ISSN 2304-0122 Ufa Mathematical Journal. Vol. 9. No 3 (2017). P. 131-136.

doi:10.13108/2017-9-3-131 UDC 512.5

ON APPLICATIONS OF FAÀ-DI-BRUNO FORMULA

A.B. SHABAT, M.Kh. EFENDIEV

Abstract. In the work we construct two modifications of the classical Faà-di-Bruno formula. We consider the applications of these formulae in the integrability theory for nonlinear partial differential equations. We discuss the problem on integration by parts in the Gelfand-Olver-Sanders formal variational calculus.

Keywords: Faà-di-Bruno formula, differential polynomials, integrability conditions.

Mathematics Subject Classification: 37K10

1. Generalization of Faà di Bruno formula

A differentiation of a composition of functions $f[u(x), v(x), \ldots, z(x)]$ leads us to a nice formula published by Francesco Faà di Bruno published in 1857. The right hand side of an extended version for a derivative of order $n \geqslant 1$ w.r.t. the independent variable x of the composition of the functions

$$\begin{gathered}\frac{f^{(n)}}{n!} = \left(\sum_{k_1+2k_2+\ldots+nk_n=n} \frac{\hat{B}_1^{k_1} \hat{B}_2^{k_2} \cdots \hat{B}_n^{k_n}}{k_1! k_2! \cdots k_n!} \right) f(u, v, \ldots z), \\ \hat{B}_j \stackrel{\text{def}}{=} \frac{u_j}{j!} \frac{\partial}{\partial u} + \frac{v_j}{j!} \frac{\partial}{\partial v} + \ldots + \frac{z_j}{j!} \frac{\partial}{\partial z}\end{gathered} \tag{1.1}$$

is written here as a partial differential operator acting on the function f with the independent variables u, v, $\ldots$, z. The coefficients of the aforementioned linear operator are expressed in terms of the variables u, v, $\ldots$, z and in terms of polynomials on their x-derivatives denoted by u_j. In formula (1.1), these so-called *differential variables*

$$u_j \stackrel{\text{def}}{=} \frac{d^j u(x)}{dx^j}, \quad v_j \stackrel{\text{def}}{=} \frac{d^j v(x)}{dx^j}, \quad \ldots, \quad z_j \stackrel{\text{def}}{=} \frac{d^j z(x)}{dx^j}, \quad j = 1,\ 2, \ldots\ , n. \tag{1.2}$$

are considered as independent variables commuting one with another and with the differentiation operators $\partial_u, \partial_v, \ldots, \partial_z$. In particular, for two variables u, v, a function f and $n \leqslant 4$, formula (1.1) gives:

$$\begin{aligned}
f^{(1)} &= (\hat{B}_1) f = (u_1 \partial_u + v_1 \partial_v) f(u, v) = \frac{df[u(x), v(x)]}{dx}, \\
f^{(2)} &= \left[u_2 \partial_u + v_2 \partial_v + u_1^2 \partial_u^2 + 2u_1 v_1 \partial_u \partial_v + v_1^2 \partial_v^2)^2\right] f(u, v) = \left[2\hat{B}_2 + \hat{B}_1^2\right] f, \\
f^{(3)} &= \left(6\hat{B}_3 + 3\hat{B}_2 \hat{B}_1 + \hat{B}_1^3\right) f, \quad f^{(4)} = \left(4!\hat{B}_4 + 4!\hat{B}_3 \hat{B}_1 + 12\hat{B}_2^2 + 12\hat{B}_2 \hat{B}_1^2 + \hat{B}_1^4\right) f.
\end{aligned}$$

The number of different monomials in these polynomials $\hat{B}_1, \ldots, \hat{B}_n$ of n variables is determined by Diophantine equation

$$k_1 + 2k_2 + 3k_3 + \ldots + nk_n = n, \quad k_j \geqslant 0 \tag{1.3}$$

A.B. Shabat, M.Kh. Efendiev, On applications of Faà di Bruno.

Submitted April 8, 2017.

and it coincides with the number $p(n)$ of different partitions of the integer number $n \geqslant 1$. This is why [1]

$$1+\sum_{n=1}^{\infty} p(n)t^n = \prod_{j=1}^{\infty}(1-t^j)^{-1} = 1+t+2t^2+3t^3+5t^4+7t^5+11t^6+15t^7+22t^8+\ldots$$

The proof of Faà di Bruno formula (1.1) in the scalar case $f[u(x)]$ can be found in above cited monograph [1]. In fact, it is reduced to proving the formula $f^{(n+1)} = D_x(f^{(n)})$, where the first order linear differential operator

$$D_x = u_1\frac{\partial}{\partial u} + \sum_{j=1}^{\infty} u_{j+1}\frac{\partial}{\partial u_j}: \quad u \mapsto u_1 \mapsto u_2 \mapsto \ldots \mapsto u_k \mapsto \ldots \tag{1.4}$$

is applied to the terms in the right hand side of formula (1.1) and these terms depend not only on u but also on addition independent variables (1.2).

In order to justify the mentioned in (1.1) generalization of Faà di Bruno formula for the "vector case" $f[u(x), v(x), \ldots, z(x)]$, we observe that for the exponential functions $f = e^{(u(x),v(x),\ldots,z(x))}$ obeying

$$\log f = w = \alpha u + \beta v + \ldots + \gamma z, \tag{1.5}$$

the scalar and vector forms of considered formula (1.1) coincides. By the linearity property this proves the general formula for a linear combination of exponential functions of form (1.5) and hence, for the general case; for the straightforward calculations see [3].

2. Darboux chains

Let us discuss briefly the applicability of Faà di Bruno formula (1.1) to the problem on integrating a linear second order hyperbolic equation for a function $\psi = \psi(x, y)$

$$[D_xD_y + b(x, y)D_y + c(x, y)]\psi = 0 \tag{2.1}$$

by the theory of Darboux transforms.

Lemma. *Let $c(n) = c(n; x, y)$, $b(n) = b(n; x, y)$ satisfy a nonlinear chain of coupled equations*

$$D_x \log c(n) = b(n+1) - b(n), \quad D_y b(n) = c(n) - c(n-1), \quad n \in \mathbb{Z}. \tag{2.2}$$

Then

i) the following Darboux equations hold:

$$[\log c(n)]_{xy} = c(n+1) + c(n-1) - 2c(n), \quad n \in \mathbb{Z}, \tag{2.3}$$

ii) the formulae

$$(D_x + b(n))\psi(n) + \psi(n-1) = 0, \quad D_y\psi(n) = c(n)\psi(n+1), \tag{2.4}$$

map the solutions of the equation into the solutions (cf. (2.1)*):*

$$[D_xD_y + b(n)D_y + c(n)]\psi(n) = 0, \quad n \in \mathbb{Z} \tag{2.5}$$

and the operators in (2.4) *commute.*

Proof. Assume that equations (2.2) hold. Since the mixed derivatives coincide

$$D_xD_y(\log c(n)) = D_yD_x(\log c(n)) = b_y(n+1) - b_y(n) = c(n+1) + c(n-1) - 2c(n),$$

this give the chain of Darboux equations.

In order to prove ii), we introduce the "shift" operators: $T^{\pm 1}: \ \psi(n) \to \psi(n \pm 1)$. Using then the formulae

$$Tc(n) = c(n+1), \quad T^{-1}c(n)T = c(n-1), \quad [D_x, D_y] = [D_x, T] = [D_y, T] = 0,$$

we obtain

$$\begin{aligned}&[D_x + b(n) + T^{-1},\, D_y - c(n)T]\\&\quad = (-c_x(n) + c(n)b(n+1) - b(n)c(n))T + c(n) - c(n-1) - b_y(n) = 0.\end{aligned} \tag{2.6}$$

□

In the theory of Darboux transforms, one succeeds to relate the solvability of linear equations (2.1)) with conditions on closing the chain on nonlinear equations of form (2.3) for the coefficients of these linear equations, see [6]. For equations (2.2), the simplest conditions of such type correspond to the Dirichlet problem and vanishing of the function $c(n) = c(n; x, y)$ as $n \leqslant 0$ and $n > N \geqslant 1$. In the case $N = 1$ this gives the scalar equation

$$[\log c(1)]_{xy} + 2c(1) = 0$$

equivalent to the *Liouville equation*

$$\alpha u_{xy} + 2e^{\alpha u} = 0$$

up to the change $c(n) = e^{\alpha u}$. A specific property indicating the integrability of the Liouville equation is formulated for $\alpha = -2$ as the following equation:

$$D_y(u_2 + u_1^2) = 0, \qquad u_1 = u_x, \qquad u_2 = u_{xx}. \tag{2.7}$$

Here $u(x, y)$ is an arbitrary solution of the considered equation valid due to the Liouville equation and its differential implications.

In the general case of the exponential systems of the form[1]

$$[\log c(n)]_{xy} = \sum_{j=1}^{N} \alpha_{n,j} c(j), \quad n \in [N], \quad \det(\alpha_{nj}) \neq 0, \tag{2.8}$$

the problem on existence of "Liouville polynomials" similar to (2.7) was solved in preprint [9] in 1981. More precisely, it was shown in this work that the presence of sufficiently many differential relations of form (2.7) leads us to the Cartan matrices (α_{nj}) corresponding to semi-simple Lie algebras. It seems interesting to relate, in addition to [9] and work by A.N. Leznov [10], the Liouville polynomials[2] for exponential systems (2.8) with Faà di Bruno formula (1.1) and to consider a possibility of their non-commutative generalizations. We restrict ourselves by two simple examples.

The scalar equation $u_{xy} = F$ with arbitrary function $F = F(u)$ allows us to determined the operator D_y acting on the set of the polynomials of differential variables (1.2):

$$D_y = F\frac{\partial}{\partial u_1} + D_x(F)\frac{\partial}{\partial u_2} + \sum_{j>2}^{\infty} F^{(j-1)}\frac{\partial}{\partial u_j}, \quad F^{(n)} = D_x^n(F \circ u). \tag{2.9}$$

The coefficients of this operator (cf. (1.1)) considered on the solutions of the equation $u_{xy} = F(u)$ are uniquely determined by the commutation condition $D_x \circ D_y = D_y \circ D_x$ of vector field (1.4) and (2.9) associated with x and y differentiations, respectively. It is easy to see that the condition $D_y(u_2 + u_1^2) = 0$ gives $F' + 2F = 0$, that is, this leads us to the Liouville equation. One can check (see [8]) that the latter condition for the function $F(u)$ remain true in all cases of the solvability of the equation

$$D_y W(u_1, \ldots, u_m) = 0, \tag{2.10}$$

[1] "integrated" in well-known works by A.M. Leznov and M.V. Savel'ev

[2] constant for varying y

and that in the scalar case $u_{xy} = F(u)$, the general polynomial solution to equation (2.10) is generated a polynomial of the minimal order $W_2 = u_2 + u_1^2$ and is expressed in terms of the Bell polynomials (see [1]).

As it has been said, the Liouville equation corresponds to the Dirichlet problem with $N = 1$ for Darboux chain (2.3). As $N = 2$, we find

$$[\log c(1)]_{xy} = c(2) - 2c(1), \quad [\log c(2)]_{xy} = c(1) - 2c(2), \tag{2.11}$$

which corresponds to the simplest among the Cartan matrices with $\alpha_{ii} = -2$ at the main diagonal. Factorization (2.2) of these equations pointed out in Lemma leads us to the additional equations for $b(1)$, $b(2)$, $b(3)$:

$$\begin{cases} D_y(b(1)) = c(1), \quad D_y(b(2)) = c(2) - c(1), \quad D_y(b(3)) = -c(2); \\ D_x \log c(1) = b(2) - b(1), \quad D_x \log c(2) = b(3) - b(2); \\ D_y(W_0) = 0, \quad W_0 = b(1) + b(2) + b(3). \end{cases} \tag{2.12}$$

By these equations being true for Dirichlet problem (2.11) and as $N \geqslant 2$, we see that while constructing Liouville polynomials W_m in formula (2.10), as the differential variables u_j we can choose

$$b_j(n) = D_x^j b(n), \quad j = 1,\ 2, \dots, \quad n \in [N+1]. \tag{2.13}$$

In particular, as $N = 2$, by means of the equations in the upper line of formula (2.12), we determine the operator D_y acting on the set of the polynomials of differential varaibles (2.13) and then we find the solutions W_m to equation (2.10) as $m = 0,\ 1$:

$$W_0 = b(3) + b(2) + b(1), \quad W_1 = b^2(1) + b^2(2) + b^2(3) + 2b_1(2) + 4b_1(1),$$

and as $W_0 = 0$, we find the following Liouville polynomial W_2:

$$\begin{aligned} &D_y b_2(1) = c(1)\left[b_1(2) - b_1(1) + (b(2) - b(1))^2\right], \\ &D_y[b_2(1) + b_1(1)b(1) - b_1(2)b(1)] = c(1)\left[b^2(2) - b^2(1)\right] - c(2)\left[b(1)b(3) - b(1)b(2)\right], \\ &W_2 = b_2(1) + b_1(1)b(1) - b_1(2)b(1) + \frac{1}{3}[b^3(1) + b^3(2) + b^3(3)]. \end{aligned}$$

We note that the aforementioned condition $W_0 = 0$ allows us to get rid of an "extra" variable $b(3)$ and to equalize in this way the number of unknowns $b(j)$ and $c(j)$. As $N = 3$, in the same way we construct (as $W_0 = 0$) three independent solutions to (2.10) of order $m = 1,\ 2,\ 3$, and so forth.

3. Integrability condition and Gelfand-Sanders problem

Let $n' = \left\lceil \frac{n}{2} \right\rceil$ be the smallest integer greater than or equal to $\frac{n}{2}$. Collecting the terms with u_j, $j \in [n']$, in the scalar Faà di Bruno formula, we obtain

$$\begin{aligned} &f^{(n)} = u_n f_u + \binom{n}{1} u_{n-1} f_u f^{(1)} + \binom{n}{2} u_{n-2} f_u f^{(2)} \dots + \binom{n}{n'} u_{n'} f_u [f^{(n')} - \varepsilon(n) u_{n'} f_u] + \dots, \\ &\varepsilon(n) = \frac{1}{2} \begin{cases} 0, & n \text{ is odd}, \\ 1, & n \text{ is even}, \end{cases} \end{aligned} \tag{3.1}$$

where "..." in the end of the formula stands for the terms in formula (1.1) corresponding to cut Diophantine equation (1.3):

$$k_1 + 2k_2 + \dots + n'' k_{n''} = n, \quad n'' = \left\lfloor \frac{n}{2} \right\rfloor \leqslant \frac{n}{2}$$

and involves only differential variables (1.2) with the indices $j \in [n'']$. For instance, as $n = 5$, we have $n' = 3$, $n'' = 2$, and formula (3.1) is written as

$$f^{(5)} = \left(u_5 + \binom{5}{1} u_4 f^{(1)} + \binom{5}{2} u_3 f^{(2)}\right) f_u + 5! \left(\sum_{k_1+2k_2=5} \frac{\hat{B}_1^{k_1} \hat{B}_2^{k_2}}{k_1! k_2!}\right) f(u).$$

We do not dwell on generalization of formula (3.1) for the vector case but we note an obvious analogy with binomial coefficients of the Leibnitz formula

$$(uv)^{(n)} = u^{(n)} v + \binom{n}{1} u^{(n-1)} v^{(1)} + \binom{n}{2} u^{(n-2)} v^{(2)} + \ldots + uv^{(n)}$$

for differentiation of the product of two functions u and v.

In what follows we discuss the applications of formulae like (3.1) to the integration by parts of arbitrary functions $f[u]$, where $[u]$ denotes a finite set of differential variables (1.2). In order to do it, we extract a linear in higher derivatives part of the function $f[u]$ and compare it with aforementioned formula (3.1). The theoretical question of representing the function $f[u]$ as the derivative $f[u] = D_x g[u]$ is equivalent to vanishing of the "variational derivative":

$$\frac{\delta f}{\delta u} = \frac{\partial f}{\partial u} - D_x \left(\frac{\partial f}{\partial u_1}\right) + D_x^2 \left(\frac{\partial f}{\partial u_2}\right) - D_x^3 \left(\frac{\partial f}{\partial u_3}\right) + \ldots = 0. \tag{3.2}$$

However, in the applications, one usually needs to find the primitive $g[u]$ of $f[u] = D_x g[u]$ and to identify the obstacles for integrating. The problem on difficulties for integrating by parts based on generalized Leibnitz formula (3.1) is the base for the symmetric approach to the integrability of nonlinear equations with two independent variables; this approach was developed in 80s in Ufa [7], [11]. Here the simplest equation is the Liouville equation and as this was shown in pioneering work [8], extra two equations should be added:

$$u_{x,y} = F(u), \quad F(u) = e^u + \beta e^{\gamma u}, \quad \gamma = -1, \quad \gamma = -2. \tag{3.3}$$

The problem on third order evolution equations $u_t = F(u, u_1, u_2, u_3)$ integrable in the same sense as the well-known Korteweg-de Vries equations is among the problems, for which the obstacles for integrating are found explicitly. Some impression on variety of these generalized Korteweg-de Vries equations gives the following list:

Equations of Korteweg-de Vries type

$$u_t = u_3 + P(u) u_1, \quad P''' = 0, \tag{3.4}$$

$$u_t = u_3 - \frac{1}{2} u_1^3 + (\alpha e^{2u} + \beta e^{-2u}) u_1, \tag{3.5}$$

$$u_t = u_3 - \frac{3}{2} \frac{u_2^2}{u_1} + \frac{r(u)}{u_1}, \quad \frac{d^5 r(u)}{du^5} = 0. \tag{3.6}$$

The general problem on polynomial equations $u_t = F([u])$ of arbitrary order possessing higher symmetries was studied in work [5], in which another approach proposed by I.M. Gelfand was employed for integrating by parts. At that, an essential role was played by the homogeneity properties of the considered polynomials $F([u])$ and theorem on divisibility related with the number theory.

BIBLIOGRAPHY

1. L.Comtet. *Advanced combinatorics. The art of finite and infinite expansions.* Reidel Publishing Company, Dordrecht (1974).
2. Faà de Bruno. *Note sur un nouvelle formulae de calcul différentiel* // Quart. J. Pure Appl. Math, **1**, 359–360 (1857).
3. R. Mishkov. *Generalization of the formula of Faà di Bruno for a composite function with vector argument* // Internat. J. Math. **24**:7, 481–491 (2000).
4. A.V. Zhiber, R.D. Murtazina, I.T. Khabibullin, A.B. Shabat. *Characteristic Lie rings and nonlinear integrable equations.* Institute of Computer Stuides, Moscow (2012). (in Russian.)
5. J.Sanders, J.P.Wang. *On the integrability of homogeneous scalar evolution equations* // J. Diff. Equat. **147**:2, 410–434 (1998).
6. A.P. Veselov, A.B. Shabat. *Dressing chains and spectral theory of the Schrödinger operator* // Funkts. Anal. Pril. **27**:2, 1–21 (1993). [Funct. Anal. Appl. **27**:2, 81–96 (1993).]
7. A.B. Shabat, V.V. Sokolov. *Classification of integrable evolution equations* // Sov. Sci. Rev. Sect. C. Math. Phys. Rev. **4**, 221–280 (1984).
8. A.V. Zhiber, A.B. Shabat. *Klein-Gordon equations with a nontrivial group* // Dokl. AN SSSR. **247**:5, 1103–1107 (1979). [Sov. Phys. Dokl. **24**:8, 607–609 (1979).]
9. A. Shabat, R. Yamilov. *Exponential systems of type I and Cartan matrices.* Preprint. Ufa (1981). (in Russian).
10. A.N. Leznov. *On the complete integrability of a nonlinear system of partial differential equations in two-dimensional space* // Teor. Matem. Fiz. **42**:3, 343–349 (1980). [Theor. Math. Phys. **42**:3, 225–229 (1980).]
11. N.Kh. Ibragimov, A.B. Shabat. *Infinite Lie-Beklund algebras* // Funkts. Anal. Pril. **14**:4, 79–80 (1980). [Funct. Anal. Appl. **14**:4, 313–315 (1980).]

Alexey Borisovich Shabat,
L.D. Landau Institute for Theoretical Physics, RAS,
Academician Semenov av. 1-A,
142432, Chernogolovka, Russia,

Karachay-Circassian State University
named after U.D. Aliev,
Lenin str. 23,
369200, Karachaevsk, Russia

Institute of Mathematics, Ufa Scientific Center, RAS,
Chernyshevsky str. 112,
450077, Ufa, Russia
E-mail: `shabatab@mail.ru`

Magomed Khochalaevich Efendiev,
Karachay-Circassian State University
named after U.D. Aliev,
Lenin str. 23,
369200, Karachaevsk, Russia
E-mail: `kchr1927@gmail.ru`