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PAVLOV-KOREVAAR-DIXON INTERPOLATION PROBLEM WITH MAJORANT IN CONVERGENCE CLASS

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Dedicated to the centenary of corresponding member of AS USSR Alexey Fedorovich Leontiev

Abstract. We study an interpolation problem in the class of entire functions of exponential type determined by some majorant in a convergence class (non-quasianalytic majorant). In a smaller class, when the majorant possessed a concavity property, similar problem was studied by B. Berndtsson with the nodes at some subsequence of natural numbers. He obtained a solvability criterion for this interpolation problem. At that, he applied first the Hörmander method for solving a $\overline{\partial}$-problem. In works by A.I. Pavlov, J. Korevaar and M. Dixon, interpolation sequences in the Berndtsson sense were applied successfully in a series of problems in the complex analysis. At that, there was found a relation with approximative properties of the system of powers $\{z^{p_n}\}$ and with the well known Polya and Macintyre problems.

In this paper we establish the criterion of the interpolation property in a more general sense for an arbitrary sequence of real numbers. In the proof of the main theorem we employ a modification of the Berndtsson method.

Keywords: interpolation sequence, entire function, convergence class.

Mathematics Subject Classification: 30E05

1. Introduction

Let L be a class of all continuous on $\mathbb{R}_+$ functions $l = l(x)$ such that $0 < l(x) \uparrow \infty$ as $x \to \infty$,

$$W = \left\{ w \in L : \int_1^\infty \frac{w(x)}{x^2} dx < \infty \right\}, \qquad \Omega = \left\{ \omega \in W : \frac{\omega(x)}{x} \downarrow \text{ as } x \to \infty \right\}.$$

The set W is called the convergence class, while the functions w in W are (non-quasi-analytic) weights.

Definition 1 ([1]). *Let $\{p_n\}$ be an increasing sequence of natural numbers. The sequence $\{p_n\}$ is called interpolating in Pavlov-Korevaar-Dixon sense if the exists a function $\omega \in \Omega$ depending only on the sequence $\{p_n\}$ such that for each sequence $\{b_n\}$ of complex numbers $|b_n| \leqslant 1$ there exists an entire function f possessing the properties:*

$$1)\ f(p_n) = b_n \quad (n \geqslant 1), \qquad 2)\ M_f(r) = \max_{|z| \leqslant r} |f(z)| \leqslant e^{\omega(r)}.$$

Let $\Lambda = \{\lambda_n\}$ be an arbitrary sequence of real numbers, $0 < \lambda_n \uparrow \infty$. The sequence Λ is called interpolating if there exists a function $w \in W$ depending only on this sequence such that for each sequence $\{b_n\}$ of complex numbers $|b_n| \leqslant 1$ there exists an entire function f possessing properties 1) and 2) but with the function w.

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The necessary and sufficient conditions for the interpolation property for a sequence $\{p_n\}$ ($p_n \in \mathbb{N}$) in the class Ω were obtained in work [1]. The aim of the present paper is to prove a criterion of the interpolation property for a sequence $\Lambda = \{\lambda_n\}$ in the class of functions W.

2. Auxiliary statements

Let

$$n(t) = \sum_{\lambda_n \leqslant t} 1$$

be the counting function of a sequence Λ and

$$N(t) = \int_0^t \frac{n(x)}{x} dx.$$

Without loss of generality we assume that $\lambda_1 = 1$. This slightly simplifies the further calculations.

The following lemma holds true.

Lemma 1. *Let* $\tau_n = \min\limits_{\substack{k \neq n \\ k \geqslant 1}} |\lambda_n - \lambda_k|$, $h_n = \min(\tau_n, 1)$,

$$K_n = \left\{ \xi : \frac{h_n}{4} \leqslant |\xi - \lambda_n| \leqslant \frac{h_n}{2} \right\} \quad (n \geqslant 1).$$

Then the estimates

$$1) \ \sup_{k \neq n} \left| \ln \left| \frac{\lambda_k - z}{\lambda_k - \lambda_n} \right| \right| \leqslant \ln 2; \qquad 2) \ \sup_{k} \left| \ln \left| \frac{\lambda_k + z}{\lambda_k + \lambda_n} \right| \right| \leqslant \ln \frac{4}{3}$$

hold true in the annuli K_n.

Proof. Let $z \in K_n$. We have

$$\left| \frac{\lambda_k - z}{\lambda_k - \lambda_n} \right| = \left| 1 + \frac{\lambda_n - z}{\lambda_k - \lambda_n} \right| \quad (k \neq n).$$

Since $|\lambda_n - z| \leqslant \frac{h_n}{2}$ for $z \in K_n$, $|\lambda_k - \lambda_n| \geqslant h_n$ $(k \neq n)$, then

$$\frac{1}{2} \leqslant \left| \frac{\lambda_k - z}{\lambda_k - \lambda_n} \right| \leqslant \frac{3}{2}.$$

Therefore,

$$-\ln 2 \leqslant \ln \left| \frac{\lambda_k - z}{\lambda_k - \lambda_n} \right| \leqslant \ln \frac{3}{2}$$

and

$$\sup_{k \neq n} \left| \ln \left| \frac{\lambda_k - z}{\lambda_k - \lambda_n} \right| \right| \leqslant \ln 2.$$

In the same way we are going to check 2). We have

$$\left| \frac{\lambda_k + z}{\lambda_k + \lambda_n} \right| = \left| 1 + \frac{z - \lambda_n}{\lambda_k + \lambda_n} \right|.$$

Since

$$\left| \frac{z - \lambda_n}{\lambda_k + \lambda_n} \right| \leqslant \frac{h_n}{2(\lambda_k + \lambda_n)} \leqslant \frac{1}{4},$$

then

$$\frac{3}{4} \leqslant \left| \frac{\lambda_k + z}{\lambda_k + \lambda_n} \right| \leqslant \frac{5}{4}.$$

Hence,

$$\sup_k \left|\ln\left|\frac{\lambda_k+z}{\lambda_k+\lambda_n}\right|\right| \leqslant \ln\frac{4}{3}.$$

The proof is complete. □

The next lemma holds.

Lemma 2. *For each z in K_n $(n \geqslant 1)$ we have*

$$\left|\ln\left|1-\frac{z^2}{\lambda_n^2}\right|\right| \leqslant \ln 10 + |\ln h_n| + \ln\lambda_n.$$

Proof. We have

$$\ln\left|1-\frac{z^2}{\lambda_n^2}\right| = \ln\left|1+\frac{z}{\lambda_n}\right| + \ln\left|\frac{\lambda_n - z}{\lambda_n}\right|.$$

Since $\operatorname{Re} z > 0$ for each $z \in K_n$ $(n \geqslant 1)$ and $\lambda_1 = 1$, then

$$0 < \ln\left|1+\frac{z}{\lambda_n}\right| \leqslant \ln\left(1+\frac{\lambda_n+\frac{h_n}{2}}{\lambda_n}\right) \leqslant \ln\frac{5}{2}.$$

Then

$$\ln\frac{h_n}{4\lambda_n} \leqslant \ln\left|\frac{\lambda_n - z}{\lambda_n}\right| \leqslant \ln\frac{h_n}{2\lambda_n} < 0.$$

Therefore,

$$\left|\ln\left|\frac{\lambda_n - z}{\lambda_n}\right|\right| \leqslant \left|\ln\frac{h_n}{4\lambda_n}\right| \leqslant |\ln h_n| + \ln 4\lambda_n \quad (n \geqslant 1).$$

Thus,

$$\left|\ln\left|1-\frac{z^2}{\lambda_n^2}\right|\right| \leqslant \ln 10 + |\ln h_n| + \ln\lambda_n$$

for each $z \in K_n$ $(n \geqslant 1)$. This proves the desired estimate. □

Assume that the sequence Λ has a finite upper density

$$\overline{\lim_{n\to\infty}} \frac{n}{\lambda_n} = \tau < \infty.$$

Then

$$q(z) = \prod_{n=1}^{\infty}\left(1-\frac{z^2}{\lambda_n^2}\right)$$

is an entire function of exponential type.

Let us estimate the function $\ln|q(z)|$ in the annuli K_n. For each fixed $n \geqslant 1$ we obtain

$$\begin{aligned}\ln|q(z)| = {}& \ln\left|1-\frac{z^2}{\lambda_n^2}\right| + \sum_{|\lambda_k-\lambda_n|\leqslant\lambda_n} \ln\left|1-\frac{z}{\lambda_k}\right| + \sum_{|\lambda_k-\lambda_n|\leqslant\lambda_n} \ln\left|1+\frac{z}{\lambda_k}\right| \\ &+ \sum_{|\lambda_k-\lambda_n|>\lambda_n} \ln\left|1-\frac{z^2}{\lambda_k^2}\right| = \ln\left|1-\frac{z^2}{\lambda_n^2}\right| + \Sigma_1+\Sigma_2+\Sigma_3;\end{aligned} \tag{1}$$

in the sums Σ_i $(i=1,2,3)$ we assume that $\lambda_k \neq \lambda_n$.

Let us estimate the sum Σ_1. For $\lambda_k \neq \lambda_n$ we have

$$\ln\left|1-\frac{z}{\lambda_k}\right| = \ln\left|\frac{\lambda_k - z}{\lambda_k}\right| = \ln\frac{1}{\lambda_k} + \ln|\lambda_k - z| = \ln\frac{1}{\lambda_k} + \ln\left|\frac{\lambda_k - z}{\lambda_k-\lambda_n}\right| + \ln|\lambda_k-\lambda_n|. \tag{2}$$

Then

$$\sum_{|\lambda_k-\lambda_n|\leqslant\lambda_n} \ln\frac{1}{\lambda_k} = \int_0^{2\lambda_n} \ln\frac{1}{t}dn_1(t) \quad (k\neq n), \tag{3}$$

where $n_1(t)$ is the counting function of the sequence $\Lambda_1=\Lambda\setminus\{\lambda_n\}$. Integrating by parts, by (3) we obtain

$$\sum_{|\lambda_k-\lambda_n|\leqslant\lambda_n} \ln\frac{1}{\lambda_k} = N_1(2\lambda_n) - n_1(2\lambda_n)\ln 2\lambda_n, \tag{4}$$

where

$$N_1(t)=\int_0^t \frac{n_1(x)}{x}dx.$$

Let us calculate the sum $\sum\limits_{|\lambda_k-\lambda_n|\leqslant\lambda_n} \ln|\lambda_k-\lambda_n|$ $(k\neq n)$. In order to do it, we observe that

$$\sum_{|\lambda_k-\lambda_n|\leqslant\lambda_n} \ln|\lambda_k-\lambda_n| = \int_0^{\lambda_n} \ln t\, d\nu(\lambda_n;\ t) \quad (k\neq n), \tag{5}$$

where $\nu(\lambda_n;\ t)$ is the amount of the points $\lambda_k\neq\lambda_n$ in the segment $\{h: |h-\lambda_n|\leqslant t\}$. Integrating by parts in Stieltjes integral (5), we write the latter identity as

$$\sum_{|\lambda_k-\lambda_n|\leqslant\lambda_n} \ln|\lambda_k-\lambda_n| = \nu(\lambda_n;\ \lambda_n)\ln\lambda_n - \int_0^{\lambda_n}\frac{\nu(\lambda_n;\ t)}{t}dt. \tag{6}$$

Taking into consideration relations (2), (4), (6), we obtain that

$$\Sigma_1 = N_1(2\lambda_n) - n_1(2\lambda_n)\ln 2 - \int_0^{\lambda_n}\frac{\nu(\lambda_n;\ t)}{t}dt + M_n^-, \tag{7}$$

where

$$M_n^- = \sum_{|\lambda_k-\lambda_n|\leqslant\lambda_n} \ln\left|\frac{\lambda_k-z}{\lambda_k-\lambda_n}\right| \quad (k\neq n).$$

We proceed to estimating Σ_2. Since

$$\ln\left|1+\frac{z}{\lambda_k}\right| = \ln\left(1+\frac{\lambda_n}{\lambda_k}\right) + \ln\left|\frac{\lambda_k+z}{\lambda_k+\lambda_n}\right|,$$

then

$$\Sigma_2 = \sum_{|\lambda_k-\lambda_n|\leqslant\lambda_n} \ln\left(1+\frac{\lambda_n}{\lambda_k}\right) + M_n^+, \tag{8}$$

where

$$M_n^+ = \sum_{|\lambda_k-\lambda_n|\leqslant\lambda_n} \ln\left|\frac{\lambda_k+z}{\lambda_k+\lambda_n}\right|.$$

But

$$\begin{aligned} 0 < \sum_{|\lambda_k-\lambda_n|\leqslant\lambda_n} \ln\left(1+\frac{\lambda_n}{\lambda_k}\right) &= \int_0^{2\lambda_n} \ln\left(1+\frac{\lambda_n}{t}\right)dn_1(t) \\ &= n_1(2\lambda_n)\ln\frac{3}{2} + \lambda_n\int_0^{2\lambda_n}\frac{n_1(t)}{t(t+\lambda_n)}dt \leqslant n_1(2\lambda_n)\ln\frac{3}{2} + N_1(2\lambda_n). \end{aligned} \tag{9}$$

Therefore, by (8), (9) we obtain that
$$|\Sigma_2| \leqslant n_1(2\lambda_n)\ln\frac{3}{2} + N_1(2\lambda_n) + |M_n^+|.$$
Then, in view of Lemma 1, we get
$$|\Sigma_2| \leqslant n_1(2\lambda_n)\ln\frac{3}{2} + N_1(2\lambda_n) + n_1(2\lambda_n)\ln\frac{4}{3},$$
that is,
$$|\Sigma_2| \leqslant n_1(2\lambda_n)\ln 2 + N_1(2\lambda_n). \tag{10}$$
It remains to estimate Σ_3. Since as $z \in K_n$, in this sum we have
$$\frac{|z|}{t} \leqslant \frac{\lambda_n + \frac{h_n}{2}}{2\lambda_n} = \frac{1}{2} + \frac{h_n}{4\lambda_n} \leqslant \frac{3}{4},$$
then $1 - \frac{r^2}{t^2} > 0$, and
$$\Sigma_3 = \int\limits_{2\lambda_n}^{\infty} \ln\left|1 - \frac{z^2}{t^2}\right| dn(t) \geqslant \int\limits_{2\lambda_n}^{\infty} \ln\left(1 - \frac{r^2}{t^2}\right) dn(t),$$
where $r = |z|$, $n(t) = \sum\limits_{\lambda_k \leqslant t} 1$. This implies that
$$\Sigma_3 \geqslant -n(2\lambda_n)\ln\left(1 - \frac{r^2}{4\lambda_n^2}\right) - 2r^2\int\limits_{2\lambda_n}^{\infty}\frac{n(t)}{t(t^2-r^2)}dt.$$
The expression is positive and neglecting it, we have
$$\Sigma_3 \geqslant -2r^2\int\limits_{2\lambda_n}^{\infty}\frac{n(t)}{t(t^2-r^2)}dt.$$
On the other hand,
$$\Sigma_3 \leqslant \int\limits_{2\lambda_n}^{\infty} \ln\left(1 + \frac{r^2}{t^2}\right) dn(t).$$
Since the substitution is negative, in the same way we obtain
$$\Sigma_3 \leqslant 2r^2\int\limits_{2\lambda_n}^{\infty}\frac{n(t)}{t(t^2+r^2)}dt.$$
This is why, finally,
$$|\Sigma_3| \leqslant 2r^2\int\limits_{2\lambda_n}^{\infty}\frac{n(t)}{t(t^2-r^2)}dt. \tag{11}$$
Since
$$r \leqslant \lambda_n + \frac{1}{2} \leqslant 2\lambda_n,$$
then
$$\frac{2r^2}{t^2-r^2} \leqslant 4\frac{2\lambda_n^2}{t^2+\lambda_n^2}\frac{t^2+\lambda_n^2}{t^2-r^2}.$$
Since
$$r \leqslant \lambda_n + \frac{1}{2} \leqslant \frac{t+1}{2},$$

then

$$t^2 - r^2 \geqslant t(t-r) \geqslant \frac{t(t-1)}{2}.$$

Therefore, taking into consideration the inequality $\lambda_n \leqslant \frac{t}{2}$, we obtain

$$\frac{t^2+\lambda_n^2}{t^2-r^2} \leqslant \frac{5}{2}\frac{t}{t-1} \leqslant \frac{5}{2}\frac{2\lambda_n}{2\lambda_n-1} \leqslant 5.$$

Thus, by (11) we finally obtain

$$|\Sigma_3| \leqslant 40\lambda_n^2 \int\limits_{2\lambda_n}^{\infty} \frac{n(t)}{t(t^2+\lambda_n^2)}dt \leqslant 20 \ln M_q(\lambda_n), \tag{12}$$

where $M_q(r) = \max\limits_{|z|=r} |q(z)|$.

In view of (1), (7) we write

$$\ln|q(z)| = \ln\left|1 - \frac{z^2}{\lambda_n^2}\right| - \int\limits_0^{\lambda_n} \frac{\nu(\lambda_n;\ t)}{t}dt + N_1(2\lambda_n) - n_1(2\lambda_n)\ln 2 + M_n^- + \Sigma_2 + \Sigma_3, \quad z \in K_n.$$

Therefore, for $z \in K_n$,

$$\left|\ln\frac{1}{|q(z)|} - \int\limits_0^{\lambda_n} \frac{\nu(\lambda_n;\ t)}{t}dt\right| \leqslant \left|\ln\left|1 - \frac{z^2}{\lambda_n^2}\right|\right| + N_1(2\lambda_n) + n_1(2\lambda_n)\ln 2 + |M_n^-| + |\Sigma_2| + |\Sigma_3|.$$

Hence, by Lemmata 1, 2 and estimates (10), (12) we finally get that

$$\left|\ln\frac{1}{|q(z)|} - \int\limits_0^{\lambda_n} \frac{\nu(\lambda_n;\ t)}{t}dt\right| \leqslant \ln 10 + |\ln h_n| + \ln\lambda_n + 2N_1(2\lambda_n) + n_1(2\lambda_n)\ln 8 + 20\ln M_q(\lambda_n).$$

We summarize the above obtained facts as the following theorem.

Theorem 1. *Assume that a sequence* $\Lambda = \{\lambda_n\}$ $(1 = \lambda_1 < \lambda_n \uparrow \infty)$ *has a finite upper density,* $h_n = \min(\min\limits_{k\neq n}|\lambda_k - \lambda_n|, 1)$,

$$q(z) = \prod_{k=1}^{\infty}\left(1 - \frac{z^2}{\lambda_k^2}\right).$$

Then in the annuli

$$K_n = \left\{\xi : \frac{h_n}{4} \leqslant |\xi - \lambda_n| \leqslant \frac{h_n}{2}\right\}$$

the estimate

$$\left|\ln\frac{1}{|q(z)|} - \int\limits_0^{\lambda_n} \frac{\nu(\lambda_n;\ t)}{t}dt\right| \leqslant m(\lambda_n)$$

holds, where $\nu(\lambda_n;\ t)$ *is the amount of the points* $\lambda_k \neq \lambda_n$ *in the segment* $\{h : |h - \lambda_n| \leqslant t\}$ *and*

$$m(\lambda_n) = \ln 10 + \ln\lambda_n + |\ln h_n| + n(2\lambda_n)\ln 8 + 2N(2\lambda_n) + 20\ln M_q(\lambda_n).$$

Corollary 1. *If* $\sum\limits_{n=1}^{\infty}\frac{1}{\lambda_n} < \infty$ *and*

$$|\ln h_n| \leqslant w_1(\lambda_n) \quad (n \geqslant 1)$$

for some function $w_1 \in W$, then

$$\left|\ln \frac{1}{|q(z)|} - \int_0^r \frac{\nu(z;\ t)}{t} dt\right| \leqslant w_2(r)$$

for $z \in K_n$, where w_2 is some function in W.

We have employed the well-known fact that the convergence of the series $\sum_{n=1}^{\infty} \frac{1}{\lambda_n}$ is equivalent to the convergence of the integrals [2], [3]:

$$\int_1^\infty \frac{n(r)}{r^2} dr, \qquad \int_1^\infty \frac{N(r)}{r^2} dr, \qquad \int_1^\infty \frac{\ln M_q(r)}{r^2} dr.$$

Let us make a remark. Since

$$-\ln \prod_{\substack{\frac{\lambda_n}{2} \leqslant \lambda_k \leqslant 2\lambda_n, \\ k \neq n}} \left|1 - \frac{\lambda_n}{\lambda_k}\right| = - \sum_{|\lambda_k - \lambda_n| \leqslant \lambda_n} \ln \left|1 - \frac{\lambda_n}{\lambda_k}\right| + \sum_{\lambda_k \leqslant \frac{\lambda_n}{2}} \left|1 - \frac{\lambda_n}{\lambda_k}\right| = -\Sigma_1(\lambda_n) + A,$$

then for $\Sigma_1(\lambda_n)$ and A the relations hold:

$$0 \leqslant A = \sum_{\lambda_k \leqslant \frac{\lambda_n}{2}} \ln \left(\frac{\lambda_n}{\lambda_k} - 1\right) \leqslant \sum_{\lambda_k \leqslant \frac{\lambda_n}{2}} \ln \left(1 + \frac{\lambda_n^2}{\lambda_k^2}\right) \leqslant \ln M_q(\lambda_n),$$

$$\Sigma_1(\lambda_n) = - \int_0^{\lambda_n} \frac{\nu(\lambda_n;\ t)}{t} dt + N_1(2\lambda_n) - n_1(2\lambda_n) \ln 2.$$

Hence, the following lemma holds.

Lemma 3. *The estimate*

$$\left| -\ln \prod_{\substack{k \neq n \\ \frac{\lambda_n}{2} \leqslant \lambda_k \leqslant 2\lambda_n}} \left|1 - \frac{\lambda_n}{\lambda_k}\right| - \int_0^{\lambda_n} \frac{\nu(\lambda_n;\ t)}{t} dt \right| \leqslant n(2\lambda_n) + N(2\lambda_n) + \ln M_q(\lambda_n)$$

holds true, where $\nu(\lambda_n;\ t)$ is the amount of the points $\lambda_k \neq \lambda_n$ in the segment $\{h : |h - \lambda_n| \leqslant t\}$.

In what follows we make use of the next lemma.

Lemma 4. *Let $w \in W$. Then the function $v(z) = w^*(|z|)$, where*

$$w^*(r) = \int_1^\infty \ln \left(1 + \frac{r^2}{t^2}\right) dw(t), \quad r = |z|,$$

is subharmonic in $\mathbb{C}$.

Proof. We observe that

$$v(z) \geqslant u(z) \equiv \int_1^\infty \ln \left|1 - \frac{z^2}{t^2}\right| dw(t),$$

and u is a subharmonic in $\mathbb{C}$ function, see, for instance, [4]. We take an arbitrary point $z_0 \in \mathbb{C}$ and choose w_0 in the imaginary axis so that $|w_0| = |z_0|$. Since $z_0 = w_0 e^{-\alpha i}$, then

$$\frac{1}{2\pi} \int_0^{2\pi} v(z_0 + \rho e^{i\varphi}) d\varphi = \frac{1}{2\pi} \int_\alpha^{2\pi + \alpha} v\left[e^{-\alpha i}(w_0 + \rho e^{i\psi})\right] d\psi,$$

$\psi = \varphi + \alpha$. Since the function $f(\psi) = v\left[e^{-\alpha i}(w_0 + \rho e^{i\psi})\right]$ is 2π-periodic and $v(z) = v(|z|)$, we have

$$\frac{1}{2\pi}\int_0^{2\pi} v(z_0 + \rho e^{i\varphi})d\varphi = \frac{1}{2\pi}\int_0^{2\pi} v(w_0 + \rho e^{i\psi})d\psi.$$

For each $\rho > 0$ (u is subharmonic in $\mathbb{C}$)

$$\frac{1}{2\pi}\int_0^{2\pi} v(w_0 + \rho e^{i\psi})d\psi \geqslant \frac{1}{2\pi}\int_0^{2\pi} u(w_0 + \rho e^{i\psi})d\psi \geqslant u(w_0) = v(z_0).$$

This implies the subharmonicity of the function v. □

3. Criterion of interpolation property for sequence Λ

Let $\Lambda = \{\lambda_n\}$, $0 < \lambda_n \uparrow \infty$, $\overline{\lim\limits_{n\to\infty}} \frac{n}{\lambda_n} = \tau < \infty$.

The next theorem holds.

Theorem 2. *The sequence Λ is interpolating if and only if there exists a function $w \in W$ such that*

$$a)\ \sum_{n=1}^{\infty} \frac{1}{\lambda_n} < \infty; \qquad b)\ -\ln \prod_{\substack{\frac{\lambda_n}{2} < \lambda_k < 2\lambda_n \\ k \neq n}} \left|1 - \frac{\lambda_n}{\lambda_k}\right| \leqslant w(\lambda_n) \quad (n \geqslant 1).$$

We observe that Lemma 3 and conditions a), b) implies that

$$\ln \frac{1}{h_n} \leqslant w_0(\lambda_n) \quad (n \geqslant 1),$$

where $h_n = \min\left(\min\limits_{\substack{k\neq n\\ k\geqslant 1}} |\lambda_n - \lambda_k|, 1\right)$, w_0 is a some function in class W.

The proof of the sufficient condition in Theorem 2 is based on one theorem by Hörmander for $\overline{\partial}$-equation. Let us formulate this theorem.

Theorem 3. *Let $\varphi = \varphi(z)$ be a function subharmonic in $\mathbb{C}$, $g \in C^\infty(\mathbb{C})$. Then there exists a solution $u \in C^\infty(\mathbb{C})$ to equation $\frac{\partial u}{\partial \bar z} = g$ satisfying the condition*

$$\int_{\mathbb{C}} |u|^2 e^{-\varphi}(1 + |z|^2)^{-2} d\lambda \leqslant \int_{\mathbb{C}} |g|^2 e^{-\varphi} d\lambda \tag{13}$$

provided the right hand is finite; here λ is the Lebesgue measure.

Proof of Theorem 2. . We begin by proving the sufficient condition. In order to do it, we take a function $\psi \in C^\infty$ such that $\psi(z) = 1$ as $|z| < \frac{1}{4}$ and $\psi(z) = 0$ as $|z| > \frac{1}{2}$. We let

$$A(z) = \sum_{n=1}^{\infty} b_n \Psi_n(z - \lambda_n), \quad \Psi_n(z) = \psi\left(\frac{z}{h_n}\right),$$

$\{b_n\}$ is an arbitrary given sequence of complex numbers $|b_n| \leqslant 1$. Since $A(z) = b_k\Psi_k(z - \lambda_k)$ for $z \in B_k = \{z : |z - \lambda_k| < \frac{h_k}{2}\}$ and $A(z) = 0$ for z in the complement of the union of the circles B_n ($n \geqslant 1$), it is obvious that $A \in C^\infty$. Since $|\lambda_k - \lambda_n| \geqslant h_n$ as $k \neq n$, then $A(\lambda_k) = b_k\psi(0) = b_k$ ($k \geqslant 1$).

Let

$$\varphi(z) = 2\ln \prod_{n=1}^{\infty} \left|1 - \frac{z^2}{\lambda_n^2}\right| + v(z),$$

where v is a subharmonic function, which will be chosen later. Since the sequence Λ has a finite upper density, then $\prod\limits_n \left(1-\frac{z^2}{\lambda_n^2}\right)$ is an entire function of exponential type and φ is a subharmonic function.

We have

$$M_\varphi(r)=\max_{|z|=r}|\varphi(z)|\leqslant 2\ln\prod_{n=1}^{\infty}\left(1+\frac{r^2}{\lambda_n^2}\right)+M_v(r).$$

Then

$$\sum_{n=1}^{\infty}\ln\left(1+\frac{r^2}{\lambda_n^2}\right)=\int_0^{\infty}\ln\left(1+\frac{r^2}{t^2}\right)dn(t). \tag{14}$$

Integrating by parts Stieltjes integral (14) and taking into consideration that $\frac{n(t)}{t}\to 0$ as $t\to\infty$, we obtain:

$$\int_0^{\infty}\ln\left(1+\frac{r^2}{t^2}\right)dn(t)=2r^2\int_1^{\infty}\frac{n(t)}{t(t^2+r^2)}dt\equiv w_1(r).$$

Let us check $w_1\in W$. Indeed, letting $t=sr$, we have

$$w_1(r)=2\int_{1/r}^{\infty}\frac{n(sr)}{s(s^2+1)}ds,$$

and we see that w_1 is an increasing function. Since

$$\frac{1}{2}\int_1^{\infty}\frac{w_1(r)}{r^2}dr=\int_1^{\infty}\frac{n(t)}{t}\left(\int_1^{\infty}\frac{dr}{t^2+r^2}\right)dt\leqslant\frac{\pi}{2}\int_1^{\infty}\frac{n(t)}{t^2}dt<\infty,$$

then $w_1\in W$.

Let us construct a subharmonic function v so that the quantity $M_v(r)$, the maximum of the absolute value of the function v, can be bounded from above by some function in the class W and at that, the right hand side in (13) for $g=\frac{\partial A}{\partial\bar z}$ is finite.

Let

$$K_n=\left\{\xi:\frac{h_n}{4}<|\xi-\lambda_n|<\frac{h_n}{2}\right\}\quad (n\geqslant 1).$$

We note that the annuli K_n $(n\geqslant 1)$ are mutually disjoint. This is implied by the fact that

$$\frac{h_n}{2}+\frac{h_{n+1}}{2}\leqslant\lambda_{n+1}-\lambda_n\quad(n\geqslant 1).$$

We have

$$\begin{aligned}\int_{\mathbb{C}}\left|\frac{\partial A}{\partial\bar\xi}\right|^2e^{-\varphi}d\lambda=&\int_{\cap_n\{\xi:|\xi-\lambda_n|>\frac{h_n}{2}\}}\left|\frac{\partial A}{\partial\bar\xi}\right|^2e^{-\varphi}d\lambda\\&+\sum_{n=1}^{\infty}\int_{K_n}\left|\frac{\partial A}{\partial\bar\xi}\right|^2e^{-\varphi}d\lambda+\sum_{n=1}^{\infty}\int_{\{\xi:|\xi-\lambda_n|<\frac{h_n}{4}\}}\left|\frac{\partial A}{\partial\bar\xi}\right|^2e^{-\varphi}d\lambda.\end{aligned}\tag{15}$$

The first and last integral in the right hand side in identity (15) obviously vanish. Then

$$A(\xi)=b_n\psi\left(\frac{\xi-\lambda_n}{h_n}\right)\quad\text{for}\quad\xi\in K_n.$$

Assuming that $\psi = \psi(w, \overline{w})$, where $w = x + iy = \frac{\xi-\lambda_n}{h_n}$, we obtain:

$$\frac{\partial\psi}{\partial\bar{\xi}} = \frac{\partial\psi}{\partial\overline{w}}\overline{\left(\frac{\partial w}{\partial\xi}\right)} = \frac{\partial\psi}{\partial\overline{w}}\frac{1}{h_n}.$$

This implies

$$\left|\frac{\partial\psi}{\partial\bar{\xi}}\right| = \frac{1}{2h_n}\left|\frac{\partial\psi}{\partial x} + i\frac{\partial\psi}{\partial y}\right| \leqslant \frac{1}{h_n}\left|\frac{\partial\psi}{\partial x}\right|, \quad \xi \in K_n.$$

Since $|b_n| \leqslant 1$, then

$$\int\limits_{\mathbb{C}}\left|\frac{\partial A}{\partial\bar{\xi}}\right|^2 e^{-\varphi}d\lambda \leqslant C_1\sum_{n=1}^{\infty}T_n,$$

where

$$T_n = \frac{1}{h_n^2}\int\limits_{K_n} e^{-v(\xi)}\prod_{k=1}^{\infty}\left|1-\frac{\xi^2}{\lambda_k^2}\right|^{-2}d\lambda(\xi)$$

and $C_1 = \max\limits_{|x|\leqslant\frac{1}{2}}\left|\frac{\partial\psi}{\partial x}\right|^2$.

For each fixed n and $\xi \in K_n$ we have

$$p(\xi) = \prod_{k=1}^{\infty}\left|1-\frac{\xi^2}{\lambda_k^2}\right| = \prod_{\lambda_k\leqslant\frac{\lambda_n}{2}}\left|1-\frac{\xi^2}{\lambda_k^2}\right|\prod_{\substack{\frac{\lambda_n}{2}<\lambda_k<2\lambda_n\\ k\neq n}}\left|1-\frac{\xi^2}{\lambda_k^2}\right|\prod_{\lambda_k\geqslant 2\lambda_n}\left|1-\frac{\xi^2}{\lambda_k^2}\right|\left|1-\frac{\xi^2}{\lambda_n^2}\right|.$$

Since $\operatorname{Re}\xi > 0$ for $\xi \in K_n$ $(n \geqslant 1)$, then

$$\left|1+\frac{\xi}{\lambda_k}\right| \geqslant 1. \tag{16}$$

As $\lambda_k \leqslant \frac{\lambda_n}{2}$,

$$\left|1-\frac{\xi^2}{\lambda_k^2}\right| \geqslant \frac{|\xi|^2}{\lambda_k^2} - 1 \geqslant 4\left[1-\frac{1}{2\lambda_n}\right]^2 - 1 \geqslant 1 \tag{17}$$

for $\lambda_n \geqslant 1 + \frac{1}{\sqrt{2}}$, that is, as $n \geqslant n_0$. Taking into consideration estimates (16), (17), we obtain that for $\xi \in K_n$, $n \geqslant n_0$, the inequality

$$p(\xi) \geqslant \prod_{\substack{\frac{\lambda_n}{2}<\lambda_k<2\lambda_n\\ k\neq n}}\left|1-\frac{\xi}{\lambda_k}\right|\prod_{\lambda_k\geqslant 2\lambda_n}\left|1-\frac{\xi^2}{\lambda_k^2}\right|\left|1-\frac{\xi^2}{\lambda_n^2}\right| \tag{18}$$

holds. Applying Lemma 1, for $\xi \in K_n$ $(n \geqslant 1)$ we have

$$\left|1-\frac{\xi}{\lambda_k}\right| = \left|1-\frac{\lambda_n}{\lambda_k}\right|\frac{|\xi-\lambda_k|}{|\lambda_n-\lambda_k|} \geqslant \frac{1}{2}\left|1-\frac{\lambda_n}{\lambda_k}\right| \quad (k \neq n). \tag{19}$$

Let us estimate the quantity $\left|1-\frac{\xi^2}{\lambda_n^2}\right|$ for $\xi \in K_n$:

$$\left|1-\frac{\xi^2}{\lambda_n^2}\right| \geqslant \frac{h_n}{4}\frac{|\xi+\lambda_n|}{\lambda_n^2} \geqslant \frac{h_n}{4\lambda_n}.$$

Above conditions a) and b) imply that

$$\frac{1}{h_n} \leqslant e^{w_0(\lambda_n)} \quad (n \geqslant 1),$$

where w_0 is some function in the class W. Therefore, for $\xi \in K_n$ $(n \geqslant 1)$, we get

$$\left|1-\frac{\xi^2}{\lambda_n^2}\right| \geqslant e^{-w_2(\lambda_n)}, \quad w_2 \in W. \tag{20}$$

The required estimate in terms of the function W is easily implied by Conditions a) and b) if we take into consideration (19). It remains to estimate the product $\prod\limits_{\lambda_k \geqslant 2\lambda_n} \left|1 - \frac{\xi^2}{\lambda_k^2}\right|$. We have[1]:

$$\ln \prod_{\lambda_k \geqslant 2\lambda_n} \left|1 - \frac{\xi^2}{\lambda_k^2}\right| = \int\limits_{2\lambda_n}^{\infty} \ln\left|1 - \frac{\xi^2}{t^2}\right| dn(t) \geqslant -C_2 \int\limits_{2\lambda_n}^{\infty} \frac{\lambda_n^2}{t^2} dn(t) \geqslant -2C_2\lambda_n^2 \int\limits_{2\lambda_n}^{\infty} \frac{n(t)}{t^3} dt. \qquad (21)$$

Let

$$w_3(r) \equiv r^2 \int\limits_{2r}^{\infty} \frac{n(t)}{t^3} dt = \int\limits_{2}^{\infty} \frac{n(sr)}{s^3} ds.$$

We have

$$\int\limits_{1}^{\infty} \frac{w_3(r)}{r^2} dr = \int\limits_{1}^{\infty} \frac{1}{r^2} \left(\int\limits_{2}^{\infty} \frac{n(sr)}{s^3} ds \right) dr = \int\limits_{2}^{\infty} \frac{1}{s^3} \left(\int\limits_{1}^{\infty} \frac{n(sr)}{r^2} dr \right) ds$$
$$= \int\limits_{2}^{\infty} \frac{1}{s^2} \left(\int\limits_{s}^{\infty} \frac{n(x)}{x^2} dx \right) ds \leqslant \frac{1}{2} \int\limits_{1}^{\infty} \frac{n(x)}{x^2} dx < \infty.$$

Since $p(\xi) \geqslant \beta > 0$ on $\cup_{n \leqslant n_0} K_n$, by estimates (18)–(21), Conditions a), b) of the theorem we finally obtain that there exists a function $w_4 \in W$ such that for all $n \geqslant 1$

$$p(\xi) \geqslant e^{-w_4(\lambda_n)}, \quad \xi \in K_n. \qquad (22)$$

We let

$$w^*(r) = \int\limits_{1}^{\infty} \ln\left(1 + \frac{r^2}{t^2}\right) dw_4^*(t) + (w_4^*(1) + 1)\ln(1 + r^2),$$

where $w_4^* = w_4 + w_0$. Then, for some $C > 0$, the function $v(z) = Cw^*(|z|)$ is the sought one. Indeed, by Lemma 4, the function v is subharmonic in $\mathbb{C}$ and the quantity $M_v(r) = Cw^*(r)$, as we saw above for the function w_1, is a function in the class W.

It remains to show that $\sum\limits_{n=1}^{\infty} T_n < \infty$. Taking into consideration estimate (22) and the definition of the function v, we get

$$T_n \leqslant \frac{1}{h_n^2} \int\limits_{K_n} e^{-Cw^*(|\xi|) + 2w_4(\lambda_n)} d\lambda(\xi) \leqslant C_3 \exp\left[2w_0(\lambda_n) + 2w_4(\lambda_n) - Cw^*(\lambda_n - \frac{1}{2})\right],$$

where $C_3 = \frac{3}{16}\pi$. We observe that

$$w^*(r) = 2r^2 \int\limits_{1}^{\infty} \frac{w_4^*(t)}{t(t^2 + r^2)} dt + \ln(1 + r^2) \geqslant 2r^2 w_4^*(r) \int\limits_{r}^{\infty} \frac{dt}{t(t^2 + r^2)} \geqslant \frac{1}{2} w_4^*(r),$$

---

[1]Since $\frac{|\xi|^2}{t^2} \leqslant \left(\frac{\lambda_n + \frac{1}{2}}{2\lambda_n}\right)^2 \leqslant \left(\frac{1}{2} + \frac{1}{4\lambda_1}\right)^2 < \frac{2}{3}$, then

$$\ln\left|1 - \frac{\xi^2}{t^2}\right| \geqslant \ln\left(1 - \frac{|\xi|^2}{t^2}\right) \geqslant -3\frac{|\xi|^2}{t^2},$$

since the function $\varphi(\alpha) = \ln(1 - \alpha) + 3\alpha$ increases as $\alpha < \frac{2}{3}$. But $\frac{|\xi|}{\lambda_n} \leqslant \frac{3}{2}$ as $\xi \in K_n$ $(n \geqslant 1)$ and this is why

$$\ln\left|1 - \frac{\xi^2}{t^2}\right| \geqslant -C_2 \frac{\lambda_n^2}{t^2}, \quad C_2 = \frac{27}{4}.$$

and also

$$\frac{w^*(\lambda_n)}{w^*(\lambda_n - \frac{1}{2})} \leqslant M \quad (n \geqslant 1).$$

Therefore,

$$\sum_{n=1}^{\infty} T_n \leqslant C_3 \sum_{n=1}^{\infty} e^{-\frac{C}{M} w^*(\lambda_n) + 2w_4^*(\lambda_n)} \leqslant C_3 \sum_{n=1}^{\infty} e^{\left(-\frac{C}{M}+4\right) w^*(\lambda_n)}.$$

The definition of the function $w^*(r)$ implies that $w^*(r) \geqslant (w_4^*(1)+1)\ln(1+r^2)$, and this is why

$$\sum_{n=1}^{\infty} T_n \leqslant C_3 \sum_{n=1}^{\infty} \frac{1}{(1+\lambda_n^2)^{C_4}},$$

where $C_4 = \left(\frac{C}{M} - 4\right)(w_4^*(1)+1)$, C is the constant in the definition of the function v. Since the sequence $\Lambda = \{\lambda_n\}$ has a finite upper density, the latter series converges provided $2C_4 > 1$. The convergence is ensured by the inequality $C > 5M$.

As we have said above, $M_v(r) = Cw^*(r)$, $w^* \in W$. Hence,

$$M_\varphi(r) \leqslant w_5(r), \tag{23}$$

where $w_5 = 2w_1 + Cw^*$ is a function in the class W.

We are going to apply Theorem 3 for $g = \frac{\partial A}{\partial \bar z}$. Since the function φ is chosen so that $e^{-\varphi}$ has a non-integrable singularity at each point λ_n, we should have $u(\lambda_n) = 0$ $(n \geqslant 1)$.

Consider the equation

$$\frac{\partial u}{\partial \bar z} = \frac{\partial A}{\partial \bar z}, \quad u(\lambda_n) = 0 \ \ (n \geqslant 1). \tag{24}$$

We let $f = A - u$, where u is the solution to equation (24); this exists by Hörmander theorem. It is clear that f is an entire function and $f(\lambda_n) = b_n$ $(n \geqslant 1)$.

Since $|f|^2$ is subharmonic in the entire plane, then for each $\rho > 0$, and, in particular, as $1 \leqslant \rho \leqslant r$, we have [5, Ch. I, Sect. 6]:

$$|f(z)|^2 \leqslant \frac{1}{\pi\rho^2} \int\limits_{|\xi - z| \leqslant \rho} |f(\xi)|^2 d\lambda(\xi) < \int\limits_{|\xi| \leqslant 2r} |f(\xi)|^2 d\lambda(\xi), \quad r = |z|.$$

Since $|f|^2 \leqslant 2(|A|^2 + |u|^2)$, we have

$$\int\limits_{|z|\leqslant 2r} |f|^2 d\lambda \leqslant 2 \int\limits_{|z|\leqslant 2r} |A|^2 d\lambda + 2 \int\limits_{|z|\leqslant 2r} |u|^2 d\lambda \leqslant 8\pi r^2 + 2 \int\limits_{|z|\leqslant 2r} |u|^2 \frac{e^{-\varphi}}{(1+|z|^2)^2} (1+|z|^2)^2 e^{\varphi} d\lambda.$$

Applying estimate (13) from the Hörmander theorem to the latter integral, we obtain

$$\int\limits_{|z|\leqslant 2r} |f|^2 d\lambda \leqslant 8\pi r^2 + 2\exp\left\{2\ln(1+4r^2) + M_\varphi(2r)\right\} \int\limits_{\mathbb{C}} |g|^2 e^{-\varphi} d\lambda.$$

In view of the convergence of latter integral and estimate (23), we conclude that

$$|f(z)| \leqslant C_5 e^{w_6(|z|)},$$

where $w_6 \in W$. The latter means that the function $f = A - u$ solves the interpolation problem. The sufficient condition is proven.

Let us prove the necessary condition. Let $\Lambda = \{\lambda_n\}$ be an interpolation sequence and $\tilde w$ be a function in the class W, existence of which is stated in Definition 1. We first choose an entire function f solving the interpolation problem for $b_1 = 1$ and $b_n = 0$ $(n > 1)$. By the Jensen inequality and by property 2) of an interpolation sequence we obtain

$$n(r) \leqslant \ln M_f(er) \leqslant \tilde w(er).$$

As we have said in Section 1, the convergence of the integral

$$\int_1^\infty \frac{n(r)}{r^2}dr$$

is equivalent to the condition

$$\sum_{n=1}^\infty \frac{1}{\lambda_n} < \infty.$$

In order to prove Condition b), we fix n and choose the entire function f solving the interpolation problem for $b_n = 1$ and $b_k = 0$ $(k \neq n)$.

The representation

$$f(z) = \prod_{\substack{\frac{\lambda_n}{2} < \lambda_k < 2\lambda_n,\\ k\neq n}} \left(1 - \frac{z}{\lambda_k}\right) G(z) \tag{25}$$

holds true, where G is an entire function; if none of λ_k $(k \neq n)$ is in the interval $\left(\frac{\lambda_n}{2}, 2\lambda_n\right)$, we assume that $G = f$. For $\frac{\lambda_n}{2} < \lambda_k < 2\lambda_n$ we have

$$\left|1 - \frac{z}{\lambda_k}\right| \geqslant \left|1 - \frac{4\lambda_n}{\lambda_k}\right| \geqslant 1, \quad |z| = 4\lambda_n.$$

This implies that $|G(z)| \leqslant |f(z)|$, $|z| = 4\lambda_n$. By the maximum modulus principle,

$$|G(\lambda_n)| \leqslant M_G(4\lambda_n) \leqslant M_f(4\lambda_n) \leqslant e^{\tilde{w}(4\lambda_n)}. \tag{26}$$

On the other hand, it follows from (25) that

$$G(\lambda_n) = \prod_{\substack{\frac{\lambda_n}{2} < \lambda_k < 2\lambda_n,\\ n\neq k}} \left(1 - \frac{\lambda_n}{\lambda_k}\right)^{-1} \tag{27}$$

since $f(\lambda_n) = 1$. By relations (26), (27) we finally obtain

$$-\ln \prod_{\substack{\frac{\lambda_n}{2} < \lambda_k < 2\lambda_n,\\ k\neq n}} \left|1 - \frac{\lambda_n}{\lambda_k}\right| \leqslant \tilde{w}(4\lambda_n),$$

where $\tilde{w}$ is a function in the class W. The proof is complete. □

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