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THIRD HANKEL DETERMINANT FOR THE INVERSE OF RECIPROCAL OF BOUNDED TURNING FUNCTIONS HAS A POSITIVE REAL PART OF ORDER ALPHA

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Abstract. Let RT be the class of functions $f(z)$ univalent in the unit disk $E = z : |z| < 1$ such that $\operatorname{Re} f'(z) > 0$, $z \in E$, and $H_3(1)$ be the third Hankel determinant for inverse function to $f(z)$. In this paper we obtain, first an upper bound for the second Hankel determinant, $|t_2t_3 - t_4|$, and the best possible upper bound for the third Hankel determinant $H3(1)$ for the functions in the class of inverse of reciprocal of bounded turning functions having a positive real part of order alpha.

Keywords: univalent function, function whose reciprocal derivative has a positive real part, third Hankel determinant, positive real function, Toeplitz determinants.

Mathematics Subject Classification: 30C45; 30C50

1. Introduction

Let A denote the class of all functions $f(z)$ of the form

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n \tag{1.1}$$

in the open unit disc $E = \{z : |z| < 1\}$. Let S be the subclass of A consisting of univalent functions. For a univalent function in the class A, it is well known that the n^{th} coefficient is bounded by n. The bounds for the coefficients of univalent functions give information about their geometric properties. In particular, the growth and distortion properties of a normalized univalent function are determined by the bound of its second coefficient. The Hankel determinant of f for $q \geqslant 1$ and $n \geqslant 1$ was defined by Pommerenke [12] as

$$H_q(n) = \begin{vmatrix} a_n & a_{n+1} & \cdots & a_{n+q-1} \\ a_{n+1} & a_{n+2} & \cdots & a_{n+q} \\ \vdots & \vdots & \vdots & \vdots \\ a_{n+q-1} & a_{n+q} & \cdots & a_{n+2q-2} \end{vmatrix}.$$

This determinant has been considered by many authors. For example, Noor [10] determined the rate of growth of $H_q(n)$ as $n \to \infty$ for the bounded functions in S. Ehrenborg [4] studied the Hankel determinant of exponential polynomials. The Hankel transform of an integer sequence and some of its properties were discussed by Layman in [7]. One can easily observe that the Fekete-Szego functional is $H_2(1)$. Fekete-Szego then further generalized the estimate for $|a_3 - \mu a_2^2|$ with μ real and $f \in S$. R. M. Ali [1] found sharp bounds for the first four coefficients

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and sharp estimate for the Fekete-Szego functional $|\gamma_3 - t\gamma_2^2|$, where t is real, for the inverse function of f defined as

$$f^{-1}(w) = w + \sum_{n=2}^{\infty} \gamma_n w^n,$$

when it belongs to the class of strongly starlike functions of order $\alpha (0 < \alpha \leqslant 1)$ denoted by $\widetilde{ST}(\alpha)$. In the recent years several authors studied bounds for the Hankel determinant of functions belonging to various subclasses of univalent and multivalent analytic functions. In particular, for $q = 2$, $n = 1$, $a_1 = 1$ and $q = 2$, $n = 2$, $a_1 = 1$, the Hankel determinant simplifies respectively to

$$H_2(1) = \begin{vmatrix} a_1 & a_2 \\ a_2 & a_3 \end{vmatrix} = a_3 - a_2^2 \quad \text{and} \quad H_2(2) = \begin{vmatrix} a_2 & a_3 \\ a_3 & a_4 \end{vmatrix} = a_2 a_4 - a_3^2.$$

For our discussion in this paper, we consider the Hankel determinant in the case of $q = 3$ and $n = 1$, denoted by $H_3(1)$, given by

$$H_3(1) = \begin{vmatrix} a_1 & a_2 & a_3 \\ a_2 & a_3 & a_4 \\ a_3 & a_4 & a_5 \end{vmatrix}. \tag{1.2}$$

For $f \in A$, $a_1 = 1$ we have

$$H_3(1) = a_3(a_2 a_4 - a_3^2) - a_4(a_4 - a_2 a_3) + a_5(a_3 - a_2^2)$$

and by applying the triangle inequality, we obtain

$$|H_3(1)| \leqslant |a_3||a_2 a_4 - a_3^2| + |a_4||a_2 a_3 - a_4| + |a_5||a_3 - a_2^2|. \tag{1.3}$$

For the second Hankel functional $H_2(2)$ for the subclass RT of S consisting of functions whose derivative has a positive real part studied by Mac Gregor [9] the sharp upper bound was obtained by Janteng [6]. It was known that if $f \in RT$ then $|a_k| \leqslant \frac{2}{k}$, for $k \in \{2, 3, \cdots\}$. Also the best possible sharp upper bound for the functional $|a_2 a_3 - a_4|$ was obtained by Babalola [2] and hence the sharp inequality for $|H_3(1)|$, for the class RT. Vamshee Krishna et al. [14] and also Venkateswarlu et al. [15] was obtained the sharp inequality $|H_3(1)|$, for the class of inverse of a function whose reciprocal derivative has a real part and of order alpha respectively. The sharp upper bound for the third Hankel determinant for the inverse of reciprocal of bounded turning functions was obtained by Venkateswarlu et al. [15].

Motivated by the above mentioned results obtained by different authors in this direction and the results by Babalola [2], in the present paper we seek an upper bound for the second Hankel determinant $|t_2 t_3 - t_4|$ and hence an upper bound to the third Hankel determinant for certain subclass of analytic functions defined as follows.

Definition 1.1. *A function $f(z) \in A$ is said to be function whose reciprocal derivative has a positive real part of order alpha (also called reciprocal of bounded turning function of order alpha) denoted by $f \in \widetilde{RT}(\alpha)$ for $0 \leqslant \alpha \leqslant 1$ if and only if*

$$\operatorname{Re}\left(\frac{1}{f'(z)}\right) > \alpha, \forall z \in E. \tag{1.4}$$

Choosing $\alpha = 0$, we obtain $\widetilde{RT}(0) = \widetilde{RT}$. Some preliminary lemmas required for proving our results are as follows.

2. Preliminary Results

Let $\mathbb{P}$ denote the class of functions denoted by p such that

$$p(z) = 1 + c_1 z + c_2 z^2 + c_3 z^3 + \ldots = 1 + \sum_{n=1}^{\infty} c_n z^n, \tag{2.1}$$

which are regular in the open unit disc E and satisfy $\operatorname{Re}\{p(z)\} > 0$ for each $z \in E$. Here $p(z)$ is called the Caratheódory function [3].

Lemma 2.1. [11, 13] *If $p \in \mathbb{P}$, then $|c_k| \leqslant 2$, for each $k \geqslant 1$ and the inequality is sharp for the function $\frac{1+z}{1-z}$.*

Lemma 2.2. [5] *The power series for $p(z)$ given in (2.1) converges in the open unit disc E to a function in $\mathbb{P}$ if and only if the Toeplitz determinants*

$$D_n = \begin{vmatrix} 2 & c_1 & c_2 & \cdots & c_n \\ c_{-1} & 2 & c_1 & \cdots & c_{n-1} \\ c_{-2} & c_{-1} & 2 & \cdots & c_{n-2} \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ c_{-n} & c_{-n+1} & c_{-n+2} & \cdots & 2 \end{vmatrix} \quad \textit{for} \quad n = 1, 2, 3, \ldots .$$

and $c_{-k} = \overline{c}_k$, are all non-negative. These determinants are strictly positive except for

$$p(z) = \sum_{k=1}^{m} \rho_k p_0(e^{it_k} z),$$

$\rho_k > 0$, t_k real and $t_k \neq t_j$, for $k \neq j$, where $p_0(z) = \frac{1+z}{1-z}$; in this case $D_n > 0$ for $n < (m-1)$ and $D_n = 0$ for $n \geqslant m$.

This necessary and sufficient condition found in [5] is due to Caratheódory and Toeplitz. We may assume without restriction that $c_1 > 0$. On using Lemma 2.2, for $n = 2$, we have

$$D_2 = \begin{vmatrix} 2 & c_1 & c_2 \\ \overline{c}_1 & 2 & c_1 \\ \overline{c}_2 & \overline{c}_1 & 2 \end{vmatrix} = [8 + 2\operatorname{Re}\{c_1^2 c_2\} - 2 \mid c_2 \mid^2 - 4|c_1|^2] \geqslant 0,$$

which is equivalent to

$$2c_2 = c_1^2 + x(4 - c_1^2) \quad \text{for some } x, \ |x| \leqslant 1. \tag{2.2}$$

For $n = 3$,

$$D_3 = \begin{vmatrix} 2 & c_1 & c_2 & c_3 \\ \overline{c}_1 & 2 & c_1 & c_2 \\ \overline{c}_2 & \overline{c}_1 & 2 & c_1 \\ \overline{c}_3 & \overline{c}_2 & \overline{c}_1 & 2 \end{vmatrix} \geqslant 0$$

and this is equivalent to

$$|(4c_3 - 4c_1c_2 + c_1^3)(4 - c_1^2) + c_1(2c_2 - c_1^2)^2| \leqslant 2(4 - c_1^2)^2 - 2|(2c_2 - c_1^2)|^2. \tag{2.3}$$

Simplifying the relations (2.2) and (2.3), we get

$$4c_3 = c_1^3 + 2c_1(4 - c_1^2)x - c_1(4 - c_1^2)x^2 + 2(4 - c_1^2)(1 - |x|^2)z$$

for some z with $|z| \leqslant 1$.

To obtain our results, we refer to the classical method initiated by Libera and Zlotkiewicz [8] and used then by several authors.

3. Main Result

Theorem 3.1. *If $f \in \widetilde{RT}(\alpha)(0 \leqslant \alpha \leqslant 1)$ and*

$$f^{-1}(w) = w + \sum_{n=2}^{\infty} t_n w^n$$

in the vicinity of $w = 0$ is the inverse function of f, then

$$|t_2 t_4 - t_3^2| \leqslant \begin{cases} \frac{(1-\alpha)^2}{144}\left[\frac{128\alpha^2-176\alpha+137}{\alpha^2-2\alpha+2}\right], & \text{for} \quad 0 \leqslant \alpha < \frac{3}{8}, \\ \left[\frac{2}{3}(1-\alpha)\right]^2, & \text{for} \quad \frac{3}{8} \leqslant \alpha \leqslant 1, \end{cases}$$

and the inequality is sharp.

Proof. For

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n \in \widetilde{RT}(\alpha),$$

there exists an analytic function $p \in \mathbb{P}$ in the open unit disc E with $p(0) = 1$ and $\operatorname{Re} p(z) > 0$ such that

$$\frac{1 - \alpha f'(z)}{(1-\alpha) f'(z)} = p(z) \Leftrightarrow 1 - \alpha f'(z) = (1-\alpha) f'(z) p(z). \tag{3.1}$$

Replacing $f'(z)$ and $p(z)$ with their equivalent series expressions in (3.1), we have

$$1 - \alpha\Big(1 + \sum_{n=2}^{\infty} n a_n z^{n-1}\Big) = (1-\alpha)\Big(1 + \sum_{n=2}^{\infty} n a_n z^n\Big)\Big(1 + \sum_{n=1}^{\infty} c_n z^n\Big).$$

Upon simplification, we obtain

$$\begin{aligned} (1-\alpha) - 2\alpha a_2 z - 3\alpha a_3 z^2 - 4\alpha a_4 z^3 - 5\alpha a_5 z^4 - \ldots &= (1-\alpha) + z(1-\alpha)(c_1 + 2a_2) \\ + z^2(1-\alpha)(c_2 + 2a_2 c_1 + 3a_3) &+ z^3(1-\alpha)(c_3 + 2a_2 c_2 + 3a_3 c_1 + 4a_4) \\ + z^4(1-\alpha)(c_4 + 2a_2 c_3 + 3a_3 c_2 + 4a_4 c_1 + 5a_5) \ldots. & \end{aligned} \tag{3.2}$$

Equating the coefficients at like powers of z, z^2, z^3 and z^4 respectively on both sides of (3.2), after simplifying, we get

$$\begin{aligned} a_2 &= -\frac{c_1(1-\alpha)}{2}; \\ a_3 &= -\frac{(1-\alpha)}{3}[c_2 - (1-\alpha)c_1^2]; \\ a_4 &= -\frac{(1-\alpha)}{4}\Big[c_3 - 2(1-\alpha)c_1 c_2 + (1-\alpha)^2 c_1^3\Big]; \\ a_5 &= -\frac{(1-\alpha)}{5}\Big[c_4 - 2(1-\alpha)c_1 c_3 + 3(1-\alpha)^2 c_1^2 c_2 - (1-\alpha)c_2^2 - (1-\alpha)^3 c_1^4\Big]. \end{aligned} \tag{3.3}$$

Since

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n \in \widetilde{RT}(\alpha),$$

by the definition of inverse function of f, we have

$$w = f(f^{-1}(w)) = f^{-1}(w) + \sum_{n=2}^{\infty} a_n \big(f^{-1}(w)\big)^n \Leftrightarrow w = w + \sum_{n=2}^{\infty} t_n w^n + \sum_{n=2}^{\infty} a_n \Big(w + \sum_{n=2}^{\infty} t_n w^n\Big)^n.$$

After simplifying, we get

$$(t_2+a_2)w^2+(t_3+2a_2t_2+a_3)w^3+(t_4+2a_2t_3+a_2t_2^2+3a_3t_2+a_4)w^4 \\ +(t_5+2a_2t_4+2a_2t_2t_3+3a_3t_3+3a_3t_2^2+4a_4t_2+a_5)w^5+\cdots=0. \tag{3.4}$$

Equating the coefficients of like powers of w^2, w^3, w^4 and w^5 on both sides of (3.4), respectively, further simplification gives

$$t_2=-a_2; \qquad t_3=-a_3+2a_2^2; \qquad t_4=-a_4+5a_2a_3-5a_2^3; \\ t_5=-a_5+6a_2a_4-21a_2^2a_3+3a_3^2+14a_2^4. \tag{3.5}$$

Using the values of a_2, a_3, a_4 and a_5 in (3.3) along with (3.5), upon simplification, we obtain

$$\begin{aligned} t_2&=\frac{c_1(1-\alpha)}{2}; \quad t_3=\frac{(1-\alpha)}{6}\Big[2c_2+(1-\alpha)c_1^2\Big];\\ t_4&=\frac{(1-\alpha)}{24}\Big[6c_3+8(1-\alpha)c_1c_2+(1-\alpha)^2c_1^3\Big];\\ t_5&=\frac{(1-\alpha)}{120}\Big[(1-\alpha)^3c_1^4+42c_1c_3(1-\alpha)+16c_2^2(1-\alpha)+22c_1^2c_2(1-\alpha)^2+24c_4\Big]. \end{aligned} \tag{3.6}$$

Substituting the values of t_2, t_3 and t_4 from (3.6) in the functional $|t_2t_4-t_3^2|$ for the function $f\in\widetilde{RT}(\alpha)$, upon simplification, we obtain

$$|t_2t_4-t_3^2|=\frac{(1-\alpha)^2}{144}\Big|18c_1c_3+8(1-\alpha)c_1^2c_2-16c_2^2-(1-\alpha)^2c_1^4\Big|$$

which is equivalent to

$$|t_2t_4-t_3^2|=\frac{(1-\alpha)^2}{144}\Big|d_1c_1c_3+d_2c_1^2c_2+d_3c_2^2+d_4c_1^4\Big|, \tag{3.7}$$

$$d_1=18; \quad d_2=8(1-\alpha); \quad d_3=-16; \quad d_4=-(1-\alpha)^2. \tag{3.8}$$

Substituting the values of c_2 and c_3 given in (2.2) and (2.4) respectively from Lemma 2.2 into the right-hand side of (3.7), and using the fact that $|z|<1$, we have

$$4|d_1c_1c_3+d_2c_1^2c_2+d_3c_2^2+d_4c_1^4|\leqslant\Big|(d_1+2d_2+d_3+4d_4)c_1^4+2d_1c_1(4-c_1^2) \\ +2(d_1+d_2+d_3)c_1^2(4-c_1^2)|x|-\{(d_1+d_3)c_1^2+2d_1c_1-4d_3\}|x|^2(4-c_1^2)\Big|. \tag{3.9}$$

From (3.8) and (3.9), we can now write

$$\begin{aligned} &d_1+2d_2+d_3+4d_4=2(-2\alpha^2-4\alpha+7);\\ &2(d_1+d_2+d_3)=4(5-4\alpha);\\ &(d_1+d_3)c_1^2+2d_1c_1-4d_3=2(c_1^2+18c_1+32); d_1=18. \end{aligned} \tag{3.10}$$

Since $c_1=c\in[0,2]$, using the result $(c_1+a)(c_1+b)\geqslant(c_1-a)(c_1-b)$, where $a,b\geqslant 0$, and applying triangle inequality, we can have

$$-(d_1+d_3)c_1^2+2d_1c_1-4d_3=-2(c_1^2-18c_1+32). \tag{3.11}$$

Substituting the calculated values from (3.10) and (3.11) on the right-hand side of (3.9), we have

$$4|d_1c_1c_3+d_2c_1^2c_2+d_3c_2^2+d_4c_1^4|\leqslant\Big|2(-2\alpha^2-4\alpha+7)c_1^4 \\ +36(4c_1-c_1^3)+4c_1^2(5-4\alpha)(4-c_1^2)|x|-2(c_1^2-18c_1+32)|x|^2(4-c_1^2)\Big|. \tag{3.12}$$

Choosing $c_1 = c \in [0,2]$, applying the triangle inequality and replacing $|x|$ by μ on the right-hand side of the above inequality

$$\begin{aligned} 4|d_1c_1c_3 + d_2c_1^2c_2 + d_3c_2^2 + d_4c_1^4| \leqslant \Big| & 2(-2\alpha^2 - 4\alpha + 7)c^4 + 36c(4-c^2) \\ & + 4c^2(5-4\alpha)(4-c^2)\mu + 2(c-16)(c-2)\mu^2(4-c^2)\Big| \\ & = F(c,\mu), \quad 0 \leqslant \mu = |x| \leqslant 1 \quad \text{and} \quad 0 \leqslant c \leqslant 2. \end{aligned} \tag{3.13}$$

Now we maximize the function $F(c,\mu)$ on the closed region $[0,2]\times[0,1]$. Differentiating $F(c,\mu)$ given in (3.13) with respect to μ, we obtain

$$\frac{\partial F}{\partial \mu} = 4\Big[(5-4\alpha)c^2 + (c-16)(c-2)\mu\Big](4-c^2) > 0. \tag{3.14}$$

For $0 < \mu < 1$ and for fixed c with $0 < c < 2$, from (3.14), we observe that $\frac{\partial F}{\partial \mu} > 0$. Therefore, $F(c,\mu)$ becomes an increasing function of μ and hence it cannot have a maximum value at any point in the interior of the closed region $[0,2]\times[0,1]$. Moreover, for a fixed $c \in [0,2]$, we have

$$\max_{0\leqslant\mu\leqslant1} F(c,\mu) = F(c,1) = G(c).$$

Therefore, replacing μ by 1 in $F(c,\mu)$, upon simplification, we obtain

$$G(c) = -4(\alpha^2 - 2\alpha + 2)c^4 + 8(3-8\alpha)c^2 + 256. \tag{3.15}$$

$$G'(c) = -16(\alpha^2 - 2\alpha + 2)c^3 + 16(3-8\alpha)c. \tag{3.16}$$

$$G''(c) = -48(\alpha^2 - 2\alpha + 2)c^2 + 16(3-8\alpha). \tag{3.17}$$

For optimum value of $G(c)$, consider $G'(c) = 0$. From (3.16), we get

$$c = 0 \quad \text{or} \quad c^2 = \Big[\frac{3-8\alpha}{\alpha^2 - 2\alpha + 2}\Big]. \tag{3.18}$$

Case 1. Suppose $0 \leqslant \alpha < \frac{3}{8}$. By (3.18), for $c = 0$, in (3.17), we get $G''(c) = 16(3-8\alpha) > 0$. Then $G(c)$ has minimum at $c = 0$. For $c^2 = \frac{3-8\alpha}{\alpha^2-2\alpha+2}$, in (3.17), we get $G''(c) = -32(3-8\alpha) < 0$. Therefore, by the second derivative test, $G(c)$ has maximum value at

$$c = \sqrt{\frac{3-8\alpha}{\alpha^2 - 2\alpha + 2}}.$$

Substituting the value of c in expression (3.15), upon simplification, we obtain the maximum value of $G(c)$:

$$G_{max}(c) = \frac{4(128\alpha^2 - 176\alpha + 137)}{\alpha^2 - 2\alpha + 2}. \tag{3.19}$$

Simplifying expressions (3.13) and (3.19), we have

$$|d_1c_1c_3 + d_2c_1^2c_2 + d_3c_2^2 + d_4c_1^4| \leqslant \frac{128\alpha^2 - 176\alpha + 137}{\alpha^2 - 2\alpha + 2}. \tag{3.20}$$

By relations (3.7) and (3.20), upon simplification, we obtain

$$|t_2t_4 - t_3^2| = \frac{(1-\alpha)^2}{144}\Big[\frac{128\alpha^2 - 176\alpha + 137}{\alpha^2 - 2\alpha + 2}\Big]. \tag{3.21}$$

Case 2. Suppose $\frac{3}{8} \leqslant \alpha \leqslant 1$. By (3.18), for

$$c^2 = \frac{3-8\alpha}{\alpha^2 - 2\alpha + 2},$$

in (3.17), we get $G''(c) = -32(3-8\alpha) > 0$. Then $G''(c)$ has minimum at

$$c^2 = \frac{3-8\alpha}{\alpha^2-2\alpha+2}.$$

For $c = 0$, in (3.17) we get $G''(c) = 16(3-8\alpha) < 0$. Therefore, by the second derivative test, $G(c)$ has maximum value at $c = 0$. Substituting the value of c in expression (3.15), upon simplification, we obtain the maximum value of $G(c)$:

$$G_{max}(c) = 256. \tag{3.22}$$

Simplifying the expressions (3.13) and (3.22), we obtain

$$|d_1c_1c_3 + d_2c_1^2c_2 + d_3c_2^2 + d_4c_1^4| \leqslant 64. \tag{3.23}$$

From the relations (3.7) and (3.23), upon simplification, we obtain

$$|t_2t_4 - t_3^2| = \left[\frac{2}{3}(1-\alpha)\right]^2.$$

By setting $c_1 = c = 0$ and choosing $x = 1$ in expressions (2.2) and (2.4), we find that $c_2 = 2$ and $c_3 = 0$ respectively. Substituting the identity is attained, which shows that our result is sharp and the extremal function in this case is given by

$$\left[\frac{1}{f'(z)}\right] = 1 + 2z^2 + 2z^4 + \cdots = \frac{1+z^2}{1-z^2}.$$

This completes the proof of our theorem. □

Remark 3.1. *For this we chose $\alpha = 0$, in (3.21) we get $|t_2t_3 - t_4^2| \leqslant \frac{137}{288}$. This result is coincides with Venkateswarlu et al.* [15] *and also Vamshee Krishna et al.* [14]*. From this we conclude that, for $\alpha = 0$, the sharp upper bound for the second Hankel determinant of a function whose derivative has a positive real part and a function whose reciprocal derivative has a positive real part is the same.*

Theorem 3.2. *If $f \in \widetilde{RT}(\alpha)(0 \leqslant \alpha \leqslant 1)$ and*

$$f^{-1}(w) = w + \sum_{n=2}^{\infty} t_n w^n$$

in the vicinity of $w = 0$ is the inverse function of f, then

$$|\, t_2t_3 - t_4 \,| \leqslant \frac{2}{3}\left[\frac{1-\alpha}{\sqrt{\alpha^2-2\alpha+4}}\Big(\frac{13-4\alpha}{6}\Big)^{\frac{3}{2}}\right].$$

Proof. Substituting the values of t_2, t_3 and t_4, from (3.3) in the determinant $|t_2t_3 - t_4|$ for the function $f \in \widetilde{RT}(\alpha)$, after simplifying, we get

$$\begin{aligned} |t_2t_3 - t_4| &= \Big|\frac{(1-\alpha)^2}{12}(2c_1c_2 + (1-\alpha)c_1^3) - \frac{(1-\alpha)}{24}(6c_3 + 8(1-\alpha)c_1c_2 + (1-\alpha)^2c_1^3)\Big| \\ &= \frac{(1-\alpha)}{24}|(1-\alpha)^2c_1^3 - 6c_3 - 4c_1c_2(1-\alpha)|. \end{aligned} \tag{3.24}$$

Substituting the values of c_2 and c_3 from (2.2) and (2.4) respectively from Lemma 2.2 into the right-hand side of (3.24) and using the fact $|z| < 1$, we have

$$\begin{aligned} 2\,|\,(1-\alpha)^2c_1^3 - 6c_3 - 4c_1c_2(1-\alpha)\,| \leqslant \Big| &-(5-2\alpha^2)c_1^3 - 6(4-c_1^2) \\ &- 2c_1(4-c_1^2)(5-2\alpha)\,|\,x\,| + 3(c_1+2)(4-c_1^2)\,|\,x\,|^2\,\Big|. \end{aligned} \tag{3.25}$$

Since $c_1 = c \in [0,2]$, using the estimate $(c_1+a) \geqslant (c_1-a)$, where $a \geqslant 0$, applying the triangle inequality and replacing $|x|$ by μ in the right-hand side of the above inequality, we have

$$\begin{aligned} 2 \mid (1-\alpha)^2 c_1^3 - 6c_3 - 4c_1c_2(1-\alpha) \mid \leqslant & \Big|(5-2\alpha^2)c^3 + 6(4-c^2) \\ & + 2c(4-c^2)(5-2\alpha)\mu + 3(c-2)(4-c^2)\mu^2\Big| \\ & = F(c,\mu), 0 \leqslant \mu = \mid x \mid \leqslant 1 \quad \text{and} \quad 0 \leqslant c \leqslant 2. \end{aligned} \tag{3.26}$$

Then we maximize the function $F(c,\mu)$ on the closed square $[0,2]\times[0,1]$. Differentiating (3.26), i.e., $F(c,\mu)$ with respect to μ, we get

$$\frac{\partial F}{\partial \mu} = 2\Big[(5-2\alpha)c + 3(c-2)\mu\Big](4-c^2) > 0.$$

As described in Theorem 3.1, further, we obtain

$$G(c) = -2c^3(\alpha^2 - 2\alpha + 4) + 4c(13 - 4\alpha). \tag{3.27}$$

$$G'(c) = -6c^2(\alpha^2 - 2\alpha + 4) + 4(13 - 4\alpha). \tag{3.28}$$

$$G''(c) = -12c(\alpha^2 - 2\alpha + 4). \tag{3.29}$$

For optimum value of $G(c)$, consider $G'(c) = 0$. From (3.28), we get

$$c^2 = \frac{2(13-4\alpha)}{3(\alpha^2 - 2\alpha + 4)} \quad \text{for} \quad 0 \leqslant \alpha \leqslant 1.$$

Using the obtained value of $c = \sqrt{\frac{2(13-4\alpha)}{3(\alpha^2-2\alpha+4)}} \in [0,2]$ in (3.29), we arrive at

$$G''(c) = -12\left[\sqrt{\frac{2(13-4\alpha)}{3(\alpha^2-2\alpha+4)}}\right](\alpha^2 - 2\alpha + 4) < 0.$$

Therefore, by the second derivative test, $G(c)$ has maximum value at c, where

$$c = \sqrt{\frac{2(13-4\alpha)}{3(\alpha^2-2\alpha+4)}}.$$

Substituting the value of c in the expression (3.27), upon simplification, we obtain the maximum value of $G(c)$ at c:

$$G_{max} = \frac{32}{\sqrt{\alpha^2 - 2\alpha + 4}}\left[\frac{13-4\alpha}{6}\right]^{\frac{3}{2}}. \tag{3.30}$$

By expressions (3.26) and (3.30), after simplifying, we get

$$\mid (1-\alpha)^2 c_1^3 - 6c_3 - 4(1-\alpha)c_1c_2 \mid \leqslant \frac{16}{\sqrt{\alpha^2 - 2\alpha + 4}}\left[\frac{13-4\alpha}{6}\right]^{\frac{3}{2}}. \tag{3.31}$$

Simplifying the relations (3.24) and (3.31), we obtain

$$|t_2t_3 - t_4| \leqslant \frac{2(1-\alpha)}{3\sqrt{\alpha^2 - 2\alpha + 4}}\left[\frac{13-4\alpha}{6}\right]^{\frac{3}{2}}. \tag{3.32}$$

This completes the proof of the theorem. □

Remark 3.2. *For the choice of* $\alpha = 0$, *from (3.32), we obtain* $|t_2t_3 - t_4| \leqslant \frac{1}{3}\left[\frac{13}{6}\right]^{\frac{3}{2}}$. *This result coincide with that by Vamshee Krishna et al.* [14] *and also by Venkateswarlu et al.* [16]. *We observe that the upper bound for* $|t_2t_3 - t_4|$ *of a function whose derivative has a positive real part* [14] *and a function whose reciprocal derivative has a positive real part is the same.*

The next theorem can be proved straightforwardly by applying the same procedure as in the proof of Theorems 3.1 and 3.2 and the result is sharp for the values $c_1 = 0$, $c_2 = 2$ and $x = 1$.

Theorem 3.3. *If $f \in \widetilde{RT}(\alpha)(0 \leqslant \alpha \leqslant 1)$ and $f^{-1}(w) = w + \sum\limits_{n=2}^{\infty} t_n w^n$ near $w = 0$ is the inverse function of f then $|t_3 - t_2^2| \leqslant \frac{2}{3}[1-\alpha]$.*

Using the fact that

$$|c_n| \leqslant 2, \quad n \in N = \{1, 2, 3, \ldots\},$$

by values of c_2 and c_3 given in (2.2) and (2.4) respectively together with the values in (3.6), we arrive at the following inequalities.

Theorem 3.4. *If $f(z) \in \widetilde{RT}(\alpha)$, $(0 \leqslant \alpha \leqslant 1)$ and*

$$f^{-1}(w) = w + \sum_{n=2}^{\infty} t_n w^n$$

in the vicinity of $w = 0$ is the inverse function of f, then the following inequalities
(i) $|t_2| \leqslant (1-\alpha)$,
(ii) $|t_3| \leqslant \frac{2}{3}(1-\alpha)(2-\alpha)$,
(iii) $|t_4| \leqslant \frac{(1-\alpha)}{6}[2\alpha^2 - 12\alpha + 13]$,
(iv) $|t_5| \leqslant \frac{(1-\alpha)}{15}[-2\alpha^3 + 28\alpha^2 - 79\alpha + 59]$
hold.

Using the results of Theorems 3.1, 3.3, 3.5 and 3.6, we arrive at the following corollary.

Corollary 1. *If $f \in \widetilde{RT}(\alpha)(0 \leqslant \alpha \leqslant 1)$ and $f^{-1}(w) = w + \sum\limits_{n=2}^{\infty} t_n w^n$ near $w = 0$ is the inverse function of f then*

$$|H_3(1)| = \begin{cases} \dfrac{(1-\alpha)^2}{1080}\Bigg[\dfrac{5(\alpha^2-3\alpha+2)(128\alpha^2-176\alpha+137)}{\alpha^2-2\alpha+2} \\ \qquad + \dfrac{120(2\alpha^2-12\alpha+13)}{\sqrt{\alpha^2-2\alpha+4}}\Big(\dfrac{13-4\alpha}{6}\Big)^{\frac{3}{2}} \\ \qquad + 48(-2\alpha^3-28\alpha^2-79\alpha+59)\Bigg], \quad \text{for} \quad 0 \leqslant \alpha < \dfrac{3}{8}, \\ \dfrac{(1-\alpha)^2}{135}\Bigg[40(\alpha^2-3\alpha+2) + \dfrac{15}{\sqrt{\alpha^2-2\alpha+4}}\Big(\dfrac{13-4\alpha}{6}\Big)^{\frac{3}{2}} \\ \qquad + 6(-2\alpha^3-28\alpha^2-79\alpha+59)\Bigg], \quad \text{for} \quad \dfrac{3}{8} \leqslant \alpha \leqslant 1. \end{cases}$$

Remark 3.3. *If we choose $\alpha = 0$ in the above expressions, we obtain $|H_3(1)| \leqslant 0.742$. These inequalities are sharp and coincide with the results by Vamshee Krishna et al.* [14] *and also by Venkateswarlu et al.* [15]. *We observe that the upper bound for the third Hankel determinant of a function whose derivative has a positive real part* [14] *and a function whose reciprocal derivative has a positive real part is the same.*

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