quad \delta \neq 0, \quad \mu = 0, \tag{9}$$

$$A^{\alpha,\mu} y^{\delta-1} \phi(\rho, \delta; -c\, y^{\rho}) = y^{\alpha\delta+\mu-1} \phi(\alpha\rho, \alpha\delta+\mu; -c\, y^{\alpha\rho}), \quad \delta > \rho. \tag{10}$$

In work [22], the formula

$$A^{\alpha,\mu} y^{\delta-1} e^{-\frac{c^2}{4y}} = y^{\alpha\delta+\mu-1} H_{1,2}^{2,0}\left[\frac{c^2}{4 y^{\alpha}} \,\middle|\, \begin{matrix} (\alpha\delta+\mu, \alpha) \\ (0,1), (\delta,1) \end{matrix} \right] \tag{11}$$

was proved. Here c is a constant, $\delta \neq 0, \pm 1, \pm 2, \dots$, $H^{\,m,n}_{\,p,q}(z)$ is the Fox H-function [14], [25], [27].

For the H-function in (11) we provide one more asymptotic estimate as $z \to \infty$ [25], [27]

$$H^{\,2,0}_{\,1,2}\left[z \,\middle|\, \begin{matrix} (\mu + \alpha\,\delta, \alpha) \\ (0,1), (\delta, 1) \end{matrix} \right] = O\left(z^{\frac{\delta(1-\alpha)-\mu}{2-\alpha}} \exp\left[-(2-\alpha)\,\alpha^{\frac{\alpha}{2-\alpha}}\, z^{\frac{1}{2-\alpha}} \right] \right). \tag{12}$$

4. Main results

We denote

$$G(x,\xi,y) = A^{\alpha}_{y}\, g(x,\xi,y), \tag{13}$$

where

$$g(x,\xi,y) = \frac{x^{\beta}\xi^{\beta}}{2y}\, e^{-\frac{x^2+\xi^2}{4y}}\, I_{\beta}\left(\frac{x\xi}{2y}\right), \qquad \beta = \frac{1-b}{2}, \tag{14}$$

$I_\nu(z)$ is the modified Bessel function of first kind of order ν defined by the series [8], [9]

$$I_\nu(z) = \sum_{n=0}^{\infty} \frac{1}{n!\,\Gamma(\nu+n+1)} \left(\frac{z}{2}\right)^{\nu+2n}.$$

Let us estimate the function $G(x,\xi,y)$. By means of the formulae for the function $I_\nu(z)$ [8], [9]

$$\frac{d}{dz}\left[z^{\nu} I_\nu(z)\right] = z^{\nu} I_{\nu-1}(z), \tag{15}$$

$$\frac{d}{dz}\left[z^{-\nu} I_\nu(z)\right] = z^{-\nu} I_{\nu+1}(z), \tag{16}$$

$$z\, I'_\nu(z) + \nu\, I_\nu(z) = z\, I_{\nu-1}(z), \tag{17}$$

by its asymptotic behavior for small positive values z [9]

$$I_\nu(z) \approx \frac{1}{\Gamma(\nu+1)}\left(\frac{z}{2}\right)^{\nu}$$

and by (14) for $x\xi \leqslant 2y$, we obtain the estimates

$$\left|\frac{\partial^n}{\partial x^n}\, g(x,\xi,y)\right| \leqslant \text{const}\cdot x^{2\beta-n}\,\xi^{2\beta}\,y^{-\beta-1}, \quad \beta \neq \frac{1}{2},$$

$$\left|\frac{\partial^{2n}}{\partial x^{2n}}\, g(x,\xi,y)\right| \leqslant \text{const}\cdot x\,\xi\,y^{-n-3/2}, \qquad \beta = \frac{1}{2},$$

$$\left|\frac{\partial^{2n+1}}{\partial x^{2n+1}}\, g(x,\xi,y)\right| \leqslant \text{const}\cdot \xi\,y^{-n-3/2}, \qquad \beta = \frac{1}{2},$$

$$\left|\frac{\partial}{\partial y}\, g(x,\xi,y)\right| \leqslant \text{const}\cdot x^{2\beta}\,\xi^{2\beta}\,y^{-\beta-2},$$

where $n = 0, 1, 2, \dots$ Applying the transform A^{α} w.r.t. the variable y by means of formula (9) to the latter estimates, by properties (6) and (8) we arrive at the estimates

$$\left|\frac{\partial^n}{\partial x^n}\, G(x,\xi,y)\right| \leqslant \text{const}\cdot x^{2\beta-n}\,\xi^{2\beta}\,y^{-\alpha\beta-1}, \quad \beta \neq \frac{1}{2}, \tag{18}$$

$$\left|\frac{\partial^{2n}}{\partial x^{2n}}\, G(x,\xi,y)\right| \leqslant \text{const}\cdot x\,\xi\,y^{-\alpha\,(2n+1)/2-1}, \quad \beta = \frac{1}{2},$$

$$\left|\frac{\partial^{2n+1}}{\partial x^{2n+1}}\, G(x,\xi,y)\right| \leqslant \text{const}\cdot \xi\,y^{-\alpha\,(2n+1)/2-1}, \quad \beta = \frac{1}{2},$$

$$\left|D_{0y}^{\alpha}\, G(x,\xi,y)\right| \leqslant \mathrm{const}\cdot x^{2\beta}\,\xi^{2\beta}\,y^{-\alpha\beta-\alpha-1}.$$

Employing formulae (15)–(17) and the asymptotic formula [8]

$$I_\nu(z) = \frac{e^{z}}{\sqrt{2\pi z}}\left[\,1+O\left(z^{-1}\right)\right], \quad |\arg z| < \frac{\pi}{2},$$

which is true for large values of z, by (14) with $x\xi > 2y$ we obtain the estimates

$$\left|\frac{\partial^n}{\partial x^n}\, g(x,\xi,y)\right| \leqslant \mathrm{const}\cdot x^{\beta+n-\frac{1}{2}}\,\xi^{\beta-\frac{1}{2}}\,y^{-n-\frac{1}{2}}\exp\left[-\frac{(x-\xi)^2}{4y}\right], \tag{19}$$

$$\left|\frac{\partial}{\partial y}\, g(x,\xi,y)\right| \leqslant \mathrm{const}\cdot x^{\beta+\frac{3}{2}}\,\xi^{\beta-\frac{1}{2}}\,y^{-\frac{5}{2}}\,\exp\left[-\frac{(x-\xi)^2}{4y}\right],$$

where $n=0,1,2,\ldots$ Applying the transformation A^{α} w.r.t. the variable y by means of formula (11) to the latter estimates and employing then formula (12), by properties (6) and (8) we get the estimates

$$\left|\frac{\partial^n}{\partial x^n}\, G(x,\xi,y)\right| \leqslant \mathrm{const}\cdot P_n(x,\xi,y)\,\exp\left[-\alpha_0\,|x-\xi|^{\frac{2}{2-\alpha}}\,y^{-\frac{\alpha}{2-\alpha}}\right], \tag{20}$$

$$\left|D_{0y}^{\alpha}\, G(x,\xi,y)\right| \leqslant \mathrm{const}\cdot P_2(x,\xi,y)\,\exp\left[-\alpha_0\,|x-\xi|^{\frac{2}{2-\alpha}}\,y^{-\frac{\alpha}{2-\alpha}}\right],$$

where $\alpha_0 = (2-\alpha)\,2^{-\frac{2}{2-\alpha}}\,\alpha^{\frac{\alpha}{2-\alpha}}$ and

$$P_n(x,\xi,y) = x^{\beta+\frac{2n-1}{2}}\,\xi^{\beta-\frac{1}{2}}\,|x-\xi|^{-\frac{(2n-1)(1-\alpha)}{2-\alpha}}\,y^{-\frac{\alpha(2n-1)}{2(2-\alpha)}-1}, \quad n=0,1,2,\ldots$$

The following theorems hold.

Theorem 1. *Let* $|b|<1$, $\varphi(x)\in C[0,\infty)$, $\varphi(0)=0$ *and the condition*

$$\lim_{x\to\infty}\varphi(x)\,\exp\left(-\,\rho\,x^{\frac{2}{2-\alpha}}\right)=0, \quad \rho < (2-\alpha)\,2^{-\frac{2}{2-\alpha}}\left(\frac{\alpha}{T}\right)^{\frac{\alpha}{2-\alpha}}$$

holds. Then the function

$$u(x,y) = \int\limits_0^{\infty} \xi^{1-2\beta}\, G(x,\xi,y)\,\varphi(\xi)\,d\xi \tag{21}$$

solves Problem 1.

Theorem 2. *There exists at most one regular solution to Problem 1 in the class of the functions satisfying the condition*

$$\lim_{x\to\infty} y^{1-\alpha}\,u(x,y)\,\exp\left(-\,k\,x^{\frac{2}{2-\alpha}}\right)=0 \tag{22}$$

for some positive k *and the convergence in* (22) *is uniform on the set* $\{y\in(0;T)\}$.

5. Proof of Theorem 1

Estimates (18) and (20) for $n=0$ imply the existence of the integral in (21). Let us prove that the function $u(x,y)$ defined by identity (21) satisfies equation (1). The possibility of swapping the differentiation and integration while differentiating w.r.t. x and calculating the fractional derivative w.r.t. y of order α follows from the above obtained estimates for the function $G(x,\xi,y)$.

We differentiate identity (21) w.r.t. x employing formula (15) for $\nu=\beta$. As a result we get

$$\frac{\partial}{\partial x}\,u(x,y) = \int\limits_0^{\infty} \xi^{1-2\beta}\,\frac{\partial}{\partial x}\,G(x,\xi,y)\,\varphi(\xi)\,d\xi, \tag{23}$$

where

$$\frac{\partial}{\partial x}\, G(x,\xi,y) = A_y^{\alpha}\, \frac{\partial}{\partial x}\, g(x,\xi,y),$$
$$\frac{\partial}{\partial x}\, g(x,\xi,y) = \left\{ \frac{x^{\beta}\xi^{\beta+1}}{(2y)^2} I_{\beta-1}\left(\frac{x\xi}{2y}\right) - \frac{x^{\beta+1}\xi^{\beta}}{(2y)^2} I_{\beta}\left(\frac{x\xi}{2y}\right) \right\} e^{-\frac{x^2+\xi^2}{4y}}.$$

We multiply both sides of (23) by $x^{1-2\beta}$ and we differentiate the obtained identity w.r.t. x employing formula (16) with $\nu = \beta - 1$. Then we use formula (17) with $\nu = \beta$ and multiply the obtained identity by $x^{2\beta-1}$. Finally we get

$$B_x\, u(x,y) = \int\limits_0^{\infty} \xi^{1-2\beta}\, B_x\, G(x,\xi,y)\, \varphi(\xi)\, d\xi, \tag{24}$$

where

$$B_x\, G(x,\xi,y) = A_y^{\alpha}\, B_x\, g(x,\xi,y), \tag{25}$$
$$\begin{aligned} B_x\, g(x,\xi,y) = &\left(\frac{x^{\beta+2}\xi^{\beta}}{(2y)^3}\, I_{\beta}\left(\frac{x\xi}{2y}\right) + \frac{x^{\beta}\xi^{\beta+2}}{(2y)^3}\, I_{\beta}\left(\frac{x\xi}{2y}\right) \right. \\ &\left. - \frac{2\, x^{\beta}\xi^{\beta}}{(2y)^2}\, I_{\beta}\left(\frac{x\xi}{2y}\right) - \frac{2\, x^{\beta+1}\xi^{\beta+1}}{(2y)^3}\, I'_{\beta}\left(\frac{x\xi}{2y}\right) \right) e^{-\frac{x^2+\xi^2}{4y}}. \end{aligned}$$

Formula (6) yields

$$D_{0y}^{\alpha}\, u(x,y) = \int\limits_0^{\infty} \xi^{1-2\beta}\, D_{0y}^{\alpha}\, G(x,\xi,y)\, \varphi(\xi)\, d\xi, \tag{26}$$

where

$$D_{0y}^{\alpha}\, G(x,\xi,y) = A_y^{\alpha}\, \frac{\partial}{\partial y}\, g(x,\xi,y), \tag{27}$$
$$\begin{aligned} \frac{\partial}{\partial y}\, g(x,\xi,y) = &\left(\frac{x^{\beta+2}\xi^{\beta}}{(2y)^3}\, I_{\beta}\left(\frac{x\xi}{2y}\right) + \frac{x^{\beta}\xi^{\beta+2}}{(2y)^3}\, I_{\beta}\left(\frac{x\xi}{2y}\right) \right. \\ &\left. - \frac{2\, x^{\beta}\xi^{\beta}}{(2y)^2}\, I_{\beta}\left(\frac{x\xi}{2y}\right) - \frac{2\, x^{\beta+1}\xi^{\beta+1}}{(2y)^3}\, I'_{\beta}\left(\frac{x\xi}{2y}\right) \right) e^{-\frac{x^2+\xi^2}{4y}}. \end{aligned}$$

Substituting (24) and (26) into equation (1), we see that it is satisfied.

Let us check condition (3). It follows from formula (7) that

$$\begin{aligned} \lim_{y\to 0} D_{0y}^{\alpha-1}\, u(x,y) &= \lim_{y\to 0} \int\limits_0^{\infty} \xi^{1-2\beta}\, g(x,\xi,y)\, \varphi(\xi)\, d\xi \\ &= \lim_{y\to 0} \left[\int\limits_0^{\infty} \xi^{1-2\beta}\, g(x,\xi,y)\, [\varphi(\xi) - \varphi(x)]\, d\xi + \varphi(x) \int\limits_0^{\infty} \xi^{1-2\beta}\, g(x,\xi,y)\, d\xi \right] \\ &= \lim_{y\to 0} \left[\, J_1(x,y) + J_2(x,y)\, \right]. \end{aligned}$$

Partition the integration segment into parts, we represent $J_1(x,y)$ as the sum of three terms

$$J_1(x,y) = \int\limits_0^{x-\varepsilon} \xi^{1-2\beta}\, g(x,\xi,y)\, [\, \varphi(\xi) - \varphi(x)\,]\, d\xi$$

$$
\begin{aligned}
&+\int\limits_{x-\varepsilon}^{x+\varepsilon} \xi^{\,1-2\beta}\, g(x,\xi,y)\,[\,\varphi(\xi)-\varphi(x)\,]\,d\xi+\int\limits_{x+\varepsilon}^{\infty} \xi^{\,1-2\beta}\, g(x,\xi,y)\,[\,\varphi(\xi)-\varphi(x)\,]\,d\xi\\
=&J_{11}(x,y)+J_{12}(x,y)+J_{13}(x,y),
\end{aligned}
$$

where ε is an arbitrary small positive number.

According (19), the formula (14) with $y\to 0$ implies the estimate

$$
|\,g(x,\xi,y)\,|\leqslant \mathrm{const}\cdot x^{\,\beta-\frac12}\,\xi^{\,\beta-\frac12}\,y^{-\frac12}\,e^{-\frac{(x-\xi)^2}{4y}}. \tag{28}
$$

This yields that

$$
\lim_{y\to 0} J_{11}(x,y)=\lim_{y\to 0} J_{13}(x,y)=0.
$$

We denote $\omega(\varepsilon)=\sup|\varphi(x)-\varphi(\xi)|$, where $\xi\in[x-\varepsilon,x+\varepsilon]$. We have $\omega(\varepsilon)\to 0$ as $\varepsilon\to 0$, since the function $\varphi(x)$ is continuous. Then by (28) we can write

$$
|J_{12}(x,y)|\leqslant \omega(\varepsilon)\,\frac{x^{\,\beta-\frac12}}{\sqrt{y}}\int\limits_{x-\varepsilon}^{x+\varepsilon}\xi^{\,\frac12-\beta}\,e^{-\frac{(x-\xi)^2}{4y}}\,d\xi.
$$

Making the change $\xi=x+2\sqrt{y}\,t$ in the latter integral, we obtain

$$
|J_{12}(x,y)|\leqslant \omega(\varepsilon)\,x^{\,\beta-\frac12}\int\limits_{-\frac{\varepsilon}{2\sqrt{y}}}^{\frac{\varepsilon}{2\sqrt{y}}}(x+2\sqrt{y}\,t)^{\,\frac12-\beta}\,e^{-t^2}\,dt. \tag{29}
$$

Applying the generalized theorem on mean value [19] to the integral in the right hand side of (29) and employing the estimate

$$
\int\limits_{-\frac{\varepsilon}{2\sqrt{y}}}^{\frac{\varepsilon}{2\sqrt{y}}} e^{-t^2}\,dt<\int\limits_{-\infty}^{\infty} e^{-t^2}\,dt=\sqrt{\pi},
$$

we find:

$$
\lim_{y\to 0} J_{12}(x,y)=\mathrm{const}\cdot\omega(\varepsilon).
$$

By the continuity of the function $\varphi(x)$ and the arbitrary choice of ε this implies

$$
\lim_{y\to 0} J_{12}(x,y)=0.
$$

Let us calculate the integral $J_2(x,y)$. In order to do this, we employ the formula [13]

$$
\int\limits_0^{\infty}\xi^{\,\delta-1}e^{-p\,\xi^{2}}I_\beta\,(c\,\xi)\;d\xi=\frac{c^\beta p^{-\frac{\delta+\beta}{2}}\,\Gamma\left(\frac{\delta+\beta}{2}\right)}{2^{\,1+\beta}\,\Gamma(\,1+\beta)}\;{}_1F_1\left(\frac{\delta+\beta}{2};1+\beta;\frac{c^{\,2}}{4p}\right), \tag{30}
$$

where $\mathrm{Re}\,p$, $\mathrm{Re}\,(\delta+\beta)>0$, $|\arg c|<\pi$, ${}_1F_1\,(a;b;z)$ is the degenerate hypergeometric function. Letting here $\delta=2-\beta$, $p=1/(4y)$, $c=x/(2y)$, we obtain

$$
J_2(x,y)=\frac{x^{\,2\beta}\,y^{\,-\beta}\,e^{-\frac{x^2}{4y}}\,\varphi(x)}{2^{\,2\beta}\,\Gamma(1+\beta)}\;{}_1F_1\left(1;1+\beta;\frac{x^2}{4y}\right).
$$

Then the asymptotic formula [9]

$$
{}_1F_1\,(a;b;z)=\frac{\Gamma(b)}{\Gamma(a)}\,e^{\,z}\,z^{-(b-a)}\left[1+O(|z|^{-1})\right],\quad z\to\infty,\quad |\arg z|<\frac{\pi}{2},\quad a,b\neq 0,-1,-2,\ldots,
$$

implies

$$\lim_{y\to 0} J_2(x,y) = \varphi(x).$$

Thus,

$$\lim_{y\to 0} D_{0y}^{\alpha-1}\, u(x,y) = \varphi(x).$$

Homogeneous condition (4) is implied by estimate (18) with $n = 0$ and condition $\beta > 0$. The proof of Theorem 1 is complete.

We observe that the solution to the inhomogeneous equation

$$\mathbf{L}\, u(x,y) = f(x,y)$$

subject to the conditions

$$\lim_{y\to 0} D_{0y}^{\alpha-1} u(x,y) = 0, \quad 0 < x < \infty, \qquad u(0,y) = \tau(y), \quad 0 < y < T,$$

can be written in terms of function (13) as

$$u(x,y) = \int_0^y \xi^{\,1-2\beta}\, G_{\xi}(x,\xi,y-\eta)\big|_{\xi=0} \tau(\eta)\, d\eta - \int_0^y \int_0^\infty \xi^{\,1-2\beta}\, G(x,\xi,y-\eta)\, f(\xi,\eta)\, d\xi\, d\eta,$$

where the functions $\tau(y)$ and $f(x,y)$ are such that $y^{1-\alpha}\tau(y) \in C[0,T]$, $y^{1-\alpha} f(x,y) \in (\bar{\Omega})$, the function $f(x,y)$ satisfies the Hölder condition w.r.t. the variable x and the conditions

$$\lim_{y\to 0} D_{0y}^{\alpha-1}\, \tau(y) = 0, \quad \lim_{x\to\infty} y^{\,1-\alpha} f(x,y)\, \exp\left(-\rho\, x^{\frac{2}{2-\alpha}}\right) = 0, \quad \rho < (2-\alpha)\, 2^{-\frac{2}{2-\alpha}} \left(\frac{\alpha}{T}\right)^{\frac{\alpha}{2-\alpha}}$$

holds.

6. Representation of solutions in particular cases

By (21), we are going to obtain the representation for the solution to Problem 1 for the diffusion equation with the Riemann-Liouville operator. By (13) and (14) for $\beta = \frac{1}{2}$ $(b = 0)$ and by the known representation [9]

$$I_{\frac{1}{2}}(z) = \frac{e^{\,z} - e^{-z}}{\sqrt{2\pi z}}$$

we have

$$G(x,\xi,y) = A_y^{\alpha}\, g(x,\xi,y), \qquad g(x,\xi,y) = \frac{1}{2\sqrt{\pi y}} \left[e^{-\frac{(x-\xi)^2}{4y}} - e^{-\frac{(x+\xi)^2}{4y}} \right].$$

Employing then another known identity [15]

$$\sqrt{\pi}\, \phi\left(-\frac{1}{2}, \frac{1}{2}; -z\right) = e^{-\frac{z^2}{4}},$$

the function $g(x,\xi,y)$ can be written as

$$g(x,\xi,y) = \frac{1}{2\sqrt{y}} \left[\phi\left(-\frac{1}{2}, \frac{1}{2}; -\frac{|x-\xi|}{\sqrt{y}}\right) - \phi\left(-\frac{1}{2}, \frac{1}{2}; -\frac{x+\xi}{\sqrt{y}}\right) \right].$$

Applying the transformation A^{α} w.r.t. the variable y by means of formula (10) to the latter identity, we obtain

$$u(x,y) = \int_0^\infty G(x,\xi,y)\, \varphi(\xi)\, d\xi, \tag{31}$$

where

$$G(x,\xi,y)=\frac{y^{\sigma-1}}{2}\left[\phi\left(-\sigma,\sigma;-\frac{|x-\xi|}{y^\sigma}\right)-\phi\left(-\sigma,\sigma;-\frac{x+\xi}{y^\sigma}\right)\right],\qquad \sigma=\frac{\alpha}{2}.$$

Function (31) coincides with the solution to the boundary value problem for the fractional diffusion equation provided in [2].

In the case $\varphi(x)$ is a power function of the coordinate x, that is $\varphi(x)=x^\nu$ with $\nu>0$, by (21) we get

$$u(x,y)=\int\limits_0^\infty \xi^{1+\nu-2\beta}G(x,\xi,y)\,d\xi.$$

Substituting the function $G(x,\xi,y)$ from (13) into the latter integral and swapping the integration order, we obtain

$$u(x,y)=A_y^\alpha\,J_3(x,y),$$

where

$$J_3(x,y)=\frac{x^\beta}{2y}\,e^{-\frac{x^2}{4y}}\int\limits_0^\infty \xi^{1+\nu-\beta}e^{-\frac{\xi^2}{4y}}I_\beta\left(\frac{x\xi}{2y}\right)\,d\xi.$$

We calculate the integral in the latter identity by formula (30). As $\delta=2+\nu-\beta$, $p=1/(4y)$ and $c=x/(2y)$, by this formula we find

$$J_3(x,y)=\frac{\Gamma(1+\frac{\nu}{2})}{\Gamma(1+\beta)}\,x^{2\beta}\,(4y)^{\frac{\nu}{2}-\beta}\,e^{-\frac{x^2}{4y}}\,{}_1F_1\left(1+\frac{\nu}{2};1+\beta;\frac{x^2}{4y}\right).$$

We substitute the found value $J_3(x,y)$ into integral (5) as $\mu=0$ and make the change $t=y^\alpha\tau$. As a result we have

$$u(x,y)=\frac{2^\nu\,\Gamma(1+\frac{\nu}{2})}{\Gamma(1+\beta)}\,y^{\alpha\frac{\nu}{2}+\alpha-1}\int\limits_0^\infty \mathcal{K}_1\left(\frac{a}{\tau}\right)\mathcal{K}_2\left(\tau\right)\frac{d\tau}{\tau},\quad a=\frac{x^2}{4y^\alpha},\tag{32}$$

where

$$\mathcal{K}_1(\tau)=\tau^\beta e^{-\tau}\,{}_1F_1\left(1+\frac{\nu}{2};1+\beta;\tau\right),\quad \mathcal{K}_2(\tau)=\tau^{1+\frac{\nu}{2}}\,\phi\left(-\alpha,0;-\tau\right).$$

We calculate the integral in (32) by the method exposed in [10]. From row 12.2(1) of the basic table in Section 10 in [10] we find the Mellin transform of the function $e^{-\tau}\,{}_1F_1\left(1+\frac{\nu}{2};1+\beta;\tau\right)$:

$$\frac{\Gamma(1+\beta)}{\Gamma(\beta-\frac{\nu}{2})}\frac{\Gamma(s)\Gamma(\beta-\frac{\nu}{2}-s)}{\Gamma(1+\beta-s)},\quad 0<\operatorname{Re}s<\beta-\frac{\nu}{2},\quad \nu<2\beta.$$

Then by Property 1.4 in Section 10 [10], the image of the function $\mathcal{K}_1(\tau)$ is

$$\mathcal{K}_1^*(s)=\frac{\Gamma(1+\beta)}{\Gamma(\beta-\frac{\nu}{2})}\frac{\Gamma(\beta+s)\Gamma(-\frac{\nu}{2}-s)}{\Gamma(1-s)},\quad -\beta<\operatorname{Re}s<-\frac{\nu}{2},\quad \nu<2\beta.$$

The Mellin transform of the function $\phi(-\alpha,0;-\tau)$ can be found by formula (9). Letting $\mu=0$, $\delta=s$ and employing definition (5), in which we make the change $t=y^\alpha\tau$, we get

$$\int\limits_0^\infty \tau^{s-1}\,\phi\left(-\alpha,0;-\tau\right)d\tau=\frac{\Gamma(s)}{\Gamma(\alpha\,s)},\quad \operatorname{Re}s>0.$$

Then by Property 1.4 in Section 10 in [10], we find the image of the second function $\mathcal{K}_2(\tau)$ if in the right hand side, we replace s by $1+\frac{\nu}{2}+s$, that is,

$$\mathcal{K}_2^*(s)=\frac{\Gamma(1+\frac{\nu}{2}+s)}{\Gamma(\alpha+\alpha\frac{\nu}{2}+\alpha\, s)},\quad \operatorname{Re} s>-1-\frac{\nu}{2}.$$

Multiplying the images $\mathcal{K}_i^*(s)$, $i=1,2$, we get

$$\mathcal{K}^*(s)=\frac{\Gamma(1+\beta)}{\Gamma(\beta-\frac{\nu}{2})}\frac{\Gamma(1+\frac{\nu}{2}+s)\Gamma(\beta+s)\Gamma(-\frac{\nu}{2}-s)}{\Gamma(\alpha+\alpha\frac{\nu}{2}+\alpha\, s)\Gamma(1-s)},$$

where $-\min\{\beta,1+\frac{\nu}{2}\}<\operatorname{Re} s<-\frac{\nu}{2}$, $\nu<2\beta$.

Calculating the pre-image of the function $\mathcal{K}^*(s)$, we obtain the needed integral in (32):

$$\begin{aligned}\frac{\Gamma(1+\beta)}{\Gamma(\beta-\frac{\nu}{2})}\frac{1}{2\pi i}\int\limits_{L_{i\infty}}&\frac{\Gamma(1+\frac{\nu}{2}+s)\Gamma(\beta+s)\Gamma(-\frac{\nu}{2}-s)}{\Gamma(\alpha+\alpha\frac{\nu}{2}+\alpha\, s)\Gamma(1-s)}\left(\frac{x^2}{4\,y^\alpha}\right)^{-s}ds\\ &=\frac{\Gamma(1+\beta)}{\Gamma(\beta-\frac{\nu}{2})}H^{2,1}_{2,3}\left[\frac{x^2}{4y^\alpha}\,\middle|\,\begin{matrix}(1+\frac{\nu}{2},1),(\alpha+\alpha\frac{\nu}{2},\alpha)\\(1+\frac{\nu}{2},1),(\beta,1),(0,1)\end{matrix}\right],\end{aligned}$$

where

$$L_{i\infty}=(\omega-i\infty,\omega+i\infty),\quad -\min\{\beta,1+\frac{\nu}{2}\}<\omega<-\frac{\nu}{2},\quad \nu<2\beta.$$

We transform the right hand side of the obtained identity by means of the formula [14], [25]

$$z^h H^{m,n}_{p,q}\left[z\,\middle|\,\begin{matrix}[a_p,A_p]\\ [b_q,B_q]\end{matrix}\right]=H^{m,n}_{p,q}\left[z\,\middle|\,\begin{matrix}[a_p+hA_p,A_p]\\ [b_q+hB_q,B_q]\end{matrix}\right],$$

letting $h=-\frac{\nu}{2}$ and we substitute the obtained expression into (32). As a result, we denote $\lambda=\Gamma(1+\frac{\nu}{2})/\Gamma(\beta-\frac{\nu}{2})$ and arrive at the function

$$u(x,y)=\lambda\, x^\nu y^{\alpha-1}H^{2,1}_{2,3}\left[\frac{x^2}{4y^\alpha}\,\middle|\,\begin{matrix}(1,1),(\alpha,\alpha)\\(1,1),(\beta-\frac{\nu}{2},1),(-\frac{\nu}{2},1)\end{matrix}\right],\tag{33}$$

which is a solution to Problem 1 in the case $\varphi(x)=x^\nu$, $0<\nu<2\beta$.

As $\alpha=1$, by means of the formula

$$\begin{aligned}H^{m,n}_{p,q}&\left[z\,\middle|\,\begin{matrix}(a_1,A_1),\dots,(a_{p-1},A_{p-1}),(b_1,B_1)\\(b_1,B_1),\dots,(b_q,B_q)\end{matrix}\right]\\ &=H^{m-1,n}_{p-1,q-1}\left[z\,\middle|\,\begin{matrix}(a_1,A_1),\dots,(a_{p-1},A_{p-1})\\(b_2,B_2),\dots,(b_q,B_q)\end{matrix}\right],\end{aligned}$$

representation (33) can be written as

$$u(x,y)=\lambda\, x^\nu H^{1,1}_{1,2}\left[\frac{x^2}{4y^\alpha}\,\middle|\,\begin{matrix}(1,1)\\(\beta-\frac{\nu}{2},1),(-\frac{\nu}{2},1)\end{matrix}\right].$$

7. Proof of Theorem 2

Let $h_r(\xi)$ be a twice continuously differentiable function possessing the following properties:

$$h_r(\xi)=\begin{cases}1, & 0\leqslant\xi\leqslant r,\\ 0, & \xi\geqslant r+1,\end{cases}\tag{34}$$

$0\leqslant h_r(\xi)\leqslant 1$, $|h_r'(\xi)|+|h_r''(\xi)|\leqslant H$, where H is a constant independent of r.

It follows from (25) and (27) that the function $G(x,\xi,y)$ as the function of the variables x and y satisfies the equation $\mathbf{L}\,G(x,\xi,y)=0$, and the function $G(x,\xi,y-\eta)$ as the function of the variables ξ and η, $0<\eta<y$, satisfies the adjoint equation

$$\mathbf{L}^*G(x,\xi,y-\eta)\equiv B_\xi\,G(x,\xi,y-\eta)-D^{\,\alpha}_{y\eta}\,G(x,\xi,y-\eta)=0. \tag{35}$$

We consider the function

$$v(x,\xi,y-\eta)=h_r(\xi)\,G(x,\xi,y-\eta).$$

Taking into consideration (35), we obtain

$$\mathbf{L}^*v(x,\xi,y-\eta)=2\,h_r'(\xi)\,G_\xi(x,\xi,y-\eta)+\frac{b}{\xi}\,h_r'(\xi)\,G(x,\xi,y-\eta)+h_r''(\xi)\,G(x,\xi,y-\eta). \tag{36}$$

We prove first that if $\varphi(x)\equiv 0$, then $u(x,y)\equiv 0$ as $0<y<\delta$ for sufficiently small δ. According theorem on general representation of solution to equation (1) [21], a regular in the domain $\Omega_r=\{(x,y):0<x<r,\,0<y<\delta\}$ solution to homogeneous problem can be represented as

$$u(x,y)=\int\limits_0^{r+1}\int\limits_0^{y}\xi^{\,1-2\beta}\,u(\xi,\eta)\,\mathbf{L}^*v(x,\xi,y-\eta)\,d\eta\,d\xi.$$

It follows from (34) and (36) that $\mathbf{L}^*v(x,\xi,y-\eta)=0$ if $0\leqslant\xi\leqslant r$, which implies

$$u(x,y)=\int\limits_r^{r+1}\int\limits_0^{y}\xi^{\,1-2\beta}\,u(\xi,\eta)\,\mathbf{L}^*v(x,\xi,y-\eta)\,d\eta\,d\xi.$$

By the properties of the function $h_r(\xi)$, estimates (20), and by (36) we obtain

$$|\,\mathbf{L}^*v(x,\xi,y-\eta)\,|\leqslant \mathrm{const}\cdot P_1(x,\xi,y-\eta)\exp\Big[-\alpha_0\,|x-\xi|^{\,\frac{2}{2-\alpha}}\,(y-\eta)^{\,-\frac{\alpha}{2-\alpha}}\Big].$$

In view of this estimate and condition (22), we find

$$|u(x,y)|\leqslant \mathrm{const}\int\limits_r^{r+1}\int\limits_0^{y}P(x,\xi,y,\eta)\exp\Big[-\alpha_0|x-\xi|^{\frac{2}{2-\alpha}}(y-\eta)^{-\frac{\alpha}{2-\alpha}}+k\,\xi^{\frac{2}{2-\alpha}}\Big]d\eta\,d\xi,$$

where $P(x,\xi,y,\eta)=\xi^{\,1-2\beta}\,\eta^{\,\alpha-1}\,P_1(x,\xi,y-\eta)$. As $\delta<(\alpha_0/k)^{(2-\alpha)/\alpha}$ and $r\to\infty$, the right hand side in the latter inequality tends to zero. This means that $u(x,y)\equiv 0$ in the domain

$$\Omega_1=\{(x,y):0<x<\infty,\,0<y<\delta\}.$$

Let us prove that $u(x,y)\equiv 0$ for each $y>0$. Let $t=y-\delta$, $\delta\leqslant y<2\delta$. We consider the function $w(x,t)=u(x,\delta+t)$. Since $u(x,y)\equiv 0$ as $0<y<\delta$, then

$$D^{\,\alpha}_{0y}\,u(x,y)=D^{\,\alpha}_{\delta y}\,u(x,y)=D^{\,\alpha}_{0t}\,w(x,t).$$

This implies that the function $w(x,t)$ satisfies the equation

$$B_x\,w(x,t)-D^{\,\alpha}_{0t}\,w(x,t)=0,\quad 0<x<\infty,\quad 0<t<\delta,$$

conditions (22) and

$$\lim_{t\to 0}D^{\,\alpha-1}_{0t}w(x,t)=0,\quad 0<x<\infty,\qquad w(0,t)=0,\quad 0<t<\delta.$$

Then in accordance with the said above, $w(x,t)\equiv 0$ in the domain

$$\Omega_2=\{(x,t):0<x<\infty,\,0<t<\delta\},$$

that is, $u(x, y) \equiv 0$ in

$$\Omega_2 = \{(x, y) : 0 < x < \infty,\ \delta < y < 2\delta\}.$$

Exactly in the same way one can prove that $u(x, y) \equiv 0$ in the strips $(n-1)\,\delta \leqslant y < n\,\delta$, $n = 3, 4, \dots$ The proof of Theorem 2 is complete.

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