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QUASI-ELLIPTIC FUNCTIONS

A.YA. KHRYSTIYANYN, DZ.V. LUKIVSKA

Abstract. We study certain generalizations of elliptic functions, namely quasi-elliptic functions.

Let $p = e^{i\alpha}$, $q = e^{i\beta}$, α, $\beta \in \mathbb{R}$. A meromorphic in $\mathbb{C}$ function g is called quasi-elliptic if there exist $\omega_1, \omega_2 \in \mathbb{C}^*$, $\operatorname{Im} \frac{\omega_2}{\omega_1} > 0$, such that $g(u+\omega_1) = pg(u)$, $g(u+\omega_2) = qg(u)$ for each $u \in \mathbb{C}$. In the case $\alpha = \beta = 0 \mod 2\pi$ this is a classical theory of elliptic functions. A class of quasi-elliptic functions is denoted by $\mathcal{QE}$. We show that the class $\mathcal{QE}$ is nontrivial. For this class of functions we construct analogues $\wp_{\alpha\beta}$, $\zeta_{\alpha\beta}$ of $\wp$ and ζ Weierstrass functions. Moreover, these analogues are in fact the generalizations of the classical $\wp$ and ζ functions in such a way that the latter can be found among the former by letting $\alpha = 0$ and $\beta = 0$. We also study an analogue of the Weierstrass σ function and establish connections between this function and $\wp_{\alpha\beta}$ as well as $\zeta_{\alpha\beta}$.

Let $q, p \in \mathbb{C}^*$, $|q| < 1$. A meromorphic in $\mathbb{C}^*$ function f is said to be p-loxodromic of multiplicator q if for each $z \in \mathbb{C}^*$ $f(qz) = pf(z)$. We obtain telations between quasi-elliptic and p-loxodromic functions.

Keywords: quasi-elliptic function, the Weierstrass $\wp$-function, the Weierstrass ζ-function, the Weierstrass σ-function, p-loxodromic function.

Mathematics subject classification: 30D30

1. Introduction

Denote $\mathbb{C}^* = \mathbb{C}\backslash\{0\}$. A meromorphic in $\mathbb{C}$ function g is called *elliptic* [1] if there exist $\omega_1, \omega_2 \in \mathbb{C}^*$ such that $\operatorname{Im} \frac{\omega_2}{\omega_1} > 0$ and for each $u \in \mathbb{C}$

$$g(u+\omega_1) = g(u), \qquad g(u+\omega_2) = g(u).$$

The theory of elliptic functions was developed by K. Jacobi, N. Abel, A. Legendre, K. Weierstrass. The following definition was introduced by A. Kondratyuk.

Definition 1. [2] *A meromorphic in $\mathbb{C}$ function f is said to be modulo-elliptic if there exist $\omega_1, \omega_2 \in \mathbb{C}^*$ such that $\operatorname{Im} \frac{\omega_2}{\omega_1} > 0$ and for each $u \in \mathbb{C}$*

$$|f(u+\omega_1)| = |f(u)|, \qquad |f(u+\omega_2)| = |f(u)|.$$

Consider the first of these identities

$$|f(u+\omega_1)| = |f(u)|, \qquad u \in \mathbb{C}. \tag{1}$$

If $f(u) \neq 0$ and $f(u) \neq \infty$, we can divide (1) by $|f(u)|$ to obtain

$$\left|\frac{f(u+\omega_1)}{f(u)}\right| = 1. \tag{2}$$

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The function $g(u) = \dfrac{f(u+\omega_1)}{f(u)}$ is meromorphic in $\mathbb{C}$. It follows from (2) that the function g is holomorphic and bounded in $\mathbb{C}$ except for a set of the zeros and poles of f. Since g is bounded, these points are removable, and relation (2) implies

$$\forall u \in \mathbb{C}: \quad |g(u)| = 1.$$

By the Liouville theorem g is constant and the latter identity implies the existence of $\alpha \in \mathbb{R}$ such that $g(u) = e^{i\alpha}$. This means that

$$\forall u \in \mathbb{C}: \qquad f(u+\omega_1) = e^{i\alpha} f(u).$$

In the same way as above, we conclude that there exists $\beta \in \mathbb{R}$ such that

$$\forall u \in \mathbb{C}: \quad f(u+\omega_2) = e^{i\beta} f(u).$$

We consider separately the following cases:
(i) $\alpha = \beta = 0 \mod 2\pi$;
(ii) $\alpha = 0 \mod 2\pi$, $\beta \neq 0 \mod 2\pi$ (or $\alpha \neq 0 \mod 2\pi$, $\beta = 0 \mod 2\pi$);
(iii) $\alpha \neq 0 \mod 2\pi$, $\beta \neq 0 \mod 2\pi$.

In the first case we obtain the classical theory of elliptic functions including the famous Weierstrass $\wp$-function

$$\wp(u) = \frac{1}{u^2} + \sum_{\omega\neq 0}\left(\frac{1}{(u-\omega)^2} - \frac{1}{\omega^2}\right), \qquad \omega = m\omega_1 + n\omega_2, \quad m,n\in\mathbb{Z}. \tag{3}$$

The Weierstrass $\wp$-function is elliptic [1] with periods ω_1, ω_2. The representations for classical Weierstrass ζ and σ functions are well-known [1], [3]:

$$\zeta(u) = \frac{1}{u} + \sum_{\omega\neq 0}\left(\frac{1}{u-\omega} + \frac{1}{\omega} + \frac{u}{\omega^2}\right), \qquad \omega = m\omega_1 + n\omega_2, \quad m,n\in\mathbb{Z}. \tag{4}$$

$$\sigma(u) = u\prod_{\omega\neq 0}\left(1 - \frac{u}{\omega}\right) e^{\frac{u}{\omega} + \frac{u^2}{2\omega^2}}, \qquad \omega = m\omega_1 + n\omega_2, \quad m,n\in\mathbb{Z}. \tag{5}$$

We also observe that the following identities

$$\wp(u) = -\zeta'(u), \qquad \zeta(u) = \frac{\sigma'(u)}{\sigma(u)}, \qquad \wp(u) = -\left(\frac{\sigma'(u)}{\sigma(u)}\right)'.$$

hold true. We note that each elliptic function can be representedby using (3), (4), (5) (see [3]). In other words, these functions play an important role in representations of elliptic functions.

In the second case we obtain so-called p-elliptic functions.

Definition 2. [4] *Let $p = e^{i\beta}$. A meromorphic in $\mathbb{C}$ function g is called p-elliptic if there exist $\omega_1, \omega_2 \in \mathbb{C}^*$ such that* $\operatorname{Im}\frac{\omega_2}{\omega_1} > 0$ *and for each $u \in \mathbb{C}$*

$$g(u+\omega_1) = g(u), \quad g(u+\omega_2) = pg(u).$$

This case was studied in [6].

The aim of this article is to consider the third case. This is a generalization of elliptic functions in some sense as the following definition says.

Definition 3. *Let $p = e^{i\alpha}$, $q = e^{i\beta}$. A meromorphic in $\mathbb{C}$ function g is called* quasi-elliptic *if there exist $\omega_1, \omega_2 \in \mathbb{C}^*$,* $\operatorname{Im}\frac{\omega_2}{\omega_1} > 0$*, such that for each $u \in \mathbb{C}$*

$$g(u+\omega_1) = pg(u), \qquad g(u+\omega_2) = qg(u).$$

We denote the class of quasi-elliptic functions by $\mathcal{QE}$.

Let $\omega = m\omega_1 + n\omega_2$, $m, n \in \mathbb{Z}$. If $f \in \mathcal{QE}$, Definition 3 implies

$$g(u + \omega) = p^m q^n g(u).$$

If $p = 1$ and $q = 1$ in Definition 3, we obtain classic elliptic function. If $p = 1$ or $q = 1$ in Definition 3, we obtain p-elliptic function.

Remark 1. *There is one special case when Definition 3 still gives an elliptic function. Namely, if* $p = e^{i\alpha}$, $q = e^{i\beta}$, *where* α, $\beta \in 2\pi\mathbb{Q}$, *then*

$$f(u + l\omega_1) = f(u), \qquad f(u + l\omega_2) = f(u),$$

where l is the least common denominator of $\dfrac{\alpha}{2\pi}$ *and* $\dfrac{\beta}{2\pi}$.

Indeed, if $\alpha = 2\pi\dfrac{a}{b}$, *using Definition 3, we have*

$$f(u + l\omega_1) = f(u + (l-1)\omega_1)e^{i2\pi\frac{a}{b}} = \cdots = f(u)e^{i2\pi\frac{al}{b}} = f(u).$$

The same conclusion can be made for β.

Remark 2. *The class $\mathcal{QE}$ of quasi-elliptic functions is not trivial. For example, consider the function*

$$f(u) = \sum_{\omega\neq 0} \frac{e^{im\alpha}e^{in\beta}}{(u-\omega)^3}, \qquad \omega = m\omega_1 + n\omega_2, \quad m, n \in \mathbb{Z}. \tag{6}$$

Consider a compact subset K from $\mathbb{C}$. Since ([1], [3]*)*

$$\sum_{\omega\neq 0} \frac{1}{|\omega|^3} < +\infty, \tag{7}$$

we obtain that the series in the right hand side of (6), or at least its remainder, is uniformly convergent on K. Therefore f is meromorphic in $\mathbb{C}$, and we have for each $u \in \mathbb{C}$

$$f(u + \omega_1) = e^{i\alpha} \sum_{m,n\in\mathbb{Z}} \frac{e^{i(m-1)\alpha}e^{in\beta}}{(u - (m-1)\omega_1 - n\omega_2)^3} = e^{i\alpha} f(u).$$

In the same way, for each $u \in \mathbb{C}$ we obtain $f(u + \omega_2) = e^{i\beta} f(u)$.

Our main aim is to construct a quasi-elliptic function $\wp_{\alpha\beta}$ being an analogue of $\wp(u)$ and also to construct corresponding analogues of ζ and σ functions.

2. Generalization of the Weierstrass $\wp$-function

Let $p = e^{i\alpha}$, $q = e^{i\beta}$. Consider the function

$$G_{\alpha\beta}(u) = \frac{1}{u^2} + \sum_{\omega\neq 0}\left(\frac{1}{(u-\omega)^2} - \frac{1}{\omega^2}\right)e^{i(m\alpha+n\beta)}, \tag{8}$$

where $\omega_1, \omega_2 \in \mathbb{C}$, $\operatorname{Im}\frac{\omega_2}{\omega_1} > 0$, $\omega = m\omega_1 + n\omega_2$, $m, n \in \mathbb{Z}$. Similarly, in view of (7), as in the case of the series from (6), we obtain that $G_{\alpha\beta}$ is meromorphic in $\mathbb{C}$.

It is obvious that, G_{00} coincides with the classical Weierstrass function $\wp$.

Consider the case $\alpha \neq 0 \mod 2\pi$ and $\beta \neq 0 \mod 2\pi$, that is, $p \neq 1$ and $q \neq 1$.

Theorem 1. *A function of the form*

$$\wp_{\alpha\beta}(u) = G_{\alpha\beta}(u) + C_{\alpha\beta},$$

where

$$C_{\alpha\beta} = \frac{G_{\alpha\beta}\left(\frac{\omega_1}{2}\right) - e^{i\alpha}G_{\alpha\beta}\left(-\frac{\omega_1}{2}\right)}{e^{i\alpha}-1} = \frac{G_{\alpha\beta}\left(\frac{\omega_2}{2}\right) - e^{i\beta}G_{\alpha\beta}\left(-\frac{\omega_2}{2}\right)}{e^{i\beta}-1}$$

belongs to $\mathcal{QE}$ *with* $p = e^{i\alpha}$, $q = e^{i\beta}$.

Proof. Consider the function $G_{\alpha\beta}$. We shall show that there exists a unique constant $C_{\alpha\beta}$ such that $(G_{\alpha\beta}(u) + C_{\alpha\beta}) \in \mathcal{QE}$, that is

$$G_{\alpha\beta}(u+\omega_1) + C_{\alpha\beta} = e^{i\alpha}(G_{\alpha\beta} + C_{\alpha\beta}),$$
$$G_{\alpha\beta}(u+\omega_2) + C_{\alpha\beta} = e^{i\beta}(G_{\alpha\beta} + C_{\alpha\beta}).$$

These properties are called multi p-periodicity with the period ω_1 and multi q-periodicity with the period ω_2, respectively.

Let us consider the derivative of $G_{\alpha\beta}$:

$$G'_{\alpha\beta}(u) = -2\sum_{\omega}\frac{e^{i(m\alpha+n\beta)}}{(u-\omega)^3}.$$

We have:

$$G'_{\alpha\beta}(u+\omega_1) = -2\sum_{m,n\in\mathbb{Z}}\frac{e^{i(m\alpha+n\beta)}}{(u+\omega_1-m\omega_1-n\omega_2)^3} = -2\sum_{m,n\in\mathbb{Z}}\frac{e^{i(m\alpha+n\beta)}}{(u-(m-1)\omega_1-n\omega_2)^3}$$
$$= -2e^{i\alpha}\sum_{m,n\in\mathbb{Z}}\frac{e^{i((m-1)\alpha+n\beta)}}{(u-(m-1)\omega_1-n\omega_2)^3} = e^{i\alpha}G'_{\alpha\beta}(u).$$

Hence, we obtain

$$G'_{\alpha\beta}(u+\omega_1) - e^{i\alpha}G'_{\alpha\beta}(u) = 0. \tag{9}$$

We note that for each $C \in \mathbb{C}$, the function $(G_{\alpha\beta} + C)$ satisfies (9). Let

$$C = C_{\alpha\beta} = \frac{G_{\alpha\beta}\left(\frac{\omega_1}{2}\right) - e^{i\alpha}G_{\alpha\beta}\left(-\frac{\omega_1}{2}\right)}{e^{i\alpha}-1}. \tag{10}$$

Then relation (9) implies

$$G_{\alpha\beta}(u+\omega_1) + C_{\alpha\beta} - e^{i\alpha}\big(G_{\alpha\beta} + C_{\alpha\beta}\big) = A,$$

where A is a constant. If we let $u = -\frac{\omega_1}{2}$, it is easy to obtain that

$$G_{\alpha\beta}\left(\frac{\omega_1}{2}\right) - e^{i\alpha}G_{\alpha\beta}\left(-\frac{\omega_1}{2}\right) + (1-e^{i\alpha})C_{\alpha\beta} = A.$$

Taking into consideration the choice of $C_{\alpha\beta}$ by formula (10), we get $A = 0$. Therefore, we have

$$G_{\alpha\beta}(u+\omega_1) + C_{\alpha\beta} = e^{i\alpha}\big(G_{\alpha\beta} + C_{\alpha\beta}\big), \tag{11}$$

that is, we have shown that the function $\big(G_{\alpha\beta} + C_{\alpha\beta}\big)$ is multi p-periodic of period ω_1.

It remains to prove the uniqueness of $C_{\alpha\beta}$. Suppose that there exists a constant C different from $C_{\alpha\beta}$ such that the function $\big(G_{\alpha\beta} + C\big)$ is multi p-periodic of period ω_1, too. Then we get

$$G_{\alpha\beta}(u+\omega_1) + C = e^{i\alpha}\big(G_{\alpha\beta}(u) + C\big).$$

Deducting this identity from (11), we obtain

$$C - C_{\alpha\beta} = e^{i\alpha}\big(C - C_{\alpha\beta}\big).$$

Since $\alpha \neq 0 \mod 2\pi$, we get $C = C_{\alpha\beta}$.

In the same way, for the period ω_2 we have

$$G_{\alpha\beta}(u+\omega_2) + C_{\alpha\beta} = e^{i\beta}\big(G_{\alpha\beta}(u) + C_{\alpha\beta}\big) + B, \tag{12}$$

where B is some constant.

Let us find B. Using identities (11) and (12), we obtain

$$\begin{aligned} G_{\alpha\beta}(u+\omega_1+\omega_2) + C_{\alpha\beta} =& e^{i\beta}(G_{\alpha\beta}(u+\omega_1) + C_{\alpha\beta}) + B \\ =& e^{i\beta}(e^{i\alpha}(G_{\alpha\beta}(u) + C_{\alpha\beta})) + B \\ =& e^{i(\alpha+\beta)}(G_{\alpha\beta}(u) + C_{\alpha\beta}) + B \end{aligned}$$

and

$$\begin{aligned} G_{\alpha\beta}(u+\omega_1+\omega_2) + C_{\alpha\beta} =& e^{i\alpha}(G_{\alpha\beta}(u+\omega_2) + C_{\alpha\beta}) \\ =& e^{i\alpha}(e^{i\beta}(G_{\alpha\beta}(u) + C_{\alpha\beta}) + B) \\ =& e^{i(\alpha+\beta)}(G_{\alpha\beta}(u) + C_{\alpha\beta}) + Be^{i\alpha}. \end{aligned}$$

Comparing the right hand sides of these relations, we get $B = Be^{i\alpha}$. Since $\alpha \neq 0 \mod 2\pi$, the previous identity implies that $B = 0$. Therefore,

$$G_{\alpha\beta}(u+\omega_2) + C_{\alpha\beta} = e^{i\beta}\big(G_{\alpha\beta}(u) + C_{\alpha\beta}\big).$$

Hence, the function $G_{\alpha\beta}$ is multi p-periodic with the period ω_1 and is multi q-periodic with period ω_2, respectively.

It is easy to see that $C_{\alpha\beta}$ can be also expressed as

$$C_{\alpha\beta} = \frac{G_{\alpha\beta}\left(\frac{\omega_2}{2}\right) - e^{i\beta}G_{\alpha\beta}\left(-\frac{\omega_2}{2}\right)}{e^{i\beta}-1}.$$

□

Definition 4. *A function of the form*

$$\wp_{\alpha\beta}(u) = G_{\alpha\beta}(u) + C_{\alpha\beta} = \frac{1}{u^2} + \sum_{\omega\neq 0}\left(\frac{1}{(u-\omega)^2} - \frac{1}{\omega^2}\right)e^{i(m\alpha+n\beta)} + C_{\alpha\beta},$$

where

$$C_{\alpha\beta} = \frac{G_{\alpha\beta}\left(\frac{\omega_1}{2}\right) - e^{i\alpha}G_{\alpha\beta}\left(-\frac{\omega_1}{2}\right)}{e^{i\alpha}-1} = \frac{G_{\alpha\beta}\left(\frac{\omega_2}{2}\right) - e^{i\beta}G_{\alpha\beta}\left(-\frac{\omega_2}{2}\right)}{e^{i\beta}-1}$$

is called the generalized Weierstrass $\wp$-function.

Remark 3. *For the sake of completeness, in the case $p = q = 1$, in other words, as $\alpha = \beta = 0 \mod 2\pi$, we efine $C_{00} = 0$. Then $\wp_{00} = \wp$.*

3. Generalization of Weierstrass ζ and σ functions

Now we consider the function

$$\zeta_{\alpha\beta}(u) = \frac{1}{u} + \sum_{\omega\neq 0}\left(\frac{1}{u-\omega} + \frac{1}{\omega} + \frac{u}{\omega^2}\right)e^{i(m\alpha+n\beta)},$$

where $\omega_1, \omega_2 \in \mathbb{C}$, $\operatorname{Im}\frac{\omega_2}{\omega_1} > 0$, $\omega = m\omega_1 + n\omega_2$, $m^2+n^2 \neq 0$, $m, n \in \mathbb{Z}$.

Differentiating $\zeta_{\alpha\beta}$, we obtain $G_{\alpha\beta}(u) = -\zeta'_{\alpha\beta}(u)$. Hence,

$$\wp_{\alpha\beta}(u) = -\zeta'_{\alpha\beta}(u) + C_{\alpha\beta}.$$

We denote

$$\chi_{mn}(u) = \left(\frac{1}{u-\omega} + \frac{1}{\omega} + \frac{u}{\omega^2}\right), \qquad m^2+n^2 \neq 0,$$

and

$$\chi_{00}(u) = \frac{1}{u}.$$

Then $\zeta_{\alpha\beta}$ can be rewritten as

$$\zeta_{\alpha\beta}(u) = \sum_{m,n\in\mathbb{Z}} e^{i(m\alpha+n\beta)}\chi_{mn}(u). \tag{13}$$

We observe that ζ_{00} coincides with the classical Weierstrass ζ function.

By A^* we denote the plane $\mathbb{C}$ with radial slits from ω to ∞. Integrating χ_{mn} and χ_{00} along a path in A^* connecting the points 0 and u, we obtain

$$\int_0^u \chi_{mn}(t)dt = \log\left(1-\frac{u}{\omega}\right) + \frac{u}{\omega} + \frac{u^2}{2\omega^2}, \quad m^2+n^2 \neq 0 \tag{14}$$

and

$$\int_0^u \chi_{00}(t)dt = \log u. \tag{15}$$

We consider entire functions

$$\sigma_{mn}(u) = \left(1-\frac{u}{\omega}\right)e^{\frac{u}{\omega}+\frac{u^2}{2\omega^2}}, \qquad m^2+n^2 \neq 0,$$

and we let

$$\sigma_{00}(u) = u.$$

Employing these functions, we can rewrite (14) as

$$\int_0^u \chi_{mn}(t)dt = \log\sigma_{mn}(u), \qquad m,n\in\mathbb{Z}.$$

Differentiating this identity and using the definitions of χ_{00} and σ_{00}, we get

$$\forall\, m,n\in\mathbb{Z}: \quad \chi_{mn}(u) = \frac{\sigma'_{mn}(u)}{\sigma_{mn}(u)}.$$

Taking into consideration this representation for χ_{mn}, we rewrite (13) as

$$\zeta_{\alpha\beta}(u) = \sum_{m,n\in\mathbb{Z}} e^{i(m\alpha+n\beta)}\frac{\sigma'_{mn}(u)}{\sigma_{mn}(u)}.$$

Hence, $\wp_{\alpha\beta}$ can be rewritten as

$$\wp_{\alpha\beta}(u) = C_{\alpha\beta} + \sum_{m,n\in\mathbb{Z}} e^{i(m\alpha+n\beta)}\frac{(\sigma'_{mn}(u))^2 - \sigma''_{mn}(u)\sigma_{mn}(u)}{\sigma^2_{mn}(u)}.$$

We note that if we consider the product $\prod\limits_{m,n\in\mathbb{Z}} \sigma_{mn}(u)$, then we obtain the Weierstrass σ-function.

4. Connection between p-loxodromic and quasi-elliptic functions

Let $q, p \in \mathbb{C}^*$, $|q| < 1$.

Definition 5. [5] *A meromorphic in $\mathbb{C}^*$ function f is said to be p-loxodromic with the multiplicator q if $f(qz) = pf(z)$ for each $z \in \mathbb{C}^*$.*

We denote by $\mathcal{L}_{qp}$ the class of p-loxodromic functions with the multiplicator q.

The case $p = 1$ was studied earlier in the works of O. Rausenberger [7], G. Valiron [8] and Y. Hellegouarch [1]. In this case the function f is called loxodromic.

Let $a_1 = e^{2\pi i \frac{\omega_2}{\omega_1}}$, $a_2 = e^{2\pi i \frac{\omega_1}{\omega_2}}$ and $f_1 \in \mathcal{L}_{a_1 q}$, $f_2 \in \mathcal{L}_{a_2 p}$. Then

$$f_1(a_1 z) = q f_1(z), \quad f_2(a_2 z) = p f_2(z).$$

We define

$$g(u) := f_1(e^{2\pi i \frac{u}{\omega_1}}) f_2(e^{2\pi i \frac{u}{\omega_2}}).$$

Then $g \in \mathcal{QE}$. Indeed,

$$\begin{aligned} g(u + \omega_1) =& f_1\left(e^{2\pi i \frac{u}{\omega_1}}\right) f_2\left(e^{2\pi i \frac{u}{\omega_2}} e^{2\pi i \frac{\omega_1}{\omega_2}}\right) \\ =& f_1\left(e^{2\pi i \frac{u}{\omega_1}}\right) f_2\left(a_2 e^{2\pi i \frac{u}{\omega_2}}\right) \\ =& p f_1(e^{2\pi i \frac{u}{\omega_1}}) f_2(e^{2\pi i \frac{u}{\omega_2}}) = pg(u), \end{aligned}$$

and

$$\begin{aligned} g(u + \omega_2) =& f_1\left(e^{2\pi i \frac{u}{\omega_1}} e^{2\pi i \frac{\omega_2}{\omega_1}}\right) f_2\left(e^{2\pi i \frac{u}{\omega_2}}\right) \\ =& f_1\left(a_1 e^{2\pi i \frac{u}{\omega_1}}\right) f_2\left(e^{2\pi i \frac{u}{\omega_2}}\right) \\ =& q f_1(e^{2\pi i \frac{u}{\omega_1}}) f_2(e^{2\pi i \frac{u}{\omega_2}}) = qg(u). \end{aligned}$$

Vice versa, let $g \in \mathcal{QE}$, $p = 1$, $q \neq 1$, that is

$$g(u + \omega_1) = g(u), \quad g(u + \omega_2) = qg(u).$$

We denote

$$f(z) := g\left(\frac{\omega_1}{2i\pi} \log z\right). \tag{16}$$

The function f is well-defined since g admits the period ω_1 and therefore, the substitution of $\log z$ by $\log z + 2\pi i k$, $k \in \mathbb{Z}$ does not change the value of g in the right hand side of (16). In other words, here the composition of a multivalent mapping with a univalent one is a univalent function. Hence, if we let $a = e^{2\pi i \frac{\omega_2}{\omega_1}}$, $\operatorname{Im} \frac{\omega_2}{\omega_1} > 0$, we obtain

$$\begin{aligned} f(az) =& g\left(\frac{\omega_1}{2i\pi} \log(az)\right) = g\left(\omega_2 + \frac{\omega_1}{2i\pi} \log z\right) \\ =& qg\left(\frac{\omega_1}{2i\pi} \log z\right) = qf(z). \end{aligned}$$

Thus, $f \in \mathcal{L}_{aq}$. The case $p \neq 1$, $q = 1$ is similar. We let

$$f(z) := g\left(\frac{\omega_2}{2i\pi} \log z\right)$$

and $a = e^{2\pi i \frac{\omega_1}{\omega_2}}$. Then $f \in \mathcal{L}_{ap}$. Indeed,

$$\begin{aligned} f(az) =& g\left(\frac{\omega_2}{2i\pi} \log(az)\right) = g\left(\omega_1 + \frac{\omega_2}{2i\pi} \log z\right) \\ =& pg\left(\frac{\omega_2}{2i\pi} \log z\right) = pf(z). \end{aligned}$$

In the case $p \neq 1$, $q \neq 1$ the functions $g\left(\frac{\omega_k}{2i\pi} \log z\right)$ are multivalent, $k = 1, 2$.

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Andriy Yaroslavovych Khrystiyanyn,
Ivan Franko National University of Lviv,
Universytetska str., 1,
79000, Lviv, Ukraine
E-mail: khrystiyanyn@ukr.net

Dzvenyslava Volodymyrivna Lukivska,
Ivan Franko National University of Lviv,
Universytetska str., 1,
79000, Lviv, Ukraine
E-mail: d.lukivska@gmail.com