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# TWO-SIDED ESTIMATES FOR THE RELATIVE GROWTH OF FUNCTIONS AND THEIR DERIVATIVES

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**Abstract.** We provide an extended presentation of a talk given at the International mathematical conference on theory of functions dedicated to centenary of corresponding member of AS USSR A.F. Leont'ev. We propose a new method for obtaining uniform two-sided estimates for the fraction of the derivatives of two real functions on the base of the information of two-sided estimates for the functions themselves. At that, one of the functions possesses certain properties and serves as a reference for measuring a growth and introduces some scale. The other function, whose growth is compared with that of the reference function, is convex, increases unboundedly or decays to zero on a certain interval. The method is also applicable to some class of functions concave on an interval. We consider examples of applications of the obtained results to the behavior of entire functions.

**Keywords:** monotone function, convex function, relative growth of two functions, uniform upper and lower estimates, entire function.

**Mathematics Subject Classification:** 26D10, 30D15

The themes of the work adjoin general Abelian and Tauberian theorems for functions of the real variable (L'Hôpital's rule and its inversion). In distinction to the classical formulation of the issue on relative asymptotic behavior of two functions, here the matter is the uniform estimates. More precisely, in the paper we establish new two-sided estimates connecting the relative growth of the derivatives of two functions with that for the functions themselves. First we provide a simple statement of "Abelian" type, in which the behavior of the quotient of two functions is determined by that for their derivatives (Theorem A and its Corollary). Then we prove more difficult results of an inverse, "Tauberian" character (Theorem 1 and 2). General facts are demonstrated by a series of particular examples. We also mention some applications to the growth of entire functions.

We adopt a natural assumption that the considered functions preserve constant coinciding signs on the considered sets.

We begin with a little known non-limiting "monotous" version of the L'Hôpital's rule, which relates the monotonicity of the quotient of function and the monotonicity of the quotient of their derivatives (see, for instance, [1]–[3]).

**Theorem A.** *Assume that functions $f(x)$ and $g(x)$ are defined and differentiable on a finite or infinite interval $(a,b)$ and they satisfy the conditions*

1) $g'(x) \neq 0$ *on* $(a,b)$,
2) $g(b-) = f(b-) = 0$ *or* $g(a+) = f(a+) = 0$.

*Then if the quotient of the derivatives* $\dfrac{f'(x)}{g'(x)}$ *is monotonous on* $(a,b)$, *then the quotient of the functions* $\dfrac{f(x)}{g(x)}$ *is monotonous in the same sense on* $(a,b)$.

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In view of the classical L'Hôpital's rule this implies immediately the following statement.

**Corollary.** *Let the assumptions of Theorem A hold. Then*

$$\sup_{x\in(a,\,b)}\frac{f(x)}{g(x)}=\sup_{x\in(a,\,b)}\frac{f'(x)}{g'(x)}\quad \text{or}\quad \inf_{x\in(a,\,b)}\frac{f(x)}{g(x)}=\inf_{x\in(a,\,b)}\frac{f'(x)}{g'(x)}$$

*depending on the character of the monotonicity of quotient* $\dfrac{f'(x)}{g'(x)}$.

We proceed to formulating the main results of the work. Consider the case of growing infinitely large functions. Hereafter the symbol $f'(x)$ stands for the right derivative of the function $f$ at the point $x$.

**Theorem 1.** *Assume that a function $f(x)$ is convex on an interval $(a,b)$, $-\infty\leqslant a<b\leqslant+\infty$, the function $g(x)$ is differentiable on this interval and $g'(x)>0$, and moreover, $g(a+)=0$, $g(b-)=+\infty$. Assume also that*

$$m\leqslant\frac{f(x)}{g(x)}\leqslant M,\qquad x\in(a,b),\tag{1}$$

*with nonnegative constants $m$, $M$, $m\leqslant M$. Then the two-sided estimate*

$$M\,c_1(\theta)\leqslant\frac{f'(x)}{g'(x)}\leqslant M\,c_2(\theta),\qquad x\in(a,b),\tag{2}$$

*holds true, where $\theta=\frac{m}{M}$, and the quantities $c_1(\theta)$, $c_2(\theta)$ are defined as*

$$c_1(\theta)=\inf_{x\in(a,\,b)}\frac{1}{g'(x)}\sup_{a<t<x}\frac{g(t)-\theta g(x)}{t-x},\tag{3}$$

$$c_2(\theta)=\sup_{x\in(a,\,b)}\frac{1}{g'(x)}\inf_{b>t>x}\frac{g(t)-\theta g(x)}{t-x}.\tag{4}$$

*Proof.* Since $f(x)$ is a convex function, for an arbitrary $x\in(a,b)$ we can write

$$f'(x)=\inf_{b>t>x}\frac{f(t)-f(x)}{t-x}\leqslant\inf_{b>t>x}\frac{Mg(t)-mg(x)}{t-x}=M\inf_{b>t>x}\frac{g(t)-\theta g(x)}{t-x}.$$

Thus,

$$f'(x)\leqslant M\inf_{b>t>x}\frac{g(t)-\theta g(x)}{t-x}.\tag{5}$$

Dividing both sides by $g'(x)$, for each $x\in(a,b)$ we obtain

$$\frac{f'(x)}{g'(x)}\leqslant M\frac{1}{g'(x)}\inf_{b>t>x}\frac{g(t)-\theta g(x)}{t-x}\leqslant M\,c_2(\theta)$$

and this completes the proof of the upper bound in (2).

The proof of the lower bound follows the same lines. Namely, for $x\in(a,b)$ we write

$$\begin{aligned}f'(x)\geqslant&f'_-(x)=\sup_{a<t<x}\frac{f(t)-f(x)}{t-x}\\&\geqslant\sup_{a<t<x}\frac{Mg(t)-mg(x)}{t-x}=M\sup_{a<t<x}\frac{g(t)-\theta g(x)}{t-x},\end{aligned}$$

or

$$f'(x)\geqslant M\sup_{a<t<x}\frac{g(t)-\theta g(x)}{t-x}.\tag{6}$$

Dividing both sides by $g'(x)$, for all $x\in(a,b)$ we obtain

$$\frac{f'(x)}{g'(x)}\geqslant M\frac{1}{g'(x)}\sup_{a<t<x}\frac{g(t)-\theta g(x)}{t-x}\geqslant M\,c_1(\theta).$$

The proof is complete. □

We observe that we have not employed the conditions $g(a+) = 0$, $g(b-) = +\infty$ in the proof of Theorem 1. However, we can show that if these conditions fail, formulae (3), (4) determining the quantities $c_1(\theta)$, $c_2(\theta)$ give $c_1(\theta) = -\infty$, $c_2(\theta) = +\infty$. One can confirm this also geometrically by assuming that the points $a$ and $b$ are finite and considering a function $f(x)$ whose graph touches the boundary lines $x = a$, $x = b$. Under the assumptions of Theorem 1, the considered quantities $c_1(\theta)$, $c_2(\theta)$ satisfy the inequalities

$$0 \leqslant c_1(\theta) \leqslant \theta, \qquad 1 \leqslant c_2(\theta) \leqslant \beta'_g := \sup_{x\in(a,b)} \frac{1}{g'(x)} \inf_{t>x} \frac{g(t)}{t-x}. \tag{7}$$

Indeed, the positivity of $c_1(\theta)$ is implied by the inequality

$$\sup_{a<t<x} \frac{g(t)-\theta g(x)}{t-x} \geqslant \frac{-\theta g(x)}{a-x} \geqslant 0.$$

Noting that

$$\theta(g(t)-g(x)) \leqslant g(t)-g(x) \leqslant g(t)-\theta g(x) \leqslant g(t),$$

we confirm that all properties in (7) hold.

Let us show that if $g(x)$ is a convex on $(a,b)$ function, then estimate (2) is sharp. Let us construct a function $f(x)$ obeying condition (1), for which the identities in both sides in (2) are attained at some points in the segment $J = (a,\, b)$. For the sake of convenience we suppose that $0 < \theta = m < M = 1$.

Let a positive differentiable function $g(x)$ be convex and increase on the segment $J$ and $\theta \in (0,\, 1)$ be a fixed number. Let $x_0 \in J$. If from the point $(x_0,\, \theta g(x_0))$ we draw to the right rays intersecting the graph $\Gamma_g$ of the function $g(x)$, then the ray with the minimal slope $k_0$ equalling to

$$k_0 = \frac{g(t_{x_0})-\theta g(x_0)}{t_{x_0}-x_0} = \min_{t>x_0} \frac{g(t)-\theta g(x_0)}{t-x_0},$$

touches $\Gamma_g$ at some point $(t_{x_0},\, g(t_{x_0}))$. We choose this ray and continue it to the intersection with the graph $\Gamma_{\theta g}$ of the function $\theta g(x)$ at a point $(x_1,\, \theta g(x_1))$. We denote by $l$ a segment of the chosen ray with the slope $k_0$. If from the point $(x_1,\, \theta g(x_1))$ we draw to the left a ray intersecting the graph $\Gamma_g$, the slope of such ray is maximal once it touches $\Gamma_g$ (at the same point $(t_{x_0},\, g(t_{x_0}))$), that is, it contains the segment $l$ and has the same slope:

$$k_0 = \frac{g(t_{x_0})-\theta g(x_1)}{t_{x_0}-x_1} = \max_{x_0<t<x_1} \frac{g(t)-\theta g(x_1)}{t-x_1}.$$

Assuming that $(x_0,\, x_1) \subset J$, on $(x_0,\, x_1)$ we define the function $f(x)$ by the equation of the ray $l$, that is, we let

$$f(x) = \theta g(x_0) + k_0(x-x_0), \qquad x \in (x_0,\, x_1).$$

On the rest of $J$ we let $f(x) = \theta g(x)$. Then such function satisfies the required conditions:

$$\begin{aligned} &\theta \leqslant \frac{f(x)}{g(x)} \leqslant 1, \qquad x \in J, \\ &\frac{f'(x)}{g'(x)} = \theta, \quad x \in J \setminus (x_0,\, x_1), \\ &\frac{k_0}{g'(x_1)} \leqslant \frac{f'(x)}{g'(x)} \leqslant \frac{k_0}{g'(x_0)}, \quad x \in (x_0,\, x_1). \end{aligned}$$

We observe that the identities can be attained in the last line: as $x = x_0$ in the right hand side and as $x = x_1 - 0$ in the left hand side.

Thus, for convex on $(a,\, b)$ functions $g(x)$ the estimate in Theorem 1 is sharp.

A reasonable direct extension of Theorem 1 for the case of concave functions $f(x)$ is impossible. Indeed, if in Theorem 1 a positive function $f(x)$ is concave on an interval $(a,\, b)$, by passing to the functions with the opposite sign we can reduce the proof of such statement to the situation when $g(x) < 0$, $g'(x) < 0$ for all $x \in (a,b)$. Formally, the arguing goes in this case. Indeed, multiplying (1) by $g(x)$, we obtain the inequality opposite to (5). Dividing this by $g'(x)$, we again change the sign of the inequality. Hence, in the proof, the sign of the inequality changes twice and it returns back to the initial one. However, now estimates (2) lose their meaning since in formulae (3), (4) for the coefficients $c_1(\theta)$, $c_2(\theta)$ we get

$$\sup_{a<t<x} \frac{g(t)-\theta g(x)}{t-x} = \lim_{t\to x-} \frac{g(t)-\theta g(x)}{t-x} = +\infty,$$
$$\inf_{x<t<b} \frac{g(t)-\theta g(x)}{t-x} = \lim_{t\to x+} \frac{g(t)-\theta g(x)}{t-x} = -\infty.$$

The problem can be resolved if the function $f(x)$ satisfies some additional conditions. For instance, if $f(x)$ is concave and $xf'(x)$ increases on $(a,\, b)$, $a \geqslant 0$, then the function $f_1(t) = f(e^t)$ is convex on $(\ln a,\, \ln b)$ since its derivative $f_1'(t) = e^t f'(e^t)$ grows. Moreover, at appropriate points the identities

$$\frac{f(x)}{g(x)} = \frac{f(e^t)}{g(e^t)} = \frac{f_1(t)}{g_1(t)}, \qquad \frac{f'(x)}{g'(x)} = \frac{e^t f'(e^t)}{e^t g'(e^t)} = \frac{f_1'(t)}{g_1'(t)}$$

hold true. Similar arguing is applicable also in the case, when the concave function $f(x)$ satisfies a slightly stronger condition: $x^\gamma f'(x)$ increases for some $\gamma \in (0,\, 1)$ (see Examples 3 and 4 below).

Let us provide some simple demonstrations of Theorem 1.

*Example 1.* Let $g(x) = x^p$, $x \in (0,\, +\infty)$, with $p > 1$. We fix $\theta \in [0,\, 1]$ and choose formulae (3), (4) to calculate constants $c_1(\theta)$, $c_2(\theta)$. The standards way of studying extrema of the function

$$\psi_{\theta,\, x}(t) = \frac{g(t)-\theta g(x)}{t-x} = \frac{t^p - \theta x^p}{t-x}$$

lead us to the equation

$$(1-p)\left(\frac{t}{x}\right)^p + p\left(\frac{t}{x}\right)^{p-1} = \theta,$$

and under the change $t = x\,\xi^{\frac{1}{p-1}}$ it becomes

$$(1-p)\,\xi^{\frac{p}{p-1}} + p\,\xi = \theta. \tag{8}$$

In the interval $(0,\, x)$, the function $\psi_{\theta,\, x}(t)$ attains the maximum at the point $t_1 = x\,\xi_1^{\frac{1}{p-1}}$ and the minimum in the interval $(x, +\infty)$ is attained at the point $t_2 = x\,\xi_2^{\frac{1}{p-1}}$, where $\xi_1$, $\xi_2$ are roots of the equation (8), and

$$0 \leqslant \xi_1 \leqslant 1 \leqslant \xi_2.$$

The associated extremal values are

$$\begin{aligned}\psi_{\theta,\, x}(t_k) =& \frac{t_k^p - \theta x^p}{t_k - x} = \frac{x^p \xi_k^{\frac{p}{p-1}} - \theta x^p}{x\xi_k^{\frac{1}{p-1}} - x} \\ =& x^{p-1}\, \frac{\xi_k^{\frac{p}{p-1}} - (1-p)\xi_k^{\frac{p}{p-1}} + p\xi_k}{\xi_k^{\frac{1}{p-1}} - 1} = p\, x^{p-1} \xi_k, \quad k = 1,\, 2.\end{aligned}$$

According to formulae (3), (4), we have

$$c_1(\theta) = \xi_1, \qquad c_2(\theta) = \xi_2,$$

where $\xi_1$, $\xi_2$ are the roots of equation (8). In particular, as $p = 2$, we obtain

$$c_1(\theta) = 1 - \sqrt{1-\theta}, \qquad c_2(\theta) = 1 + \sqrt{1-\theta}.$$

In this case Theorem 1 states that the derivative $f'(x)$ of a convex function $f(x)$ with the condition

$$m\, x^2 \leqslant f(x) \leqslant M\, x^2, \qquad x \in (0,\, +\infty),$$

satisfies the two-sided estimate

$$2\, M\left(1 - \sqrt{1 - m/M}\right) x \leqslant f'(x) \leqslant 2\, M\left(1 + \sqrt{1 - m/M}\right) x, \qquad x \in (0,\, +\infty),$$

where $0 \leqslant m \leqslant M$.

*Example 2.* Let $g(x) = e^{\rho x}$, $x \in (-\infty, +\infty)$, $\rho > 0$. We calculate the quantity $c_1(\theta)$ by formula (3). For fixed $x$ and $\theta \in [0, 1]$ we have

$$\sup_{-\infty<t<x} \frac{e^{\rho t} - \theta e^{\rho x}}{t - x} = e^{\rho x} \max_{-\infty<t<x} \frac{e^{\rho(t-x)} - \theta}{t - x} =: K_{x,\theta}.$$

The standard methods in analysis for finding the maximum point $t = t_0 < x$ lead us to the equation

$$e^{\rho(t-x)} (1 - \rho(t - x)) = \theta.$$

Under the change $\xi = e^{\rho(t-x)}$, the equation

$$\xi \ln \frac{e}{\xi} = \theta \tag{9}$$

arises and the sought maximum point is found by the formula $t_0 = x + \frac{1}{\rho} \ln \xi_1$. Here $\xi_1 = \xi_1(\theta)$ stands for the smaller root of equation (9). Hence,

$$K_{x,\theta} = \frac{e^{\rho x}(\xi_1 - \theta)}{\frac{1}{\rho} \ln \xi_1} = e^{\rho x} \frac{\xi_1 - \xi_1 \ln \frac{e}{\xi_1}}{\frac{1}{\rho} \ln \xi_1} = \rho e^{\rho x} \xi_1,$$

$$c_1(\theta) = \inf_{x\in(-\infty,+\infty)} \frac{1}{\rho e^{\rho x}} K_{x,\theta} = \xi_1.$$

In the same way we find that

$$c_2(\theta) = \xi_2,$$

where $\xi_2 = \xi_2(\theta)$ is the greater root of equation (9).

Thus, Theorem 1 states that the derivative of each convex function $f(x)$ with the condition

$$m\, e^{\rho x} \leqslant f(x) \leqslant M\, e^{\rho x}, \qquad x \in (-\infty, +\infty), \qquad \rho > 0,$$

satisfies the two-sided estimate

$$\xi_1\, \rho\, M\, e^{\rho x} \leqslant f'(x) \leqslant \xi_2\, \rho\, M\, e^{\rho x}, \qquad x \in (-\infty, +\infty),$$

where $\xi_1$, $\xi_2$ are the roots of equation (9), $\xi_1 \leqslant 1 \leqslant \xi_2$.

As it has been mentioned, Theorem 1 does not work directly for concave functions $f(x)$. However, if for some $\gamma \in (0, 1]$ the function $x^\gamma f'(x)$ increase, the situation changes.

*Example 3.* Let $f(x)$ be a concave function on an interval $(0, +\infty)$ obeying the conditions

$$m\, x^\rho \leqslant f(x) \leqslant M\, x^\rho, \qquad \rho \in (0, 1),$$

and $xf'(x)$ increases. Then applying Theorem 1 to the convex on the entire axis $(-\infty, +\infty)$ function $f_1(t) = f(e^t)$, we obtain two-sided estimate (see Example 2):

$$\xi_1\, \rho\, M\, x^{\rho-1} \leqslant f'(x) \leqslant \xi_2\, \rho\, M\, x^{\rho-1},$$

where $\xi_1$, $\xi_2$ are the roots to equation (9):

$$\xi \ln \frac{e}{\xi} = \frac{m}{M}.$$

*Example 4.* Let $f(x)$ be a concave on the interval $(0, +\infty)$ function obeying the conditions

$$m\, x^\rho \leqslant f(x) \leqslant M\, x^\rho, \qquad \rho \in (0, 1),$$

and $x^\gamma f'(x)$ increases for some $\gamma \in (1 - \rho, 1)$. Applying Theorem 1 to the convex on the interval $(0, +\infty)$ function $f_1(t) = f(t^\delta)$ with the exponent $\delta = \dfrac{1}{1-\gamma} > \dfrac{1}{\rho}$, we obtain that the condition

$$m \leqslant \frac{f(x)}{x^\rho} = \frac{f(t^\delta)}{t^{\delta\rho}} = \frac{f_1(t)}{t^{\delta\rho}} \leqslant M$$

yields the two-sided estimate

$$M\zeta_1 \leqslant \frac{f_1'(t)}{\delta\rho\, t^{\delta\rho-1}} \leqslant M\zeta_2.$$

But since

$$\frac{f_1'(t)}{\delta\rho\, t^{\delta\rho-1}} = \frac{\delta t^{\delta-1} f'(t^\delta)}{\delta\rho\, t^{\delta\rho-1}} = \frac{f'(t^\delta)}{\rho\, t^{\delta(\rho-1)}} = \frac{f'(x)}{\rho\, x^{\rho-1}},$$

the inequalities hold:

$$\zeta_1\, \rho\, M\, x^{\rho-1} \leqslant f(x) \leqslant \zeta_2\, \rho\, M\, x^{\rho-1}.$$

Here $\zeta_1$, $\zeta_2$ are the roots of equation (8) in Example 1 with the parameter $p = \delta\rho > 1$. Due to the relation

$$(1-p)\, \xi^{\frac{p}{p-1}} + p\xi \leqslant \xi \ln \frac{e}{\xi}, \quad \xi \in (0,\, e),$$

the inclusion $(\zeta_1,\, \zeta_2) \subset (\xi_1,\, \xi_2)$ holds and the estimates for the quotients of the derivatives in Example 4 are sharper than those in Example 3.

Now we consider the issue on comparing decreasing infinitesimal function.

**Theorem 2.** *Assume that a function $f(x)$ is convex on an interval $(a,b)$, $-\infty \leqslant a < b \leqslant +\infty$, a function $g(x)$ is differentiable on this interval and $g'(x) < 0$ and moreover, $g(a+) = +\infty$, $g(b-) = 0$. Assume also that condition* (1) *holds with non-negative constants $m$, $M$, $m \leqslant M$, that is,*

$$m \leqslant \frac{f(x)}{g(x)} \leqslant M, \qquad x \in (a,b).$$

*Then the two-sided estimate*

$$M\, d_1(\theta) \leqslant \frac{f'(x)}{g'(x)} \leqslant M\, d_2(\theta), \qquad x \in (a,b), \tag{10}$$

*holds true, where $\theta = \frac{m}{M}$, while the quantities $d_1(\theta)$, $d_2(\theta)$ are defined as*

$$d_1(\theta) = \inf_{x\in(a,\,b)} \frac{1}{g'(x)} \inf_{b>t>x} \frac{g(t)-\theta g(x)}{t-x}, \tag{11}$$

$$d_2(\theta) = \sup_{x\in(a,\,b)} \frac{1}{g'(x)} \sup_{a<t<x} \frac{g(t)-\theta g(x)}{t-x}. \tag{12}$$

*Proof.* The proof is similar to that of Theorem 1. We point out slight differences. Dividing (5) by $g'(x)$, for all $x \in (a,b)$ we obtain

$$\frac{f'(x)}{g'(x)} \geqslant M \frac{1}{g'(x)} \inf_{b>t>x} \frac{g(t)-\theta g(x)}{t-x} \geqslant M\, d_1(\theta),$$

which proves the left estimate in (10). Dividing (6) by $g'(x)$, for all $x \in (a,b)$ we obtain

$$\frac{f'(x)}{g'(x)} \leqslant M \frac{1}{g'(x)} \sup_{a<t<x} \frac{g(t)-\theta g(x)}{t-x} \leqslant M\, d_2(\theta).$$

The proof is complete. □

We observe that the proof can be done in a different way by applying Theorem 1 to the functions $f(-x)$ and $g(-x)$. For case $g(x) < 0$, $g'(x) > 0$ as $x \in (a,b)$, see the comments after Theorem 1.

*Example 5.* Let $g(x) = x^{-q}$, $x \in (0,\, +\infty)$, $q > 0$. We find quantities $d_1(\theta)$, $d_2(\theta)$ by formulae (11), (12). Since in our case

$$\inf_{t>x} \frac{t^{-q} - \theta x^{-q}}{t-x} = x^{-q-1} \inf_{t>x} \frac{\left(\frac{t}{x}\right)^{-q} - \theta}{\frac{t}{x} - 1} = x^{-q-1} \inf_{\xi<1} \frac{\xi - \theta}{\xi^{-\frac{1}{q}} - 1},$$

$$\left(\frac{\xi-\theta}{\xi^{-\frac{1}{q}}-1}\right)' = q^{-1} \left(\xi^{-\frac{1}{q}} - 1\right)^{-2} \xi^{-\frac{q+1}{q}} \left[\xi(q+1) - q\xi^{\frac{q+1}{q}} - \theta\right],$$

then

$$d_j(\theta) = \xi_j^{\frac{q+1}{q}}, \qquad j = 1,\, 2,$$

where $\xi_j$ are roots of equation (8) with $p = q+1$. In particular, for $q = 1$ we have

$$d_1(\theta) = \left(1 - \sqrt{1-\theta}\right)^2, \qquad d_2(\theta) = \left(1 + \sqrt{1-\theta}\right)^2,$$

and Theorem 2 ensures that the derivative of each convex function $f(x)$ with the condition

$$m \leqslant xf(x) \leqslant M, \qquad x \in (0, +\infty),$$

satisfies the estimate

$$\left(\sqrt{M} - \sqrt{M-m}\right)^2 \leqslant \left(-x^2\right) f'(x) \leqslant \left(\sqrt{M} + \sqrt{M-m}\right)^2, \qquad x \in (0, +\infty).$$

Here we still have $0 \leqslant m \leqslant M < +\infty$.

*Example 6.* Let $g(x) = e^{-\rho x}$, $x \in (-\infty, +\infty)$, $\rho > 0$. As in the previous example we find

$$\inf_{t>x} \frac{e^{-\rho t} - \theta e^{-\rho x}}{t-x} = e^{-\rho x} \inf_{t>x} \frac{e^{-\rho(t-x)} - \theta}{t-x} = \rho e^{-\rho x} \inf_{\xi<1} \frac{\xi - \theta}{\ln \frac{1}{\xi}},$$

$$\left(\frac{\xi-\theta}{\ln\frac{1}{\xi}}\right)' = \frac{\xi \ln \frac{e}{\xi} - \theta}{\xi \ln^2 \xi}, \qquad d_j(\xi) = -\frac{\xi_j - \theta}{\ln \frac{1}{\xi_j}} = -\frac{\xi_j - \xi_j \ln \frac{e}{\xi_j}}{\ln\frac{1}{\xi_j}} = \xi_j.$$

Thus,

$$d_1(\theta) = \xi_1, \quad d_2(\theta) = \xi_2, \quad \xi_1 \leqslant 1 \leqslant \xi_2,$$

where $\xi_1$, $\xi_2$ are roots of equation (9).

According to Theorem 2 we can state that the derivative of each convex function $f(x)$ with the condition

$$m \leqslant e^{\rho x} f(x) \leqslant M, \qquad x \in (-\infty, +\infty),$$

satisfies the two-sided estimate

$$\xi_1 \rho M \leqslant (-e^{\rho x}) f'(x) \leqslant \xi_2 \rho M, \qquad x \in (-\infty, +\infty),$$

where $\xi_1$, $\xi_2$ are the roots of equation (9), $\xi_1 \leqslant 1 \leqslant \xi_2$.

We consider some applications.

Let $f(z)$ be an entire function, $M_f(r)$ be the maximum of its absolute value in the circle of a radius $r$ centered at the origin and $0 \leqslant \sigma_0 < \sigma < +\infty$, $\rho > 0$. We assume that the inequalities

$$e^{\sigma_0 r^\rho} \leqslant M_f(r) \leqslant e^{\sigma r^\rho}, \qquad r > 0,$$

hold true. Then

$$\xi_1 \sigma\rho r^{\rho-1} M_f(r) \leqslant M_f'(r) \leqslant \xi_2 \sigma\rho r^{\rho-1} M_f(r), \qquad r > 0,$$

where $\xi_1$, $\xi_2$ are the roots of the equation

$$\xi \ln \frac{e}{\xi} = \frac{\sigma_0}{\sigma},$$

see (9).

Indeed, we rewrite initial inequalities as

$$\sigma_0 \leqslant \frac{\ln M_f(e^t)}{e^{\rho t}} \leqslant \sigma, \qquad t = \ln r \in \mathbb{R}.$$

Since the function $\ln M_f(e^t)$ is convex on $\mathbb{R}$, by Theorem 1 we obtain

$$\xi_1 \sigma \leqslant \frac{\left(\ln M_f(e^t)\right)'}{(e^{\rho t})'} = \frac{e^t M_f'(e^t)}{M_f(e^t) \rho\, e^{\rho t}} \leqslant \xi_2 \sigma.$$

Returning back to the initial variable, we arrive at the desired result.

By this, in particular, we conclude that if an entire function $f(z)$ with non-negative Taylor coefficients obeying $f(0) = 1$ satisfies the inequality

$$f(x) \leqslant e^{\sigma x}, \quad x > 0,$$

for some $\sigma > 0$, then its derivative satisfies the estimate

$$f'(x) \leqslant \sigma e f(x), \quad x > 0.$$

Indeed, as $\sigma_0 = 0$, the roots of equation (9), that is, of the equation

$$\xi \ln \frac{e}{\xi} = \frac{\sigma_0}{\sigma} = 0$$

are the numbers $\xi_1 = 0$, $\xi_2 = e$.

It is reasonable to compare the obtained fact with the well-known Bernstein theorem stating that an entire function of exponential type $\sigma$ bounded on the real axis, that is, obeying the condition

$$|f(x)| \leqslant M, \qquad x \in \mathbb{R},$$

satisfies the estimate

$$|f'(x)| \leqslant \sigma M, \qquad x \in \mathbb{R}.$$

At the same time, as it has been shown before, replacing the boundedness condition of a function on the real axis by the condition $|f(x)| \leqslant e^{\sigma x}$ on the half-axis and keeping the positivity of the Taylor coefficients, we arrive at the estimate

$$f'(x) \leqslant \sigma e f(x), \quad x > 0.$$

Similar estimates can be given for the functions of zero order.

For instance, consider entire functions of logarithmic growth. Let $0 \leqslant \sigma_0 < \sigma < +\infty$ and $\gamma > 1$. Assume that an entire function satisfies the conditions

$$e^{\sigma_0 (\ln r)^\gamma} \leqslant M_f(r) \leqslant e^{\sigma (\ln r)^\gamma}, \qquad r > 0.$$

Then

$$\xi_1\, \sigma\gamma\, (\ln r)^{\gamma-1} M_f(r) \leqslant r\, M_f'(r) \leqslant \xi_2\, \sigma\gamma\, (\ln r)^{\gamma-1} M_f(r), \qquad r > 0,$$

where $\xi_1$, $\xi_2$ are the roots of the equation

$$(1-\gamma)\, \xi^{\frac{\gamma}{\gamma-1}} + \gamma\, \xi = \frac{\sigma_0}{\sigma}$$

of form (8) with the parameters $p = \gamma$ and $\theta = \frac{\sigma_0}{\sigma}$.

Indeed, as above, we can write

$$\sigma_0 \leqslant \frac{\ln M_f(e^t)}{t^\gamma} \leqslant \sigma, \qquad t = \ln r \in \mathbb{R}.$$

In this case Theorem 1 ensures the estimate

$$\xi_1\, \sigma \leqslant \frac{\big(\ln M_f(e^t)\big)'}{(t^\gamma)'} = \frac{e^t M_f'(e^t)}{M_f(e^t)\, \gamma\, t^{\gamma-1}} \leqslant \xi_2\, \sigma,$$

which for $t = \ln r$ gives the desired statement.

In conclusion we note that some results of asymptotic character related to the L'Hôpital's rule and its inversion were exposed in [4]–[6].

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