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REPRESENTATION OF FUNCTIONS IN LOCALLY CONVEX SUBSPACES OF $A^\infty(D)$ BY SERIES OF EXPONENTIALS

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Abstract. Let D be a bounded convex domain in the complex plane, $\mathcal{M}_0 = (M_n)_{n=1}^\infty$ be a convex sequence of positive numbers satisfying the "non-quasi-analyticity" condition:

$$\sum_n \frac{M_n}{M_{n+1}} < \infty,$$

$\mathcal{M}_k = (M_{n+k})_{n=1}^\infty$, $k = 0, 1, 2, 3, \ldots$ be the sequences obtained from the initial ones by removing first k terms. For each sequence $\mathcal{M}_0 = (M_n)_{n=1}^\infty$ we consider the Banach space $H(\mathcal{M}_0, D)$ of functions analytic in a bounded convex domain D with the norm:

$$\|f\|^2 = \sup_n \frac{1}{M_n^2} \sup_{z\in D} |f^{(n)}(z)|^2.$$

In the work we study locally convex subspaces in the space of analytic functions in D infinitely differentiable in $\overline{D}$ obtained as the inductive limit of the spaces $H(\mathcal{M}_k, D)$. We prove that for each convex domain there exists a system of exponentials $e^{\lambda_n z}$, $n \in \mathbb{N}$, such that each function in the inductive limit $f \in \lim ind\, H(\mathcal{M}_k, D) := \mathcal{H}(\mathcal{M}_0, D)$ is represented as the series over this system of exponentials and the series converges in the topology of $\mathcal{H}(\mathcal{M}_0, D)$. The main tool for constructing the systems of exponentials is entire functions with a prescribed asymptotic behavior. The characteristic functions L with more sharp asymptotic estimates allow us to represent analytic functions by means of the series of the exponentials in the spaces with a finer topology. In the work we construct entire functions with gentle asymptotic estimates. In addition, we obtain lower bounds for the derivatives of these functions at zeroes.

Keywords: analytic functions, entire functions, subharmonic functions, series of exponentials.

Mathematics Subject Classification: 30B50, 30D20, 30D60

1. Introduction

In the work we consider subspaces $A \subset A^\infty(D) = H(D) \bigcap C^\infty(\overline{D})$ for a bounded convex domain D in the plane and we are interesting in representing the functions $f \in A$ in these subspaces by the exponentials series

$$f(z) = \sum_{n=1}^\infty f_n e^{\lambda_n z}$$

converging in the topology of the subspace A. In the classical theory of exponentials series exposed in details in monograph [1] by A.F. Leontiev, one the main theorems is the theorem on representation by exponentials series in $H(D)$ with the topology of uniform convergence on compact sets in D [1, Thm. 5.3.2]

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Theorem A. *Let D be a bounded convex domain. Then there exists a sequence $\{\lambda_n\}$ depending only on the domain D such that each function $F(z)$ analytic in the domain D can be expanded into the Dirichlet series in D:*

$$F(z) = \sum_{n=1}^{\infty} a_n e^{\lambda_n z}, \quad z \in D.$$

As the main tool in constructing the exponentials series, entire functions with a prescribed asymptotic behavior serve. For instance, in the proof of Theorem A, the exponents $\{\lambda_n\}$ are chosen are simple zeroes of an entire function $L(\lambda)$ of exponential type and of completely regular growth with the property: for each $\varepsilon > 0$,

$$|L(\lambda)| \prec e^{H_D(\lambda)+\varepsilon|\lambda|}, \quad \lambda \in \mathbb{C}, \qquad |L'(\lambda_n)| \succ e^{H_D(\lambda_n)-\varepsilon|\lambda_n|}, \quad n \in \mathbb{N}. \tag{1}$$

Here $H_D(\lambda) = \max_{z\in\overline{D}} \operatorname{Re} \lambda\overline{z}$ is the support function of the domain D. In view of this fact, convex polygons are in a special place in the theory of representation by exponentials series. The matter is that in this case the characteristic entire function L can be chosen as the quaispolynomial:

$$L(\lambda) = \sum_j a_j e^{\gamma_j \lambda}, \quad \lambda \in \mathbb{C},$$

where γ_j are the vertices of the polygon and required property (1) holds in much more specified form

$$|L(\lambda)| \prec e^{H_D(\lambda)}, \quad \lambda \in \mathbb{C}, \qquad |L'(\lambda_n)| \succ e^{H_D(\lambda_n)}, \quad n \in \mathbb{N}. \tag{2}$$

By means of such entire functions, it was proven in [1, Thm. 4.7.4] that a function analytic in a polygon D and continuous together with its first derivative in $\overline{D}$ can be represented as a series over the system $e^{\lambda_n z}$ and this series converges everywhere in $\overline{D}$ and converges uniformly in $\overline{D} \setminus \bigcup B(\gamma_j, \varepsilon)$. Here γ_j are the vertices of the polygon and $\varepsilon > 0$ is arbitrary. It was proven in work [2] that this system forms an unconditional basis in Smirnov space $E_2(D)$.

Theorem B. *Let a function $L(\lambda)$ with simple zeroes λ_n satisfy the conditions*

$$|L(\lambda)| \prec e^{H_D(\lambda)}, \quad \lambda \in \mathbb{C}, \qquad |L(\lambda)| \succ e^{H_D(\lambda)}, \quad z \notin \bigcup_n B(\lambda_n, \delta),$$

and the balls $B(\lambda_n, \delta)$ are mutually disjoint. Then each function $f \in E_2(D)$ can be represented uniquely as the series

$$f(z) = \sum_n f_n e^{\lambda_n z},$$

and the relation

$$\|f\|^2 \asymp \sum_n |f_n|^2 e^{-2H_D(\lambda_n)}, \quad f \in E_2(D),$$

holds true.

Thus, the characteristic functions L admitting more precise asymptotic estimates, allow one to represent analytic functions by exponentials series in spaces with a finer topology.

The problem on existence and constructing entire functions with prescribed asymptotic properties arose as an inner problem of theory of entire functions. In the most general formulation such problem was solved in work [3].

Theorem C. *For each subharmonic function u on the plane having a finite type at an order $\rho > 0$, there exists an entire function f satisfying the relation*

$$|u(\lambda) - \ln|f(\lambda)|| = o(|\lambda|^\rho), \quad \lambda \notin E, \quad |\lambda| \to \infty.$$

The exceptional set E is a C_0-set, that is, it can be covered by the balls $B(w_k, r_k)$ so that

$$\sum_{|w_k|\leqslant R} r_k = o(R), \quad R \to \infty.$$

In work [4] this theorem was specified in the sense of the estimates for the difference and for the size of the exceptional set.

Theorem D. *For each subharmonic function u on the plane having a finite growth order, there exists an entire function f satisfying the relation*

$$|u(\lambda) - \ln|f(\lambda)|| = O(\ln(|\lambda| + 1)), \quad \lambda \notin E, \quad |\lambda| \to \infty.$$

For each $\beta > 0$, the exceptional set E can be covered by a system of balls $B(w_k, r_k)$ so that

$$\sum_{|w_k|\geqslant R} r_k = O(R^{-\beta}), \quad R \to \infty.$$

Theorems C and D can not be applied directly in issues on expansions into the exponentials series. One has to obtain additionally the lower bounds for $|L'(\lambda_k)|$, and in order to do this, one needs to have not only the estimates for the size of the exceptional set but mainly the information on the structure of this set.

In the present work we prove the following theorem.

Theorem 1. *Let u be a subharmonic function on the plane having a finite growth order ρ, μ is the measure associated in the Riesz set. If for some a, $\alpha > 0$, for all points $z \in \mathbb{C}$ the condition*

$$\mu(B(z,t)) \leqslant a(|z|+1)^{\alpha} t, \quad t \in (0; (|z|+1)^{-\alpha}), \tag{3}$$

holds, then there exists an entire function f with simple zeroes λ_n such that for some $\delta, \beta > 0$ the balls $B_n = B(\lambda_n, \delta(|\lambda_n|+1)^{-\beta})$ are mutually disjoint and for some constants A, B, C the function satisfies the relations

$$\ln|f(\lambda)| \leqslant u(\lambda) + A_1 \ln(|\lambda|+1), \quad \lambda \in \mathbb{C},$$

$$\ln|f(\lambda)| \geqslant u(\lambda) - A_2 \ln(|\lambda|+1), \quad \lambda \notin \bigcup_n B_n,$$

$$\ln|f'(\lambda_n)| \geqslant u(\lambda_n) - A_3 \ln(|\lambda_n|+1), \quad n \in \mathbb{N}.$$

The constants A_1, A_2, A_3 depend on ρ, α, a and are independent on the particular form of the function u.

For each sequence $\mathcal{M}_0 = (M_n)_{n=1}^{\infty}$ we consider the Banach space $H(\mathcal{M}_0, D)$ of functions analytic in a bounded convex domain D with the norm

$$\|f\|^2 = \sup_n \frac{1}{M_n^2} \sup_{z\in D} |f^{(n)}(z)|^2.$$

Let

$$T_k(r) = \sup_n \frac{r^n}{M_{n+k}}, \quad k = 0, 1, 2, \dots$$

be the trace functions of the "shifted" sequences $\mathcal{M}_k = (M_{n+k})_{n=1}^{\infty}$. By $P_k(D)$ we denote the Banach spaces of entire functions F with the norm

$$\|F\|^2 = \sup_{\lambda\in\mathbb{C}} \frac{|F(\lambda)|^2 e^{-2H_D(\lambda)}}{T_k(|\lambda|)},$$

while the symbol $\mathcal{P}(\mathcal{M}_0, D)$ stands for the projective limit of the spaces P_k, $k = 0, 1, 2, \dots$. It was shown in [5, Thm. 1] that each function $f \in H(\mathcal{M}_k, D)$, $k \in \mathbb{N}$, is the Fourier-Laplace transform of some linear continuous functional S on the space $\mathcal{P}(\mathcal{M}_0, D)$, that is,

$$f(z) = S_\lambda(e^{\lambda z}), z \in D.$$

Employing this fact and the idea of sufficient set, we prove the following theorem.

Theorem 2. *Let $\mathcal{M}_0 = (M_n)_{n=1}^{\infty}$ be a convex sequence satisfying the "non-quasi-analyticity" condition*

$$\sum_n \frac{M_n}{M_{n+1}} < \infty.$$

For each convex domain there exists a system of exponentials $e^{\lambda_n z}$, $n \in \mathbb{N}$, such that each function in the inductive limit

$$f \in \lim ind\, H(\mathcal{M}_k, D) := \mathcal{H}(\mathcal{M}_0, D)$$

is represented by a series over this system of exponentials

$$f(z) = \sum_n f_n e^{\lambda_n z}$$

and this series converges in the topology of $\mathcal{H}(\mathcal{M}_0, D)$.

In what follows such systems will be called representing systems.

2. Entire functions with prescribed asymptotic behavior. Proof of Theorem 1

The proof of Theorem 1 is a minor modification of the proof of Theorem 4' in work [4].

Lemma 1. *Let a function u satisfies $u(0) = 0$, is subharmonic in the entire plane and obeys the condition*

$$u(\lambda) \leqslant \sigma(|\lambda| + 1)^{\rho}, \quad \lambda \in \mathbb{C}. \tag{4}$$

Assume that its associated measure satisfies condition (3). Then there exists a subharmonic function $v \in C^{\infty}(\mathbb{C})$ satisfying conditions (3), (4) and

$$u(\lambda) \leqslant v(\lambda) \leqslant u(\lambda) + O\big(\ln(|\lambda| + 1)\big), \quad \lambda \to \infty, \qquad \Delta v(\lambda) = O\big((|\lambda| + 1)^{3(\rho+\alpha)}\big), \quad \lambda \to \infty.$$

Proof. By the Jensen formula

$$\frac{1}{2\pi}\int_0^{2\pi} u(Re^{i\varphi})d\varphi = \int_0^R \frac{\mu(t)}{t}dt$$

it follows from condition (4) that

$$\mu\left(\frac{R}{e}\right) \leqslant \sigma(R+1)^{\rho}, \quad R > 0,$$

or

$$\mu(R) \leqslant \sigma e^{\rho}(R+1)^{\rho}, \quad R > 0. \tag{5}$$

We partition the segment $[2^{n-1}; 2^n]$ into $N_n = [\sigma e^{\rho} 2^{(n+1)(\rho+1)}] + 1$ segments I_k of the same length; here $[x]$ denotes the integer part of x. Then by the Dirichlet principle there exists at least one annulus $S_n := \{z = te^{i\varphi}, t \in I_{k_n}, \varphi \in [0; 2\pi]\}$, whose measure μ satisfies the estimate

$$\mu(S_n) \leqslant \frac{\mu(2^n)}{N_n} < 2^{-n}. \tag{6}$$

We denote the width of these annuli by h_n:

$$S_n = \{z : R_n \leqslant |z| < R_n + h_n\}, \quad n \in \mathbb{N},$$

at that,

$$h_n \geqslant \frac{1}{8\sigma e^{\rho}} 2^{-(n+1)\rho}, \quad R_n \in [2^n; 2^{n+1}]. \tag{7}$$

Let
$$\nu = \sum_{n=1}^{\infty} \mu\Big|_{S_n}$$
be the sum of the restrictions of the measure μ on the annuli S_n. It follows from (6) that $\nu(\mathbb{C}) \leqslant 1$. In the usual way, this condition and (3), we prove the estimate
$$\pi(\lambda) := \int_{\mathbb{C}} \ln\left|1 - \frac{\lambda}{w}\right| d\nu(w) = O(\ln(|\lambda| + 1), \quad \lambda \to \infty.$$
The function $u_0(\lambda) = u(\lambda) - \pi(\lambda)$

1) is subharmoninc in the entire plane and is harmonic in annuli S_n;
2) the associated measure μ_0 satisfy conditions (3), (4);
3) satisfies the estimate
$$|u_0(\lambda) - u(\lambda)| = |\pi(\lambda)| = O(|\lambda| + 1), \quad \lambda \to \infty. \tag{8}$$
We apply the regularization procedure to the function u_0. Let $\alpha(t) \in C^\infty(\mathbb{R})$ be a non-zero compactly supported function with the support in $[-1; 1]$ such that
$$\int_{\mathbb{C}} \alpha(|\lambda|) dm(\lambda) = 1,$$
where dm is the plane Lebesgue measure. We take a sequence of numbers
$$\delta_n = \min\left(\frac{h_n}{4}, 2^{-\alpha(n+2)}\right)$$
and we let $\alpha_n(\lambda) = \delta^{-2}\alpha(\delta^{-1}(\lambda - w))$. Then the functions
$$u_n(\lambda) = \int_{\mathbb{C}} \alpha_n(\lambda - w) u_0(w) dm(w), \ \lambda \in \mathbb{C}, \ n \in \mathbb{N},$$
possess general properties of the regularizations. They are

1) subharmonic, infinitely differentiable and $u_n(\lambda) \geqslant u_0(\lambda)$, $\lambda \in \mathbb{C}$, $n \in \mathbb{N}$.

And they possess the property implied by the properties of u_0:

2) $u_n(\lambda) \equiv u_0(\lambda)$ in the annulus $\widetilde{S} = \left\{\lambda : |\lambda| \in [R_n + \frac{h_n}{4}; R_n + \frac{3h_n}{4}]\right\}$.

We define the function v as
$$v(\lambda) = u_n(\lambda), \ |\lambda| \in \left[R_n + \frac{h_n}{4}; R_{n+1} + \frac{3h_{n+1}}{4}\right], \ n \in \mathbb{N}.$$
By the second property of the functions u_n, the function v is "glued" into a function subharmonic in the annuli $\widetilde{S}_n$ equalling to the function u_0 in the annuli $\widetilde{S}$. It is obvious that $v \in C^\infty(\mathbb{C})$ and this function satisfies conditions (3), (4). If λ lies between the annuli $\widetilde{S}_n$ and $\widetilde{S}_{n+1}$, then
$$v(\lambda) - u_0(\lambda) = u_n(\lambda) - u_0(\lambda) = \int_{\mathbb{C}} (u_0(w) - u_0(\lambda))\alpha_n(\lambda - w) dm(w).$$
Passing to the polar coordinates and employing the Jensen formula, we obtain
$$v(\lambda) - u_0(\lambda) = 2\pi \int_0^{\delta_n} \alpha_n(t) \left(\int_0^t \frac{\mu_0(\lambda, s)}{s} ds\right) t dt.$$
By the definition
$$\delta_n \leqslant 2^{-\alpha(n+1)} \leqslant (|\lambda| + 1)^{-\alpha},$$
and by property (3) we get
$$v(\lambda) - u_0(\lambda) \leqslant a \int_{\mathbb{C}} \alpha_n(\lambda) dm(\lambda) = a.$$
By estimate (8) this implies the first statement of Lemma 1.

Let us estimate Δv. If λ lies between the annuli $\widetilde{S}_n$ and $\widetilde{S}_{n+1}$, we consider u_0 as a generalized function to obtain

$$\Delta v(\lambda)=\int \Delta_\lambda \alpha_n(\lambda-w)u_0(w)dm(w)=\int \Delta_w \alpha_n(\lambda-w)u_0(w)dm(w)=\pi\int \alpha_n(\lambda-w)d\mu(w).$$

If $\alpha=\max_t \alpha(t)$, then in view of (4) we have

$$\Delta v(\lambda)\leqslant \delta_n^{-2}2\mu(|\lambda|+1)=O((|\lambda|+1)^{3(\rho+\alpha)}).$$

The proof is complete. □

Lemma 2. *Let u be a smooth subharmonic function, μ be the associated measure satisfying conditions (3), (4), and for some β, the estimate*

$$\Delta u(\lambda)=O((|\lambda|+1)^{\beta}),\quad |\lambda|\to\infty,$$

holds true. Then there exists a subharmonic function $\widetilde{u}$ such that

$$|\widetilde{u}(\lambda)-u(\lambda)|=O(|\lambda|),\quad \lambda\to\infty.$$

At that, the associated measure $\widetilde{\mu}$ of the function $\widetilde{u}$ satisfies conditions (3), (4). Moreover, there exist measures μ_n and rectangles P_n, $n\in\mathbb{N}$, such that

1) $\sum_n \mu_n=\mu$;
2) the interiors of convex hulls of the supports of the measures μ_n are mutually disjoint;
3) the support of the measure μ_n is located in P_n, $n\in\mathbb{N}$;
4) the quotient of the sides of the rectangle P_n is in the interval $[3^{-1};3]$;
5) each point of the plane belongs to at most four rectangles P_n;
6) if F_n is a convex hull of the support of the measure μ_n, then

$$\operatorname{diam} F_n\leqslant 2\sqrt{2}\min_{\lambda\in F_n}|\lambda|.$$

7) Inside the supports F_n, the function $\widetilde{u}$ is smooth and the estimate

$$\Delta\widetilde{u}(\lambda)=O((|\lambda|+1)^{\beta}),\quad |\lambda|\to\infty,$$

holds true.

Proof. Let Q_n, $n\in\mathbb{N}$, be the square centered at the origin with the sides of length 3^n and being parallel to the axes. Then $Q_{n+1}\setminus Q_n=\bigcup_1^8 Q_{nj}$, $n\in\mathbb{N}$, where Q_{nj} are the squared obtained by the shift of the square Q_n by the vectors $(\pm 3^n,0)$, $(0,\pm 3^n)$, $(\pm 3^n,\pm 3^n)$. We let $\mu(Q_{nj}):=m_{nj}+q_{nj}$, $j=1,2,\ldots,8$, $n\in\mathbb{N}$, where $q_{nj}=\{\mu(Q_{nj})\}\in[0;1)$ is the fractional part of $\mu(Q_{nj})$. We let

$$q_n^+=\sum_j q_{nj}\in[0;8),\quad q_n^-=\sum_j (q_{nj}-1)\in[-8;0).$$

We define a sequence q_n as follows: we let $q_0=\{\mu(Q_1)\}$, if q_j are defined for $j\leqslant k-1$ then as $\sum_{j\leqslant k-1} q_j\geqslant 0$, we let $q_k:=q_k^-$ and $q_k:=q_k^+$ otherwise. Thus,

$$\sum_{k=0}^{n} q_k\in(-8;8),\ n\in\mathbb{N}. \tag{9}$$

Then we define the sequence of natural numbers N_0, N_{nj}, $j=1,\ldots 8$, $n\in\mathbb{N}$. We let $N_0=[\mu(Q_1)]$, if $q_n=q_n^-$, then $N_{nj}=\mu(Q_{nj})-(q_{nj}-1)$, and if $q_n=q_n^+$, then $N_{nj}=\mu(Q_{nj})-q_{nj}$. Thus, either $N_{nj}=m_{nj}+1$ or $N_{nj}=m_{nj}$. The restriction of the measure μ on the square Q_{nj} is denoted by μ_{nj}, $\mu_0=\mu\Big|_{Q_0}$ and we let

$$\widetilde{\mu}_0=\frac{N_0}{\mu(Q_0)}\mu_0,\qquad \widetilde{\mu}_{nj}=\frac{N_{nj}}{\mu(Q_{nj})}\mu_{nj},\quad j=1,\ldots,8,\quad n\in\mathbb{N}.$$

If $\mu(Q_{nj}) = 0$, then $\widetilde{\mu}_{nj} = 0$. Then $\widetilde{\mu}_{nj}(\mathbb{C}) = N_{nj}$ are non-negative integer numbers and if we let $\nu_{nj} = \mu_{nj} - \widetilde{\mu}_{nj}$, then

$$\nu_{nj}(\mathbb{C}) \in (-1;1), \ \left(\sum_{j=1}^{8} \nu_{nj}\right)(\mathbb{C}) \in (-8;8).$$

Let

$$\nu = \nu_0 + \sum_{n=1}^{\infty}\sum_{j=1}^{8} \nu_{nj}, \quad \nu^+ = \nu_0 + \sum_{q_n=q_n^+}\sum_{j=1}^{8} \nu_{nj}, \quad \nu^- = -\sum_{q_n=q_n^-}\sum_{j=1}^{8} \nu_{nj},$$

then $\nu^\pm$ are non-negative measures and $\nu = \nu^+ - \nu^-$. At that,

$$\nu^\pm\left(\bigcup_j Q_{nj}\right) = q^\pm \in (-8;8).$$

Proposition 1. *The identity*

$$\pi(\lambda) := \int \left|1 - \frac{\lambda}{w}\right| d\nu(w) = O(\ln(|\lambda|+1)), \quad |\lambda| \to \infty.$$

holds true.

Proof. Let us prove that

$$|\nu(t)| = |\nu(B(0,t))| \leqslant 17, \quad t \geqslant 0, \tag{10.1}$$

and that for $|\nu| = \nu^+ + \nu^-$, the identity

$$|\nu|(t) = |\nu|(B(0,t)) \leqslant 17\ln(t+e), \quad t \geqslant 0. \tag{10.2}$$

holds.

If $t < \frac{9}{\sqrt{2}}$, then $B(0,t) \subset Q_2$ and therefore

$$|\nu(t)| \leqslant \nu_0(t) + \sum_{j=1}^{8} |\nu_{1j}(\mathbb{C})| \leqslant 9.$$

For $t \geqslant \frac{9}{\sqrt{2}}$ by n we denote the maximal natural number, for which $\frac{3^n}{\sqrt{2}} \leqslant t$, then $Q_n \subset B(0,t)$ and

$$\frac{3^{n+2}}{2} \geqslant \frac{3}{\sqrt{2}}\frac{3^{n+1}}{\sqrt{2}} \geqslant \frac{3}{\sqrt{2}}t > t.$$

Hence, $Q_{n+2} \supset B(0,t)$. Thus, in view of (8) we obtain

$$|\nu(t)| \leqslant |\nu(Q_n)| + \sum_{i=n}^{n+1}\sum_{j=1}^{8} |\nu_{ij}(\mathbb{C})| \leqslant 17.$$

We can also estimate $\nu^\pm(t)$ and summing up the obtained estimate, we can prove inequality (10.2).

We choose arbitrary $\lambda \in \mathbb{C}$ and partition the plane into the sets

$$E_1 = \mathbb{C} \setminus B(0;2|\lambda|), \quad E_2 = B(0;1), \quad E_3 = B\left(0;\frac{|\lambda|}{2}\right) \setminus B(0;1),$$
$$E_4 = B(\lambda,(|\lambda|+1)^{-\alpha}), \quad E' = \mathbb{C} \setminus \bigcup_k E_k.$$

On the set E_1, we obtained the needed estimate by employing (10.2) and the inequality: for some constant C, for all $|z| \leqslant \frac{1}{2}$

$$|\ln|1-z|| \leqslant C|z|, \tag{$*$}$$

$$\left|\int_{E_1} \ln\left|1-\frac{\lambda}{w}\right| d\nu(w)\right| = O(\ln|\lambda|),\ |\lambda|\to\infty.$$

On E_2, the estimate for the integral is obvious:

$$\left|\int_{E_2} \ln\left|1-\frac{\lambda}{w}\right| d\nu(w)\right| = O(\ln|\lambda|), \quad |\lambda|\to\infty.$$

The estimate of the first integral in the right hand side of the inequality

$$\left|\int_{E_3} \ln\left|1-\frac{\lambda}{w}\right| d\nu(w)\right| \leqslant \left|\int_{E_3} \ln\left|1-\frac{w}{\lambda}\right| d\nu(w)\right| + \left|\int_{E_3} \ln\left|\frac{w}{\lambda}\right| d\nu(w)\right|$$

is implied by (10.2) and (*). In the second integral we integrate by parts:

$$\left|\int_{E_3} \ln\left|\frac{w}{\lambda}\right| d\nu(w)\right| = \left|\int_1^{\frac{|\lambda|}{2}} \ln\frac{|\lambda|}{t} d\nu(t)\right| \leqslant \ln 2\left|\nu\left(\frac{|\lambda|}{2}\right)\right| + \left|\int_1^{\frac{|\lambda|}{2}} \frac{\nu(t)dt}{t}\right|.$$

We obtain the needed estimate by employing (10.1):

$$\left|\int_{E_3} \ln\left|1-\frac{\lambda}{w}\right| d\nu(w)\right| = O(\ln|\lambda|),\ |\lambda|\to\infty.$$

By property (3) we have

$$\left|\int_{E_4} \ln\left|1-\frac{\lambda}{w}\right| d\nu(w)\right| \leqslant a, \quad \lambda\in\mathbb{C}.$$

Let n be the smallest natural number, for which $B(0,2|\lambda|)\subset Q_n$, that is,

$$\frac{3^{n-1}}{2} < 2|\lambda| \leqslant \frac{3^n}{2}.$$

Then

$$E' \subset B(0,2|\lambda|)\setminus B(0,\frac{|\lambda|}{2}) \subset Q_n\setminus Q_{n-3}$$

and this is why

$$|\nu|(E')\leqslant 24.$$

On the other hand,

$$\max_{w\in E'}\left|\ln\left|1-\frac{\lambda}{w}\right|\right| = O(\ln|\lambda|), \quad |\lambda|\to\infty.$$

Two latter estimates imply the estimate for the integral over the set E'. □

We return back to the proof of Lemma 2. We let $\widetilde{u}(\lambda) = u(\lambda)-\pi(\lambda)$, where $\pi(\lambda)$ is the potential of the measure ν defined in Proposition 1. Then $\widetilde{u}$ is a subharmonic function with the associated measure $\widetilde{\mu}$. By construction, conditions (3) and (4) are satisfied. And in the interior of each square Q_{nj} we have

$$\Delta\widetilde{u}(\lambda)dm(\lambda) = \pi\widetilde{\mu}(\lambda) = \pi\frac{N_{nj}}{\mu(Q_{nj})}\mu_{nj} = \frac{N_{nj}}{\mu(Q_{nj})}\Delta u(\lambda),$$

and this is why the estimate

$$\Delta\widetilde{u}(\lambda) = O((|\lambda|+1)^\beta), \quad |\lambda|\to\infty \tag{11}$$

holds true. Proposition 1 implies the relation

$$|\widetilde{u}(\lambda)-u(\lambda)| = O((|\lambda|), \quad |\lambda|\to\infty.$$

By construction, $\widetilde{\mu}(Q_{nj}) = N_{nj}$ are non-negative integer numbers and we can apply Theorem 1 in work [4] on partition of the measures to the restrictions $\widetilde{\mu}_{nj}$: there exists a set of pairs (μ^k_{nj}, P^k_{nj}) of rectangles P^k_{nj} and unit measures μ^k_{nj} such that Statements 1–5 of Lemma 2 hold. In addition, the rectangles P^k_{nj} are located in squares Q_{nj}. It remains to renumber them by a single index. Statement 7 holds thanks to estimate (11), while Statement 6 is implied by the corresponding property of the squares Q_{nj}. □

We continue proving Theorem 1.

We denote by λ_n the gravity center of the unit measures μ_n constructed in Lemma 2:

$$\int w d\mu(w) = \lambda_n, \quad n \in \mathbb{N}.$$

By $\widetilde{\mu}_n$ we denote the restriction of the measure $\widetilde{\mu}$ on the set $Q_n \setminus Q_0$ and by π_n we denote the potential of this measure:

$$\pi_n(\lambda) = \int \ln \left|1 - \frac{\lambda}{w}\right| d\widetilde{\mu}_n(\lambda).$$

Then the measure $\widetilde{\mu}_n$ satisfies the assumptions of Theorem 3 in work [4] and by this theorem, in view of condition (3), we obtain the existence of the polynomial P such that outside a set of the balls $B_n(\varepsilon) = B(\lambda_k, \varepsilon(|\lambda_k| + 1)^{-\gamma})$, where λ_k are the zeroes of the polynomial P, $\gamma \geqslant \alpha$ and $\varepsilon > 0$ is a sufficiently small number, the inequality

$$|\pi(\lambda) - \ln |P(\lambda)|| \leqslant A \ln(|\lambda| + 1)$$

holds. At that, the constant A is independent of n. Thanks to the latter fact, in the usual way we justify the passage to the limit. As a result we get that there exists an entire function f with simple zeroes at the points λ_n satisfying the condition

$$|\widetilde{u}(\lambda) - \ln |f(\lambda)|| \leqslant A \ln(|\lambda| + 1), \quad \lambda \notin \bigcup_n B_n(\varepsilon). \tag{12}$$

We need to show that for sufficiently small $\varepsilon > 0$, the balls $B_n(\varepsilon)$ are mutually disjoint. Let us estimate the distance d_n from the point λ_n to the boundary of the convex hull F_n of the support of the measure μ_n. Let w_n be a point, at which this distance is attained:

$$|\lambda_n - w_n| = \inf \big\{|\lambda_n - w|, \ w \notin F_n\big\}.$$

Let $\lambda_n - w_n = e^{i\varphi_n}|\lambda_n - w_n|$ and $z = Tw = e^{-\varphi_n}(\lambda_n - w)$. under such transformation, the image F^* of the hull F_n is located in $\{\operatorname{Re} z \leqslant d_n\}$ and the image of the measure $d\mu^*(z) = d\mu_n(\lambda - e^{i\varphi_n} z)$ satisfies the conditions

$$\int d\mu^*(z) = 1, \quad \int z d\mu^*(z) = 0, \quad d\mu^*(z) = \frac{1}{\pi}\chi_n(z)\Delta\widetilde{u}(\lambda_n - e^{i\varphi_n} z) dm(z),$$

where $\chi_n(z)$ is the characteristic function of F^*. Let

$$\delta(x) = \frac{1}{\pi}\int \chi_n(x + iy)\Delta\widetilde{u}(\lambda_n - e^{i\varphi_n}(x + iy))dy.$$

Then $\delta(x)$ is a compactly supported function on the segment $[a; d_n]$ and by Statements 6, 7 in Lemma 2,

$$0 \leqslant \delta(x) \leqslant C(|\lambda_n| + 1)^{\beta+1} := M_n.$$

Moreover, it follows from the properties of μ^* that

$$\int_a^{d_n} \delta(x)dx = 1, \quad \int_a^{d_n} x\delta(x)dx = 0.$$

Proposition 2. *Let $\delta(x)$ be a non-negative continuous compactly supported function satisfying the conditions*

1) $\operatorname{conv}\operatorname{supp}\delta = [a; d]$, 2) $\sup_x \delta(x) \leqslant M < \infty$,

3) $\int \delta(x)dx = 1$, 4) $\int x\delta(x)dx = 0.$

Then
$$d \geqslant \frac{1}{6M}.$$

Proof. We introduce a number $c > 0$ by the identity
$$\int_{-c}^{c} \delta(x)dx = \frac{1}{3}.$$
It follows from Condition 2) that $c \geqslant \frac{1}{6M}$. Assume that $d < c$. Then taking into consideration 3), we have
$$\int_{-\infty}^{-c} \delta(x)dx = 1 - \int_{-c}^{d} \delta(x)dx = 1 - \int_{-c}^{c} \delta(x)dx = \frac{2}{3}.$$
This is why
$$\int_{-\infty}^{0} |x|\delta(x)dx \geqslant \frac{2c}{3} + \int_{-c}^{0} |x|\delta(x)dx \geqslant \frac{2c}{3}.$$
On the other hand,
$$\int_{0}^{d} |x|\delta(x)dx = \int_{0}^{c} |x|\delta(x)dx \leqslant c\int_{-c}^{c} \delta(x)dx = \frac{c}{3}.$$
By Condition 4),
$$\frac{2c}{3} \leqslant \int_{-\infty}^{0} |x|\delta(x)dx = \int_{0}^{1} |x|\delta(x)dx =\leqslant \frac{c}{3}.$$
By the obtained contradiction we get
$$d \geqslant c \geqslant \frac{1}{6M}.$$
The proof is complete. □

The proven proposition implies that for $\gamma = 3(\rho+\alpha)+1$ and sufficiently small $\varepsilon > 0$ the ball
$$B_n(\varepsilon) = B(\lambda_n, \varepsilon(|\lambda|+1)^{-\gamma})$$
is located inside the hull F_n and thus, these balls are mutually disjoint. By usual tricks and by the Cauchy formula
$$\frac{1}{f'(\lambda_n)} = \frac{1}{2\pi i}\int \frac{dz}{f(z)(z-\lambda_n)},$$
one can obtain the needed estimates for the derivatives at the points λ_n.

The proof of Theorem 1 is complete.

3. Representing systems of exponentials in the space $\mathcal{H}(\mathcal{M}_0, D)$. Proof of Theorem 2

Let $\mathcal{M}_0 = (M_n)_{n=1}^{\infty}$ be an increasing convex sequence of positive numbers. The convexity is understood in the sense that if
$$T_0(r) = \sup_n \frac{r^n}{T_n}, \quad r > 0,$$
is the trace function for this sequence, then
$$M_n = \sup_{r>0} \frac{r^n}{T(r)}, \quad n \in \mathbb{N}.$$
In Introduction we have introduced the notations: given $k \in \mathbb{N}$, by $\mathcal{M}_k$ we have denoted the shift of the sequence $\mathcal{M}_k = (M_{n+k})_{n=1}^{\infty}$ and we introduced the subspaces
$$H(\mathcal{M}_k, D) = \left\{ f \in H(D) : \ \|f\|_k = \sup_n \frac{\sup_{z\in D}|f^{(z)}|}{M_{n+k}} \right\}, \quad k \in \mathbb{N},$$

and the inductive limit of these spaces

$$\mathcal{H}(\mathcal{M}_0, D) = \bigcup_k H(\mathcal{M}_k, D).$$

By $P_k(d)$ we have denoted the Banach space of entire functions with the norm

$$\|F\|_k = \sup_{\lambda\in\mathbb{C}} \frac{|F(\lambda)|e^{-H_D(\lambda)}}{T_k(|\lambda|)},$$

where T_k is the trace function for the sequence $\mathcal{M}_k$ and the inductive limit of these spaces was denoted by $\mathcal{P}(\mathcal{M}_0, D)$. The Fourier-Laplace transform of the linear continuous functional S on $\mathcal{P}(\mathcal{M}_0, D)$ is introduced as

$$S \to \widehat{S}(z) = S_\lambda(e^{\lambda z}).$$

It was shown in [5] that for a sequence $\mathcal{M}_0$ satisfying the condition

$$\sum_n \frac{M_n}{M_{n+1}} < \infty, \tag{13}$$

the Fourier-Laplace transform defines the isomorphism between the strongly dual space $\mathcal{P}^*(\mathcal{M}_0, D)$ and the space $\mathcal{H}(\mathcal{M}_0, D)$.

To each subset $S \subset \mathbb{C}$, we associate the semi-norm in $\mathcal{P}(\mathcal{M}_0, D)$:

$$\|F\|_{k,S} = \sup_{\lambda\in S} \frac{|F(\lambda)|e^{-H_D(\lambda)}}{T_k(|\lambda|)}.$$

If the topology defined by the system of these semi-norms coincides with the initial topology of the space $\mathcal{P}(\mathcal{M}_0, D)$, then the set S is called sufficient for the space $\mathcal{P}(\mathcal{M}_0, D)$ (see [6], [7]).

Theorem 3. *Assume that an entire function $L(\lambda)$ with zeroes λ_n, $n \in \mathbb{N}$, satisfies the conditions*

1) for some $A > 0$,

$$|L(\lambda)| \prec e^{H_D(\lambda)} T_0(|\lambda|)(|\lambda| + 1)^A, \ \lambda \in \mathbb{C},$$

2) for some $B > 0$ and some sequence $R_k \to \infty$,

$$|L(\lambda)| \succ e^{H_D(\lambda)} T_0(|\lambda|)(|\lambda| + 1)^{-B}, \quad |\lambda| = R_k, \quad k \in \mathbb{N},$$

3) for some $C > 0$,

$$|L'(\lambda_n)| \succ e^{H_D(\lambda_n)} T_0(|\lambda_n|)(|\lambda_n| + 1)^{-C}, \quad n \in \mathbb{N}.$$

Then the set $S = \{\lambda_n, \ n \in \mathbb{N}\}$ is sufficient for the space $\mathcal{P}(\mathcal{M}_0, D)$.

Proof. In the estimate we shall employ the following relation between the trance functions introduced in work [5, Lm 1.2]: for each natural k, m, $m \leqslant k$, there exists $r_{k,m}$ such that as $r \geqslant r_{k,m}$,

$$T_k(r) = r^{m-k} T_m(r). \tag{14}$$

So, we choose and fix a natural number k, take a number $N \geqslant A + k$ and a number $m \geqslant N + C + B + 2$, where the constants A, C come from the assumptions of the theorem. Let $F \in \mathcal{P}(\mathcal{M}_0, D)$, then

$$|F(\lambda)| \leqslant \|F\|_m e^{H_D(\lambda)} T_m(|\lambda|), \ \lambda \in \mathbb{C}, \tag{15}$$

and by relation (14),

$$|\lambda^N F(\lambda)| \prec e^{H_D(\lambda)} T_0(|\lambda|)(|\lambda| + 1)^{-B-2}.$$

By the second condition for the function L this implies that on the circles $|\lambda| = R_n$, the estimate

$$\frac{|\lambda^N F(\lambda)|}{|L(\lambda)|} \prec (|\lambda| + 1)^{-2}$$

holds true. Therefore, the Lagrange series

$$\lambda^N F(\lambda) = \sum_n \frac{\lambda_n^N F(\lambda_n)}{(\lambda - \lambda_n)L'(\lambda_n)} L(\lambda)$$

converges uniformly in the plane. By the choice of m and by (15), for $|\lambda - \lambda_n| \geqslant 1$ we have the estimate

$$\frac{|\lambda_n^N F(\lambda_n)|}{|L'(\lambda_n)|} \leqslant \|F\|_{m,S} |\lambda_n|^{-B-2}.$$

This is why, as $|\lambda - \lambda_n| \geqslant 1$, we get

$$|F(\lambda)| \prec \|F\|_{m,S} |L(\lambda)| |\lambda|^{-N}.$$

In view of Condition 1) of the theorem we obtain

$$\frac{|F(\lambda)| e^{-H_D(\lambda)}}{T_k(|\lambda|)} \prec \|F\|_{m,S} \frac{T_0(|\lambda|)(|\lambda|+1)^A}{T_k(|\lambda|)} \prec \|F\|_{m,S}$$

or

$$\|F\|_k \prec \|F\|_{m,S}, \ \ f \in \mathcal{P}(\mathcal{M}_0, D).$$

The proof is complete. □

Theorem 2 follows Theorem 3 thanks to the well-known relation between the sufficient sets and representing systems. The existence of entire functions L with required properties is implied by Theorem 1.

It is obvious that if the system $e^{\lambda_n z}$ is representing for the space $\mathcal{H}(\mathcal{M}_0, D)$, after removing finitely many elements, the remaining part of the system is again representing. But after removing infinitely many elements, the system is, generally speaking, no longer representing.

Proposition 3. *Let λ_n, $n \in \mathbb{N}$, be the zeroes of an entire function L satisfying the assumptions of Theorem 3 and μ_k, $k \in \mathbb{N}$, be a subset of the zeroes. Then the system*

$$E = \{e^{\lambda_n z}, \ n \in \mathbb{N}\} \setminus \{e^{\mu_n z}, \ n \in \mathbb{N}\}$$

is not representing in the space $\mathcal{H}(\mathcal{M}_0, D)$.

Proof. Passing if needed to a subset, we suppose that the set $\{\mu_k\}$ satisfies the condition: each segment $[2^n; 2^{n+1}]$¡ $n \in \mathbb{N}$ contains at most one $r_k = |\mu_k|$ and $r_k > 1$. By $m(t)$ denote the counting function of this set. Then the function

$$g(\lambda) = \prod_k \left(1 - \frac{\lambda}{\mu_k}\right), \quad \lambda \in \mathbb{C},$$

is entire.

Let $\lambda \in [2^n; 2^{n+1}]$. Then

$$\left| \sum_{|\mu_k| \geqslant 2^{n+2}} \ln \left| 1 - \frac{\lambda}{\mu_k} \right| \right| \leqslant |\lambda| \sum_{k=n+2}^{\infty} \frac{1}{2^k} \leqslant 1, \quad \left| \sum_{|\mu_k| \leqslant 2^{n-1}} \ln \left| 1 - \frac{\mu_k}{\lambda} \right| \right| \leqslant \sum_{k=1}^{n-1} \frac{1}{2^k} \leqslant 1.$$

Since as $|\lambda - \mu_k| \geqslant 1$, we have

$$\left| \sum_{2^{n-1} < |\mu_k| < 2^{n+2}} \left| 1 - \frac{\lambda}{\mu_k} \right| \right| = O(\ln |\lambda|), \quad |\lambda| \to \infty,$$

then

$$
\begin{aligned}
|\ln|g(\lambda)|| &= \sum_{|\mu_k|\leqslant 2^{n-1}} \ln\left|\frac{\lambda}{\mu_k}\right| + O(\ln|\lambda|) \\
&= \int_1^{|\lambda|} \frac{m(t)dt}{t} + O(\ln|\lambda|), \quad \lambda\to\infty, \quad |\lambda-\mu_n|\geqslant 1, \quad n\in\mathbb{N}.
\end{aligned}
$$

And since $m(t)\to\infty$ as $t\to\infty$, we get

$$\ln|\lambda| = o(\ln|g(\lambda)|), \ |\lambda-\mu_k|\geqslant 1, \quad k\in\mathbb{N}, \quad |\lambda|\to\infty.$$

Thus, the function $L_1(\lambda)=L(\lambda)/g(\lambda)$ belongs to the space $\mathcal{P}(\mathcal{M}_0,D)$ and the system E is not representing in $\mathcal{H}(\mathcal{M}_0,D)$. The proof is complete. □

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