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DICRETE HÖLDER ESTIMATES FOR A CERTAIN KIND OF PARAMETRIX. II

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Abstract. In the first paper of this series we have introduced a certain parametrix and the associated potential. The parametrix corresponds to a uniformly elliptic second order differential operator with locally Hölder continuous coefficients in the half-space. Here we show that the potential is an approximate left inverse of the differential operator modulo hyperplane integrals, with the error estimated in terms of the local Hölder norms. As a corollary, we calculate approximately the potential whose density and differential operator originate from the straightening of a special Lipschitz domain. This corollary is aimed for the future derivation of approximate formulae for harmonic functions.

Keywords: cubic discretization, Lipschitz domain, local Hölder norms, parametrix, potential, straightening.

Mathematics Subject Classification: 35A17

1. Introduction

Let $\mathcal{A}_\lambda^\mu$ be the family of all second order uniformly elliptic operators in the upper half-space $\mathbb{R}_+^n$ ($n \geqslant 2$) with an ellipticity constant $\lambda \geqslant 1$ and locally μ-Hölder coefficients, $0 < \mu < 1$. In work [1] a Z-parametrix $E(A; x, y)$ (shortly: parametrix) was proposed for an operator $A \in \mathcal{A}_\lambda^\mu$ and for the corresponding potential

$$\Phi_f(x) = \int_{y_n > 0} E(A; x, y) f(y)\, dy, \quad x \in \mathbb{R}_+^n,$$

estimates for local Hölder norms $\|D^\alpha \Phi_f\|_I$ ($|\alpha| \leqslant 2$) and $\|Rf\|_I$ were established in terms of the same norms $\|f\|_J$, where $f \mapsto Rf = f - A\Phi_f$ is the error operator.

The parametrix $E(A; x, y)$ and the potential Φ_f were introduced in order to study a special harmonic function. Let Ω be the overgraph of a Lipschitz function $\omega : \mathbb{R}^{n-1} \to \mathbb{R}$. Lemma 3.7 in [2] and the properties of the Kelvin transform imply the existence and the uniqueness up to a positive multiplicative constant of a function U with the following properties:

$$U \in C^\infty(\Omega) \cap C\big(\overline{\Omega}\big), \quad \Delta U = 0 \text{ and } U > 0 \text{ in domain } \Omega, \quad U\big|_{\partial\Omega} = 0.$$

Up to the equivalence, the function U determines the behavior of arbitrary positive harmonic functions vanishing continuously on a part of the boundary of a Lipschitz domain. Indeed, roughly speaking, each two such functions are comparable by the boundary Harnack principle. As an example see [3, Thm. 5.1].

Let us outline the plan of studying the function U. Denoting

$$u = (U \circ g)\varphi$$

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for an appropriate straightening diffeomorphism $g : \mathbb{R}^n_+ \to \Omega$ and a cut-off function $\varphi \in C_0^\infty\big(\overline{\mathbb{R}^n_+}\big)$, by the Laplace equation $\Delta U = 0$ we obtain the differential equation

$$Au = LD_n u + L'$$

for some operator $A \in \mathcal{A}^\mu_\lambda$ and functions $L, L' \in C^\infty(\mathbb{R}^n_+)$. Here A and L depend on Ω and g, but not on U and φ. If the function ω is compactly supported and its Lipschitz constant is sufficiently small, the Neumann series

$$Q = \sum_{k=0}^{\infty} R^k$$

makes sense. The boundary condition $u\big|_{\partial\mathbb{R}^n_+} = 0$ and the boundedness of the support of the function u are preconditions for the validity of the integral representation

$$u = \Phi_F, \quad F = QAu = Q(vL + L'), \quad v = D_n u = D_n \Phi_F.$$

For the function $\mathbf{x}_n^{-1}(x) = x_n^{-1}$ and a number v_0 we write

$$v = \mathbf{x}_n^{-1}\Phi_F\{\underbrace{D_n\Phi_L - \mathbf{x}_n^{-1}\Phi_L + 1}_{\Theta}\} + \underbrace{D_n\Phi_{F-v_0L} - \mathbf{x}_n^{-1}\Phi_{F-v_0L}}_{\Theta_1} + \{\underbrace{D_n\Phi_L - \mathbf{x}_n^{-1}\Phi_L}_{\Theta_2}\}\{\underbrace{v_0 - \mathbf{x}_n^{-1}\Phi_F}_{\Theta_3}\}.$$

It turns out that $\Theta \approx \mathbf{x}_n D_n(S \circ g)$, where

$$S(x) = \lim_{r\to\infty}\left\{\ln r - \frac{\Gamma(n/2)}{\pi^{n/2}}\int_{y\in\mathbb{R}^n\setminus\Omega:\ |x-y|<r} |x-y|^{-n}\,dy\right\}, \quad x \in \Omega,$$

and the approximate error is quadratic in approximation numbers b_I expressing how close locally the surface $\partial\Omega$ is to a hyperplane. We can choose v_0 so that the term Θ_1 is estimated quadratically in b_I, while the expressions Θ_2 and Θ_3 are estimated linearly. This implies that $\frac{D_n u}{u} \approx D_n(S \circ g)$ with a quadratic error. Generalizing the arguments and the definition of the function S to the case of a not necessarily compactly supported function ω with an arbitrary Lipschitz constant, by means of rotations of the coordinate system we obtain the approximate formula

$$\frac{\nabla U}{U} \approx \nabla S. \tag{1}$$

The integration of this formula gives rise to the exponential asymptotic formula (EAF)

$$U \approx U_0 \mathrm{e}^S.$$

For known EAFs for conformal mappings, EAFs for solutions to elliptic systems and asymptotics for positive harmonic functions see works [4]–[8].

The present paper is devoted to realizing a part of the outlined plan, namely, to justifying, for error term in the formula $\Theta \approx \mathbf{x}_n D_n(S \circ g)$, an estimate quadratic in approximating numbers of the function ω. The paper consists of the introduction and two sections. In Section 2 we find approximately the potential Φ_{Af}. The main definitions are given in Subsections 2.1 and 2.2. In Subsection 2.3, the discrete Hölder estimates from [1] for the functions $D^\alpha\Phi_f$ and Rf are completed by an estimate for the expression $D_n\Phi_f - \mathbf{x}_n^{-1}\Phi_f$, which is more precise than the independent estimates for the functions Φ_f and $D_n\Phi_f$. In Subsection 2.4, the derivatives $D^\alpha\Phi_{Af}$ and the expression $D_n\Phi_{Af} - \mathbf{x}_n^{-1}\Phi_{Af}$ are found up to the errors majorized by local Hölder seminorms $|A|_J$ of the coefficients of the operator A and by the norms $\|D^2 f\|_J$.

In Section 3, to a pair (ω, θ), where $\theta \geqslant \|\omega\|_{\mathrm{Lip}}$, we associate the *standard set*

$$\big(\{\gamma_K\}, w, W, g, \mathfrak{g}, \mathfrak{G}, A, \lambda, L\big)$$

relating to a straightening of the domain Ω, after that the formula $\Theta \approx \mathbf{x}_n D_n(S \circ g)$ and its analogue for the derivatives $D_{ij}\Phi_L$ are established by a reduction to Subsection 2.4. We observe that the formula for the derivatives $D_{ij}\Phi_L$ can be used while obtaining an analogue of formula (1) for the derivatives $D_{ij}U$.

Convention. The letter c (with a possible subscript or superscript) stands for various positive constants and always equipped by the brackets with all numerical parameters, on which these constants depend. For $t > 0$ and a cube or a ball $X \subset \mathbb{R}^d$ centered at $\mathfrak{c}_X$ and of the edge or radius of an arbitrary length we let

$$tX = \{\mathfrak{c}_X + t(\xi - \mathfrak{c}_X)\colon \xi \in X\}.$$

If $\xi \in \mathbb{R}^d$, then $|\xi|_\infty = \max_i |\xi_i|$ and (if ξ is not a multi-index) $|\xi|^2 = \sum_i |\xi_i|^2$. For multi-indices $\alpha \in \mathbb{N}_0^d$, by $D^\alpha f$ we denote partial derivatives of a real function f, at that, $D_i f \equiv D^{e_i} f$ and $D_{ij} f \equiv D^{e_i+e_j} f$, where $\{e_i\}_1^d$ is the canonical basis in $\mathbb{R}^d$. For a semi-norm p and a number $q \in \mathbb{N}_0$ we let

$$p(D^q f) = \max_{|\alpha|=q} p(D^\alpha f).$$

For instance, $|Df| = \max_{1\leqslant i\leqslant d} |D_i f| = |\nabla f|_\infty$, where ∇f is the gradient of the function f. By $\overline{X}$ and X° we denote the closure and the interior of a set $X \subset \mathbb{R}^d$.

2. Approximate calculations with potential Φ_{Af}

2.1. Basic information on a dyadic family. Given an integer $n \geqslant 2$, we introduce a dyadic family $\mathcal{D}$ in $\mathbb{R}^{n-1}$:

$$\mathcal{D} = \bigcup_{k\in\mathbb{Z}} \mathcal{D}_k, \quad \mathcal{D}_k = \bigl\{I\colon I = [0, 2^k)^{n-1} + 2^k a \text{ for some } a \in \mathbb{Z}^{n-1}\bigr\}.$$

For the sets $I_i \subset \mathbb{R}^{n-1}$ with a bounded non-empty union we let

$$[I_1, I_2] = \sup_{\xi,\eta\in I_1\cup I_2} |\xi - \eta|_\infty.$$

We denote $l_I = [I, \varnothing]$ as $I \in \mathcal{D}$ (the side-length). For $\alpha, \beta \in \mathbb{R}$ we let

$$\Gamma_{IJ}^{(\alpha,\beta)} = l_I^\alpha l_J^\beta [I, J]^{-\alpha-\beta}, \quad I, J \in \mathcal{D}.$$

The following statements is Theorem 2(a) proved in [9]. Hereinafter, unless otherwise said, the summation are taken over the set $\mathcal{D}$.

Lemma 1. *If $\alpha > 0$ and $\beta > n - 1$, then*

$$\sum_J \Gamma_{IJ}^{(\alpha,\beta)} \leqslant c(n, \alpha, \beta), \quad I \in \mathcal{D}.$$

For $I, J \in \mathcal{D}$ we say $I \odot J$ if $l_I = l_J$ and $\overline{I} \cap \overline{J} \neq \varnothing$. By $\{I^J, J^I\}$ we denote a pair of cubes $\{H_1, H_2\} \subset \mathcal{D}$ with the smallest possible value of $l_{H_1} = l_{H_2}$ and the property

$$I \subset H_1 \odot H_2 \supset J.$$

The cubes I and J can be connected by the chain

$$\widehat{IJ} = \bigl\{H \in \mathcal{D}\colon I \subset H \subset I^J \text{ or } J \subset H \subset J^I\bigr\}.$$

We fix $\mu \in (0, 1)$. For a function f on a set $X \subset \mathbb{R}^d$ containing more than one point, we let

$$|f|_{C^\mu(X)} = \sup_{x,y\in X\colon x\neq y} \frac{\bigl|f(x) - f(y)\bigr|}{|x - y|^\mu}, \quad \|f\|_{\mathrm{Lip}} = \sup_{x,y\in X\colon x\neq y} \frac{\bigl|f(x) - f(y)\bigr|}{|x - y|}.$$

We denote

$$\mathbb{R}^n_+ = \{x = (x', x_n) \in \mathbb{R}^n : x_n > 0\},$$
$$I^{\boxdot} = \overline{I} \times [l_I, 2l_I], \quad I \in \mathcal{D},$$
$$\mathfrak{c}^{\boxdot}_I = (\mathfrak{c}_I, 3l_I/2) \quad \text{for the center } \mathfrak{c}_I \text{ of the cube } I.$$

Let $\mathcal{C} = C^\mu_{\mathrm{loc}}(\mathbb{R}^n_+)$, that is, $\mathcal{C}$ consists of all real functions f on $\mathbb{R}^n_+$ such that $|f|_{C^\mu(I^{\boxdot})} < \infty$ for each $I \in \mathcal{D}$. We let

$$|f|_I = l_I^\mu |f|_{C^\mu(I^{\boxdot})}, \quad \|f\|_I = \|f\|_{L^\infty(I^{\boxdot})} + |f|_I.$$

The estimate

$$|f|_I \leqslant n l_I \|Df\|_{L^\infty(I^{\boxdot})}, \quad f \in C^1(I^{\boxdot}) \tag{2}$$

is obvious.

2.2. We introduce main notations related to the Z-parametrix $E(A; x, y)$ of an arbitrary operator $A \in \mathcal{A}^\mu_\lambda$.

Let δ_{ij} be the Kronecker delta, $\Gamma(\cdot)$ be the Euler Gamma function, $\mathcal{A}$ be the set of all differential operators

$$A = \sum_{i,j=1}^n a_{ij} D_{ij} \tag{3}$$

with constant coefficients $a_{ij} = a_{ji} \in \mathbb{R}$. For $\lambda \geqslant 1$ we let

$$\mathcal{A}_\lambda = \left\{ A \in \mathcal{A} \colon (\forall \zeta \in \mathbb{R}^n)\ \lambda^{-1}|\zeta|^2 \leqslant \sum_{i,j=1}^n a_{ij}\zeta_i\zeta_j \leqslant \lambda|\zeta|^2 \right\}.$$

We denote by $\mathfrak{z}_I$ the unique vertex of a cube $I \in \mathcal{D}$ possessing the property $\mathfrak{z}_I/(2l_I) \in \mathbb{Z}^{n-1}$. Let

$$I^\times = \{\xi \in \overline{3I} \colon |\xi - \mathfrak{z}_I|_\infty \leqslant 3l_I/2\} \quad (\Rightarrow I \subset \overline{2I} \subset I^\times \subset \overline{3I}),$$
$$I^{\boxtimes} = I^\times \times [3l_I/4, 3l_I] \qquad (\Rightarrow I^{\boxdot} \subset I^{\boxtimes}).$$

The symbol $\mathcal{A}^\mu$ stands for all operators (3) with real coefficients $a_{ij} = a_{ji} \in \mathcal{C}$. Hereafter a_{ij} always stand for the coefficients of the operator $A \in \mathcal{A}$ or $A \in \mathcal{A}^\mu$. If $A \in \mathcal{A}^\mu$, then

$$|A|_I = l_I^\mu \max_{i,j} |a_{ij}|_{C^\mu(I^{\boxtimes})},$$
$$A[x] = \sum_{i,j=1}^n a_{ij}(x) D_{ij}, \quad x_n > 0,$$
$$Af\big|_x = A[x]f\big|_x, \quad f \in C^2(\mathbb{R}^n_+).$$

We let $\mathcal{A}^\mu_\lambda = \{A \in \mathcal{A}^\mu \colon A[x] \in \mathcal{A}_\lambda \text{ for all } x\}$.

For $I \in \mathcal{D}$ and $k \in \mathbb{N}_0$ by $I^{(k)}$ we denote the unique cube in $\mathcal{D}$ with the properties $I \subset I^{(k)}$ and $l_{I^{(k)}} = 2^k l_I$. It is easy to construct the functions $\varphi_k : \mathbb{R}^n_+ \to [0, 1]$ in the class C^∞ such that $\varphi_0 \equiv 0$ and, as $k \geqslant 1$,

$$\varphi_k \equiv 1 \text{ on the set } \mathfrak{P}_k \equiv \overline{3I^{(k-1)}} \times (0, 3l_{I^{(k-1)}}], \tag{4a}$$
$$\operatorname{supp} \varphi_k \subset \mathfrak{P}^*_k \equiv (5I^{(k-1)})^\circ \times (0, 4l_{I^{(k-1)}}), \tag{4b}$$
$$|D^\alpha \varphi_k| \leqslant c(\alpha) l_{I^{(k)}}^{-|\alpha|}, \quad \alpha \in \mathbb{N}^n_0. \tag{4c}$$

We also let $\mathfrak{Q}_{-1} = \mathfrak{P}_0 = \mathfrak{P}^*_0 = \varnothing$ and

$$\mathfrak{Q}_k = \overline{3I^{(k)}} \times (0, 2l_{I^{(k)}}], \quad k \geqslant 0. \tag{5}$$

It is obvious that

$$\mathfrak{Q}_{k-1} \subset \mathfrak{P}_k \subset \mathfrak{P}_k^* \subset \mathfrak{Q}_k^\circ. \tag{6}$$

It is easy to check the existence of C^∞-functions $\psi_K \colon \mathbb{R}^n_+ \to [0,1]$ with the properties

$$\operatorname{supp} \psi_K \subset \overline{\frac{3}{2}K} \times \left[\frac{3}{4}l_K, \frac{5}{2}l_K\right], \quad K \in \mathcal{D}, \tag{7a}$$

$$\sum_K \psi_K(x) = 1, \quad x_n > 0, \tag{7b}$$

$$|D^\alpha \psi_K| \leqslant c(\alpha) l_K^{-|\alpha|}, \quad \alpha \in \mathbb{N}_0^n. \tag{7c}$$

For $A \in \bigcup_{\lambda \geqslant 1} \mathcal{A}_\lambda$ and $x \neq 0$ we denote

$$\det{}_A = \det(a_{ij}), \quad (b_{ij}) = (a_{ij})^{-1}, \quad Q_A(x) = \sum_{i,j=1}^n b_{ij} x_i x_j,$$

$$E_A(x) = \begin{cases} \dfrac{1}{4\pi\sqrt{\det_A}} \ln Q_A(x), & n = 2, \\ \dfrac{\Gamma(n/2)}{(2-n)2\pi^{n/2}\sqrt{\det_A}} Q_A^{\frac{2-n}{2}}(x), & n \geqslant 3. \end{cases}$$

For $x, y \in \mathbb{R}^n_+$, $x \neq y$, we let

$$\begin{aligned} &e_n^A = a_{nn}^{-1}\{a_{1n}e_1 + a_{2n}e_2 + \dots + a_{nn}e_n\}, \\ &\widetilde{y}^A = y - y_n e_n^A, \quad \widetilde{y}_A = y - 2y_n e_n^A, \\ &G_A(x,y) = E_A(x-y) - E_A(x - \widetilde{y}_A). \end{aligned}$$

For $A \in \mathcal{A}_\lambda^\mu$ and $x, y \in \mathbb{R}^n_+$, $x \neq y$, we let

$$E(A; x, y) = \sum_K G_{A[\mathfrak{c}_K^{\boxdot}]}(x,y)\psi_K\big(Z(x,y)\big),$$

where

$$Z(x,y) = x + \kappa|x - \widetilde{y}|e_n, \quad \kappa = \frac{1}{3\sqrt{4n+9}}, \quad \widetilde{y} = (y', -y_n).$$

In [1] the parametrix $E(A; x, y)$ was introduced with the constant $\kappa_0 = \frac{1}{3\sqrt{n+15}}$ instead of κ.

2.3. Let us write down the potential Φ_f and discrete Hölder estimates for it.

Theorem 1. *Let $\lambda \geqslant 1$, $0 < \mu < 1$ and $A \in \mathcal{A}_\lambda^\mu$. Then for each function $f \in \mathrm{VL}(0)$, where*

$$\begin{aligned} \mathrm{VL}(0) &= \Big\{ f \in \mathcal{C} \colon (\exists I \in \mathcal{D}) \ \sum_J \Gamma_{IJ}^{(0,n)} l_J \|f\|_J < \infty \Big\} \\ &= \Big\{ f \in \mathcal{C} \colon (\forall I \in \mathcal{D}) \ \sum_J \Gamma_{IJ}^{(0,n)} l_J \|f\|_J < \infty \Big\}, \end{aligned}$$

the integral

$$\Phi_f(x) = \int_{y_n > 0} E(A; x, y) f(y)\, dy$$

converges absolutely and is twice continuously differentiable in x. We have $D^\alpha\Phi_f \in \mathcal{C}$ ($|\alpha| \leqslant 2$) and for each $I \in \mathcal{D}$

$$l_I^{-1}\|\Phi_f\|_I + \|D\Phi_f\|_I \leqslant c(n,\lambda,\mu)\sum_J \Gamma_{IJ}^{(0,n)} l_J \|f\|_J, \tag{8}$$

$$l_I^{-1}\|D_n\Phi_f - \mathbf{x}_n^{-1}\Phi_f\|_I + \|D^2\Phi_f\|_I \leqslant c(n,\lambda,\mu)\sum_J \Gamma_{IJ}^{(0,n+1)} \|f\|_J, \tag{9}$$

$$\|f - A\Phi_f\|_I \leqslant c(n,\lambda,\mu)\sum_J \Gamma_{IJ}^{(0,n+1)} \|f\|_J \min\left\{1 + |A|_I, \sum_{H:\, I\subset H\subset I^J} |A|_H\right\}. \tag{10}$$

Remark. *Here $\mathbf{x}_n^{-1}$ is the function $x \mapsto x_n^{-1}$.*

Proof. All statements of the theorem, except the estimate for the norm $\|D_n\Phi_f - \mathbf{x}_n^{-1}\Phi_f\|_I$, were checked in [1, Thm. 5] for the parametrix $E(A;x,y)$ defined by the constant κ_0 instead of κ. Due to the property $\kappa \leqslant \kappa_0$, the arguments can be extended to our parametrix with minor changes. This is why it remains to check the inequality:

$$l_I^{-1}\|D_n\Phi_f - \mathbf{x}_n^{-1}\Phi_f\|_I \leqslant c(n,\lambda,\mu)\sum_J \Gamma_{IJ}^{(0,n+1)}\|f\|_J. \tag{11}$$

By (2), (4b) and (4c), for the function φ_1 in (4) we have

$$\|\varphi_1\|_J + \|1-\varphi_1\|_J \leqslant 2 + 2nl_J\|D\varphi_1\|_{L^\infty(J^\boxdot)} \leqslant c_1(n),$$
$$\|\varphi_1 f\|_J + \big\|(1-\varphi_1)f\big\|_J \leqslant c_1\|f\|_J, \quad J \in \mathcal{D},$$

and hence, $\varphi_1 f \in \mathrm{VL}(0)$ and $(1-\varphi_1)f \in \mathrm{VL}(0)$. In the same way,

$$\|\mathbf{x}_n^{-1}\Phi_{\varphi_1 f}\|_I \leqslant \|\mathbf{x}_n^{-1}\|_I\|\Phi_{\varphi_1 f}\|_I \leqslant c_2(n) l_I^{-1}\|\Phi_{\varphi_1 f}\|_I.$$

If $\|\varphi_1 f\|_J \neq 0$, then $J^\boxdot \cap \mathfrak{Q}_1^\circ \neq \varnothing$ in view of (4b) and (6), which implies $J^\boxdot \subset \mathfrak{Q}_1$ and

$$l_I^{-1}\Gamma_{IJ}^{(0,n)} l_J = l_I^{-1}\Gamma_{IJ}^{(0,n+1)}[I,J] \leqslant l_I^{-1}\Gamma_{IJ}^{(0,n+1)}[I^{(1)},J] \leqslant 4\Gamma_{IJ}^{(0,n+1)}. \tag{12}$$

By (8) we conclude that

$$\begin{aligned} l_I^{-1}\|D_n\Phi_{\varphi_1 f} - \mathbf{x}_n^{-1}\Phi_{\varphi_1 f}\|_I &\leqslant l_I^{-1}\|D_n\Phi_{\varphi_1 f}\|_I + c_2 l_I^{-2}\|\Phi_{\varphi_1 f}\|_I \\ &\leqslant c_3(n,\lambda,\mu) l_I^{-1}\sum_J \Gamma_{IJ}^{(0,n)} l_J\|\varphi_1 f\|_J \leqslant 4c_1c_3\sum_J \Gamma_{IJ}^{(0,n+1)}\|f\|_J. \end{aligned}$$

We assume that for each $x \in I^\boxdot$ and $y \in J^\boxdot \setminus \mathfrak{P}_1$ ($J \in \mathcal{D}$), the functions

$$\zeta_K(x,y) = G_{A[\mathfrak{c}_K^\boxdot]}(x,y)\psi_K\big(Z(x,y)\big), \quad \zeta_K^*(x,y) = D_{x_n}\zeta_K(x,y) - x_n^{-1}\zeta_K(x,y)$$

satisfy the inequalities

$$\big|\zeta_K(x,y)\big| \leqslant c(n,\lambda) l_I \Gamma_{IJ}^{(0,n)} l_J^{1-n}, \tag{13}$$

$$\big|D_x^\alpha\zeta_K^*(x,y)\big| \leqslant c(\alpha,\lambda) l_I^{1-|\alpha|}\Gamma_{IJ}^{(0,n+1)} l_J^{-n}, \quad |\alpha| \leqslant 1. \tag{14}$$

Then by (4a), (7a) and the belonging $(1-\varphi_1)f \in \mathrm{VL}(0)$, the formula

$$\Phi_{(1-\varphi_1)f}(x) = \int_{y_n>0}\left(\sum_K \zeta_K(x,y)\right)\big(1-\varphi_1(y)\big)f(y)\,dy$$

leads us to the formula with an absolutely convergent series

$$D^\alpha(D_n\Phi_{(1-\varphi_1)f} - \mathbf{x}_n^{-1}\Phi_{(1-\varphi_1)f})(x) = \sum_{J,K}\int_{J^\boxdot} D_x^\alpha\zeta_K^*(x,y)\big(1-\varphi_1(y)\big)f(y)\,dy,$$

which together with (2), (7a) and the property $\int_{J^{\boxdot}} |f|\,dy \leqslant l_J^n \|f\|_J$ yield

$$l_I^{-1}\|D_n\Phi_{(1-\varphi_1)f} - \mathbf{x}_n^{-1}\Phi_{(1-\varphi_1)f}\|_I \leqslant c(n,\lambda)\sum_J \Gamma_{IJ}^{(0,n+1)}\|f\|_J.$$

In view of the result in the previous paragraph we obtain (11).

Let us check (13) and (14). For $x, y \in \mathbb{R}_+^n$, $x \neq y$, estimate (23) in [1] is of the form:

$$\bigl|D_x^\alpha G_B(x,y)\bigr| \leqslant c(\alpha,\lambda) y_n |x-y|^{1-n-|\alpha|}, \quad (\alpha, B) \in \mathbb{N}_0^n \times \mathcal{A}_\lambda. \tag{15}$$

Let $x \in I^{\boxdot}$ and $y \in J^{\boxdot} \setminus \mathfrak{P}_1$ ($J \in \mathcal{D}$). Then

$$[I,J] \leqslant 4\bigl|(x', \tau x_n) - y\bigr|_\infty \quad \text{as } 0 < \tau \leqslant 1, \tag{16}$$

which is implied easily by the inequality $\bigl|(x', \tau x_n) - y\bigr|_\infty > l_I$. Hence,

$$\bigl|G_{A[\mathfrak{c}_K^{\boxdot}]}(x,y)\bigr| \leqslant \left|\int_0^1 \frac{\partial}{\partial\tau} G_{A[\mathfrak{c}_K^{\boxdot}]}\bigl((x',\tau x_n), y\bigr)\,d\tau\right| \leqslant c(n,\lambda) l_I l_J [I,J]^{-n},$$

which yields (13). Let $\alpha \in \{0, e_1, \ldots, e_{n-1}\}$. By the Taylor formula,

$$\zeta_K^*(x,y) = x_n \int_0^1 \tau D^{2e_n} \zeta_K\bigl((x', \tau x_n), y\bigr)\,d\tau, \tag{17a}$$

$$D_x^\alpha \zeta_K^*(x,y) = x_n \int_0^1 \tau D^{\alpha+2e_n} \zeta_K\bigl((x', \tau x_n), y\bigr)\,d\tau, \tag{17b}$$

$$D_{x_n} \zeta_K^*(x,y) = \int_0^1 \{\tau D^{2e_n} + x_n \tau^2 D^{3e_n}\} \zeta_K\bigl((x', \tau x_n), y\bigr)\,d\tau, \tag{17c}$$

where the derivatives D^β are taken w.r.t. the first vector independent variable. By (15), (16), (7a), (7c) and the Leibnitz formula for $\bar{x} = (x', \tau x_n)$ we have

$$\bigl|D_{\bar{x}}^\beta G_{A[\mathfrak{c}_K^{\boxdot}]}(\bar{x},y)\bigr| \leqslant c(\beta,\lambda) l_J [I,J]^{1-n-|\beta|}, \quad |\beta| \leqslant 3, \tag{17d}$$

$$\bigl|D_{\bar{x}}^\beta \psi_K(Z(\bar{x},y))\bigr| \leqslant c(\beta) |\bar{x} - \widetilde{y}|^{-|\beta|} \leqslant c(\beta)[I,J]^{-|\beta|}, \tag{17e}$$

$$\bigl|D_{\bar{x}}^\beta \zeta_K(\bar{x},y)\bigr| \leqslant c(\beta,\lambda) l_J [I,J]^{1-n-|\beta|}. \tag{17f}$$

Hence,

$$\bigl|D_x^\alpha \zeta_K^*(x,y)\bigr| \leqslant c_4(\alpha,\lambda) l_I l_J [I,J]^{-n-1-|\alpha|} \leqslant c_4 l_I^{1-|\alpha|} l_J [I,J]^{-n-1}, \tag{17g}$$

$$\bigl|D_{x_n} \zeta_K^*(x,y)\bigr| \leqslant c_5(n,\lambda)\bigl\{l_J[I,J]^{-n-1} + l_I l_J [I,J]^{-n-2}\bigr\} \leqslant 2c_5 l_J [I,J]^{-n-1}, \tag{17h}$$

which coincides with (14). This completes the proof of inequality (11) and Theorem 1. □

2.4. Calculation of Φ_{Af}. The proof of the following lemma is trivial.

Lemma 2. *If $d \in \mathbb{N}$, $f \in C(\mathbb{R}^d)$ and $\sup_{\xi\in\mathbb{R}^d}\bigl|f(\xi)\bigr||\xi|^d < \infty$, then the limit*

$$\lim_{r\to\infty} \int_{|\xi - \mathfrak{x}| < r} f(\xi)\,d\xi$$

either exists for all $\mathfrak{x} \in \mathbb{R}^d$ or does not exist for all $\mathfrak{x} \in \mathbb{R}^d$. In the former case, its value is independent of $\mathfrak{x}$.

We say that $D^2 f \in \mathrm{VL}(0)$ if $f \in C^{2,\mu}_{\mathrm{loc}}(\mathbb{R}_+^n)$ and

$$\sum_J \Gamma_{IJ}^{(0,n)} l_J \|D^2 f\|_J < \infty \text{ for some } I \in \mathcal{D}.$$

Let $\mathbb{P}_1^n$ be the space of all polynomials in $\mathbb{R}^n$ of degree at most one. By $\langle x, y\rangle$ we denote the scalar product $\sum_{i=1}^n x_i y_i$ in $\mathbb{R}^n$.

Lemma 3. *Let $D^2 f \in \mathrm{VL}(0)$. Then*

$$t\nabla f(\cdot,t) \to 0 \text{ in } L^1_{\mathrm{loc}}(\mathbb{R}^{n-1}) \text{ as } t\downarrow 0, \tag{18}$$

$$\text{in } L^1_{\mathrm{loc}}(\mathbb{R}^{n-1}) \text{ there exists the limit } f(\cdot,0+). \tag{19}$$

If $f(x)=\gamma(x)$ for large $|x|$ for some polynomial $\gamma\in\mathbb{P}^n_1$, then for each operator $A\in\bigcup_{\lambda\geqslant 1}\mathcal{A}_\lambda$ and points $x\in\mathbb{R}^n_+$ and $\mathfrak{x}\in\mathbb{R}^{n-1}$ we have

$$\begin{aligned} f(x) &- \int_{y_n>0} G_A(x,y)Af(y)\,dy \\ &= x_n\frac{\Gamma(n/2)}{\pi^{n/2}\sqrt{\det_A}}\lim_{r\to\infty}\int_{|\xi-\mathfrak{x}|<r} Q_A^{-n/2}\big(x-(\xi,0)\big)f(\xi,0+)\,d\xi + \langle\nabla\gamma, e^A_n\rangle x_n. \end{aligned}\tag{20}$$

Remark. *With no pedantry we write $f(\cdot,t)$ instead of $f\big((\cdot,t)\big)$. The first integral in* (20) *exists by Theorem 1 since $G_A(x,y)=E(A;x,y)$.*

Proof. The condition $D^2 f\in\mathrm{VL}(0)$ implies immediately that

$$\int_{\Xi\times(0,1)} x_n\big|D^2 f(x)\big|\,dx<\infty$$

for each compact set $\Xi\subset\mathbb{R}^{n-1}$. For $0<t<1$ we have

$$\begin{aligned} t\big\|Df(\cdot,t)\big\|_{L^1(\Xi)} \leqslant & t\big\|Df(\cdot,1)\big\|_{L^1(\Xi)} + t\int_{\Xi\times(t,1)}\big|D^2 f(x)\big|\,dx \\ \leqslant & t\big\|Df(\cdot,1)\big\|_{L^1(\Xi)} + \int_{\Xi\times(t,\sqrt{t})} x_n\big|D^2 f(x)\big|\,dx + \sqrt{t}\int_{\Xi\times(\sqrt{t},1)} x_n\big|D^2 f(x)\big|\,dx, \end{aligned}$$

and as $t\to 0$, this proves (18). As $0<t_1<t_2<1$, the relations

$$\begin{aligned} \big\|f(\cdot,t_1)-f(\cdot,t_2)\big\|_{L^1(\Xi)} &\leqslant \int_{\Xi\times(t_1,t_2)}\big|D_n f(x)\big|\,dx \\ &\leqslant t_2\big\|D_n f(\cdot,1)\big\|_{L^1(\Xi)} + \int_{\Xi\times(t_1,t_2)} x_n\big|D_{nn}f(x)\big|\,dx + t_2\int_{\Xi\times(t_2,1)}\big|D_{nn}f(x)\big|\,dx \end{aligned}$$

hold. These relations and the Cauchy convergence criterion give (19).

Let us prove that if the support $\operatorname{supp} f$ is bounded under the assumptions of formula (20), then

$$f(x)-\int_{y_n>0} G_A(x,y)Af(y)\,dy = x_n\frac{\Gamma(n/2)}{\pi^{n/2}\sqrt{\det_A}}\int_{\mathbb{R}^{n-1}} Q_A^{-n/2}\big(x-(\xi,0)\big)f(\xi,0+)\,d\xi. \tag{21}$$

In view of [1, (19)] and the formula

$$\int_{\mathbb{R}^n} E_A(y-z)A\varphi(y)\,dy=\varphi(z),\quad \varphi\in C_0^\infty(\mathbb{R}^n),$$

we get

$$Q_A(x-y)-Q_A(x-\widetilde{y}_A) = -\frac{4x_n y_n}{a_{nn}} = Q_A(y-x)-Q_A(y-\widetilde{x}_A), \tag{22}$$

$$Q_A(x-\widetilde{y}_A)=Q_A(y-\widetilde{x}_A), \tag{23}$$

$$G_A(x,y)=G_A(y,x), \tag{24}$$

$$\int_{y_n>0} G_A(x,y)A\varphi(y)\,dy = \int_{y_n>0} G_A(y,x)A\varphi(y)\,dy=\varphi(x),\quad \varphi\in C_0^\infty(\mathbb{R}^n_+).$$

If the function f is concentrated near the point x, by the regularization we have

$$f(x) - \int_{y_n>0} G_A(x,y) A f(y)\, dy = 0.$$

Hence, while checking (21), we can assume that $f \equiv 0$ in the vicinity of the point x. In this case, considering the integrals over the set $\{y\colon y_n > t\}$ and employing relations $(15)\big|_{\alpha=0}$, (18), (19) as well as the boundedness of $\operatorname{supp} f$, we obtain

$$\begin{aligned}
-\int_{y_n>0} G_A(x,y) A f(y)\, dy &= \sum_{i,j=1}^{n} a_{ij} \int_{y_n>0} D_{y_i} G_A(x,y) D_{y_j} f(y)\, dy \\
&= \sum_{i,j=1}^{n} a_{ij} \int_{y_n>0} D_{y_j} \{ D_{y_i} G_A(x,y) f(y) \}\, dy \\
&= -\sum_{i=1}^{n} a_{in} \int_{\mathbb{R}^{n-1}} D_{y_i} G_A\big(x,(\xi,0)\big) f(\xi, 0+)\, d\xi.
\end{aligned}$$

In view of (22) we obtain

$$\begin{aligned}
D_{y_i} G_A\big(x,(\xi,0)\big) &= C_A Q_A^{-n/2}(x-y) D_{y_i} \{ Q_A(x-y) - Q_A(x-\widetilde{y}_A) \}\Big|_{y=(\xi,0)} \\
&= -\frac{4x_n \delta_{in}}{a_{nn}} C_A Q_A^{-n/2}\big(x-(\xi,0)\big), \quad C_A = \frac{\Gamma(n/2)}{4\pi^{n/2}\sqrt{\det_A}}.
\end{aligned}$$

This implies (21), that is, formula (20) with $\gamma = 0$.

As $\gamma \neq 0$, we apply formula (21) to the function $f - \gamma$. Thanks to the identity

$$\gamma(x) - \gamma(\widetilde{x}^A) = \langle \nabla\gamma, x_n e_n^A \rangle$$

formula (20) will be proved if we establish the relation

$$\gamma(\widetilde{x}^A) = x_n \frac{\Gamma(n/2)}{\pi^{n/2}\sqrt{\det_A}} \lim_{r\to\infty} \int_{|\xi-\mathfrak{x}|<r} Q_A^{-n/2}\big(x-(\xi,0)\big)\gamma(\xi,0)\, d\xi. \tag{25}$$

Expanding $\gamma(\xi,0)$ into the powers of the variable $\eta \equiv \xi - (\widetilde{x}^A)'$, we see that we need to check the identities

$$\begin{aligned}
&x_n \frac{\Gamma(n/2)}{\pi^{n/2}\sqrt{\det_A}} \int_{\mathbb{R}^{n-1}} Q_A^{-n/2}\big(x-(\xi,0)\big)\, d\xi = 1, \\
&\lim_{r\to\infty} \int_{|\eta+(\widetilde{x}^A)'-\mathfrak{x}|<r} Q_A^{-n/2}\big(x_n e_n^A - (\eta,0)\big)\eta_i\, d\eta = 0, \quad i = \overline{1, n-1}.
\end{aligned}$$

The former identity is obtained by substituting the function $f = \varphi(\cdot/r)$ into (21), where $\varphi \in C_0^\infty(\mathbb{R}^n)$ and $\varphi \equiv 1$ in the vicinity of the origin and by passing then to limit as $r \to \infty$ taking into consideration the inequality

$$\big|G_A(x,y)\big| \leqslant c(n,\lambda) x_n |x-y|^{1-n}$$

implied by (15) and (24). The second needed identity are yielded by the relations

$$Q_A\big(x_n e_n^A - (\eta,0)\big) \overset{(23)}{=} Q_A\Big((\eta,0) - \widetilde{x_n e_n^A}_A\Big) = Q_A\big(x_n e_n^A + (\eta,0)\big)$$

and Lemma 2. This completes the proof of (25), (20) and the lemma. □

The next result allows us to find approximately the derivatives $D^\alpha \Phi_{Af}$ of the potential Φ_{Af} and the expression $D_n \Phi_{Af} - \mathbf{x}_n^{-1} \Phi_{Af}$.

Theorem 2. *Let* $\lambda \geqslant 1$, $0 < \mu < 1$, $A \in \mathcal{A}^{\mu}_{\lambda}$, $D^2 f \in \mathrm{VL}(0)$, $I \in \mathcal{D}$ *and*

$$\Theta = \sum_J \Gamma^{(0,n)}_{IJ} l_J \left(\sum_{H \in \overleftarrow{IJ} \cup \overrightarrow{IJ}} |A|_H \right) \mathcal{F}_J < \infty,$$

where

$$\begin{aligned}
\overleftarrow{IJ} &= \{H \in \mathcal{D} \colon I \subset H \subset I^J \ \&\ 12 l_H > \kappa l_{I^J}\},\\
\overrightarrow{IJ} &= \{H \in \mathcal{D} \colon J \subset H \subset J^I\},\\
\mathcal{F}_J &= \|D^2 f\|_J + [I,J]^{-1} \sum_{H \in \{I^J\} \cup \overrightarrow{IJ}} l_H \|D^2 f\|_H.
\end{aligned}$$

Then $Af \in \mathrm{VL}(0)$ *and the integral*

$$\Phi_{Af}(x) = \int_{y_n > 0} E(A; x, y) Af(y)\, dy, \quad x \in \mathbb{R}^n_+$$

converges absolutely. For $k \in \mathbb{N}_0$ *and the functions* $\{\varphi_k\}$ *in* (4) *we denote*

$$\begin{aligned}
&\mathfrak{c}_k = \mathfrak{c}^{\boxdot}_{I(k)}, \quad A_k = A[\mathfrak{c}_k], \quad \gamma_k(x) = f(\mathfrak{c}_k) + \langle \nabla f(\mathfrak{c}_k), x - \mathfrak{c}_k \rangle,\\
&f_k = \varphi_k f + (1 - \varphi_k)\gamma_k, \quad f_{(k)} = f_{k+1} - f_k.
\end{aligned}$$

Then $D^2 f_k \in \mathrm{VL}(0)$, *the limits*

$$F_k(x) = \frac{\Gamma(n/2)}{\pi^{n/2}\sqrt{\det_{A_k}}} \lim_{r \to \infty} \int_{|\xi - \mathfrak{c}'_k| < r} Q^{-n/2}_{A_k}\big(x - (\xi, 0)\big) f_{(k)}(\xi, 0+)\, d\xi$$

exist as $x \in I^{\boxdot}$, *the functions* F_k *belong to* $C^{2,\mu}(I^{\boxdot})$, *the series*

$$F = \sum_{k=0}^{\infty} F_k$$

converges absolutely in $C^{2,\mu}(I^{\boxdot})$, *the scalar series*

$$\gamma' = \sum_{k=0}^{\infty} \langle \nabla(\gamma_{k+1} - \gamma_k), e^{A_k}_n \rangle$$

converges absolutely and the inequalities

$$l_I^{-1} \|\mathcal{R}_f\|_I + \|D\mathcal{R}_f\|_I \leqslant c(n, \lambda, \mu)\Theta, \tag{26a}$$

$$l_I^{-1} \|D_n \mathcal{R}_f - \mathbf{x}_n^{-1} \mathcal{R}_f\|_I + \|D^2 \mathcal{R}_f\|_I \leqslant c(n, \lambda, \mu)\Theta^* \tag{26b}$$

hold true, where

$$\begin{aligned}
&\mathcal{R}_f = \Phi_{Af} - \Psi, \quad \Psi = f - \gamma_0 - \mathbf{x}_n F - \gamma' \mathbf{x}_n,\\
&\Theta^* = \sum_J \Gamma^{(0,n+1)}_{IJ} \left(\sum_{H \in \overleftarrow{IJ} \cup \overrightarrow{IJ}} |A|_H \right) \mathcal{F}_J.
\end{aligned}$$

Remark. *The limits* $f_{(k)}(\cdot, 0+)$ *are treated in the sense of* (19). *The inequality*

$$\Theta^* \leqslant \Theta / l_I$$

implies the finiteness of Θ^*.

Proof. It is obvious that $Af \in \mathcal{C}$. For each $J \in \mathcal{D}$ we have

$$\big\|a_{ij} - a_{ij}(\mathfrak{c}_J^{\boxdot})\big\|_J = \big\|a_{ij} - a_{ij}(\mathfrak{c}_J^{\boxdot})\big\|_{L^\infty(J^{\boxdot})} + l_J^\mu |a_{ij}|_{C^\mu(J^{\boxdot})} \leqslant c(n)|A|_J, \tag{27}$$
$$\|Af\|_J \leqslant \big\|Af - A[\mathfrak{c}_J^{\boxdot}]f\big\|_J + \big\|A[\mathfrak{c}_J^{\boxdot}]f\big\|_J \leqslant c(n)|A|_J\|D^2f\|_J + c(n,\lambda)\|D^2f\|_J.$$

In view of the conditions $D^2f \in \mathrm{VL}(0)$ and $\Theta < \infty$ we obtain the belonging $Af \in \mathrm{VL}(0)$ and hence, the absolute convergence of the integral $\Phi_{Af}(x)$ by Theorem 1.

Let $k \geqslant 0$ and $J^{\boxdot} \subset \mathfrak{Q}_k$ (see (5)). Considering Taylor polynomials for the functions f at the touching points for the cubes in the set $\big\{H^{\boxdot} \colon H \in \widehat{I^{(k)}J}\big\}$, by the inequalities

$$\|\gamma\|_{L^\infty(J^{\boxdot})} \leqslant \|\gamma\|_{L^\infty(3(H^{\boxdot}))} \leqslant c(n)\|\gamma\|_{L^\infty(H^{\boxdot})}, \quad \gamma \in \mathbb{P}_1^n,$$

and the Taylor formula we obtain the estimates

$$\|f - \gamma_k\|_{L^\infty(J^{\boxdot})} \leqslant c(n) \sum_{H \in \widehat{I^{(k)}J}} l_H^2 \|D^2f\|_{L^\infty(H^{\boxdot})},$$
$$\big\|D(f - \gamma_k)\big\|_{L^\infty(J^{\boxdot})} \leqslant c(n) \sum_{H \in \widehat{I^{(k)}J}} l_H \|D^2f\|_{L^\infty(H^{\boxdot})}.$$

In view of relations (2), $|D^2\gamma_k| \equiv 0$ and $(I^{(k)})^J = I^{(k)}$ we conclude that

$$\|f - \gamma_k\|_J \leqslant c(n) \sum_{H \in \widehat{I^{(k)}J}} l_H^2 \|D^2f\|_H,$$
$$\big\|D(f - \gamma_k)\big\|_J \leqslant c(n) \sum_{H \in \widehat{I^{(k)}J}} l_H \|D^2f\|_H,$$
$$l_H \leqslant l_{I^{(k)}} \quad \& \quad [I,J] \leqslant [I^{(k)}, J] \leqslant 2l_{I^{(k)}},$$
$$l_{I^{(k)}}^{-2}\|f - \gamma_k\|_J + l_{I^{(k)}}^{-1}\big\|D(f - \gamma_k)\big\|_J \leqslant c(n)[I,J]^{-1} \sum_{H \in \widehat{I^{(k)}J}} l_H \|D^2f\|_H. \tag{28}$$

Let us estimate $\big\|D^2(f - f_k)\big\|_J$ and $\|D^2f_{(k)}\|_J$. We write

$$f - f_k = (1 - \varphi_k)(f - \gamma_k), \quad f_{(k)} = f_{k+1} - f_k = \{f - f_k\} - \{f - f_{k+1}\}.$$

If $J^{\boxdot} \subset \mathfrak{Q}_k \setminus \mathfrak{Q}_{k-1}^\circ$, then $l_J \leqslant l_{I^{(k)}}$ and $\widehat{I^{(k)}J} = \{I^J\} \cup \overrightarrow{IJ}$. By (2), (4c), (28) and the Leibnitz formula, this implies

$$l_{I^{(k)}}^2\big\|D^2(1 - \varphi_k)\big\|_J + l_{I^{(k)}}\big\|D(1 - \varphi_k)\big\|_J + \|1 - \varphi_k\|_J \leqslant c(n), \quad \big\|D^2(f - f_k)\big\|_J \leqslant c_1(n)\mathcal{F}_J.$$

By (4a), (4b) and (6),

$$f - f_k \equiv 0 \quad \text{on the set } \mathfrak{Q}_{k-1},$$
$$|D^2 f_k| \equiv 0 \quad \text{on the set } \mathbb{R}_+^n \setminus \mathfrak{Q}_k^\circ,$$
$$|D^2 f_{(k)}| \equiv 0 \quad \text{on the set } \mathbb{R}_+^n \setminus \mathfrak{Q}_{k+1}^\circ.$$

Therefore,

$$\big\|D^2(f - f_k)\big\|_J \leqslant \begin{cases} 0, & J^{\boxdot} \subset \mathfrak{Q}_{k-1}, \\ c_1\mathcal{F}_J, & J^{\boxdot} \subset \mathfrak{Q}_k \setminus \mathfrak{Q}_{k-1}^\circ, \\ \|D^2f\|_J \leqslant \mathcal{F}_J, & J^{\boxdot} \subset \mathbb{R}_+^n \setminus \mathfrak{Q}_k^\circ, \end{cases} \tag{29}$$

$$\|D^2 f_{(k)}\|_J \leqslant \begin{cases} (c_1 + 1)\mathcal{F}_J, & J^{\boxdot} \subset \mathfrak{Q}_{k+1} \setminus \mathfrak{Q}_{k-1}^\circ, \\ 0 & \text{otherwise.} \end{cases} \tag{30}$$

In view of [9, Eq. (25e)], we have $[I^J, J] \leqslant 3[I, J]$ and hence, for each $\alpha, \beta \in \mathbb{R}$

$$\Gamma_{IJ}^{(\alpha,\beta)} \leqslant \max\{1, 3^{\alpha+\beta}\}\Gamma_{IH}^{(\alpha,\beta)}\Gamma_{HJ}^{(\alpha,\beta)}, \quad H \in \widehat{IJ}. \tag{31}$$

Let $\alpha \in \mathbb{R}$, $\beta > n-1$ and $H \in \overleftarrow{IJ} \cup \overrightarrow{IJ}$. Then

$$[H, J] \leqslant [I^J, J^I] \leqslant 2l_{I^J} < 24\kappa^{-1}l_H \quad \text{as } H \in \overleftarrow{IJ}, \quad [H, J] = l_H \quad \text{as } H \in \overrightarrow{IJ},$$
$$\Gamma_{HJ}^{(\alpha,\beta)} \leqslant c(n,\alpha)\Gamma_{HJ}^{(\alpha_1,\beta)}, \quad \alpha_1 = \max\{\alpha, 1\}.$$

By (31) and Lemma 1, for each $H \in \mathcal{D}$ we obtain

$$\sum_{J\colon H \in \overleftarrow{IJ} \cup \overrightarrow{IJ}} \Gamma_{IJ}^{(\alpha,\beta)} \leqslant c(n,\alpha,\beta)\Gamma_{IH}^{(\alpha,\beta)} \sum_J \Gamma_{HJ}^{(\alpha_1,\beta)} \leqslant c(n,\alpha,\beta)\Gamma_{IH}^{(\alpha,\beta)}. \tag{32}$$

Hence,

$$\sum_J \Gamma_{IJ}^{(0,n)} l_J [I,J]^{-1} \sum_{H \in \{I^J\} \cup \overrightarrow{IJ}} l_H \|D^2 f\|_H \leqslant \sum_H l_H \|D^2 f\|_H \sum_{J\colon H \in \overleftarrow{IJ} \cup \overrightarrow{IJ}} \Gamma_{IJ}^{(0,n+1)}$$
$$\leqslant c(n) \sum_H \Gamma_{IH}^{(0,n+1)} l_H \|D^2 f\|_H,$$
$$\theta := \sum_J \Gamma_{IJ}^{(0,n)} l_J \mathcal{F}_J \leqslant c(n) \sum_J \Gamma_{IJ}^{(0,n)} l_J \|D^2 f\|_J < \infty.$$

In view of (29) we conclude that $D^2(f - f_k), D^2 f_k, D^2 f_{(k)} \in \mathrm{VL}(0)$.

In what follows we suppose that $x \in I^{\boxdot}$. Due to (30) we have

$$\|A f_{(k)}\|_J \leqslant c(n,\lambda)(|A|_J + 1)\|D^2 f_{(k)}\|_J, \quad \|A_k f_{(k)}\|_J \leqslant c(n,\lambda)\|D^2 f_{(k)}\|_J,$$
$$\sum_J \Gamma_{IJ}^{(0,n)} l_J \big\{\|A f_{(k)}\|_J + \|A_k f_{(k)}\|_J\big\} \leqslant c_2(n,\lambda) \sum_J \Gamma_{IJ}^{(0,n)} l_J (|A|_J + 1)\mathcal{F}_J \leqslant c_2\{\Theta + \theta\} < \infty.$$

Hence, $A f_{(k)} \in \mathrm{VL}(0)$ and $A_k f_{(k)} \in \mathrm{VL}(0)$, so, the potentials

$$\Phi_k(x) = \int_{y_n > 0} E(A; x, y) A f_{(k)}(y)\, dy,$$
$$\Phi_k'(x) = \int_{y_n > 0} E(A; x, y) A_k f_{(k)}(y)\, dy,$$
$$\Phi_k''(x) = \int_{y_n > 0} G_{A_k}(x, y) A_k f_{(k)}(y)\, dy$$

are well-defined. In view of (8), (9), (29) and (30), the series

$$\Phi = \sum_{k=0}^{\infty} \Phi_k, \qquad \Phi' = \sum_{k=0}^{\infty} \Phi_k', \qquad \Phi'' = \sum_{k=0}^{\infty} \Phi_k''$$

converge absolutely in $C^{2,\mu}(I^{\boxdot})$, and the potential $\Phi_{A(f-f_k)}$ tends to zero as $k \to \infty$ in $C^{2,\mu}(I^{\boxdot})$. Bearing in mind the relation $f_0 = \gamma_0 \in \mathbb{P}_1^n$, on $I^{\boxdot}$ we get

$$\Phi_{Af} = \Phi_{A(f-f_q)} + \sum_{k=0}^{q-1} \Phi_k = \lim_{q\to\infty} \left(\Phi_{A(f-f_q)} + \sum_{k=0}^{q-1} \Phi_k\right) = \Phi.$$

By Lemma 3, the limits $F_k(x)$ exist and

$$f_{(k)} - \Phi_k'' = \mathbf{x}_n F_k + \langle \nabla(\gamma_{k+1} - \gamma_k), e_n^{A_k} \rangle \mathbf{x}_n.$$

In particular, $F_k \in C^{2,\mu}(I^{\boxdot})$. By the Taylor formula

$$\big|\nabla(\gamma_{k+1}-\gamma_k)\big| \leqslant c(n) l_{I^{(k)}} \big\{\|D^2 f\|_{I^{(k)}} + \|D^2 f\|_{I^{(k+1)}}\big\},$$
$$\sum_{k=0}^{\infty}\big|\langle\nabla(\gamma_{k+1}-\gamma_k), e_n^{A_k}\rangle\big| \leqslant c(n,\lambda)\sum_J \Gamma_{IJ}^{(0,n)} l_J \|D^2 f\|_J < \infty,$$

that is, the series γ' converges absolutely. The absolute convergence of the series F follows the relation $f_{(k)}\big|_{I^{\boxdot}} \equiv 0$ $(k \geqslant 1)$ and the absolute convergence of the series Φ'' and γ'. On the cube $I^{\boxdot}$, the identities

$$\Psi = f - \gamma_0 - \sum_{k=0}^{\infty}\big\{f_{(k)} - \Phi_k''\big\} = f - \gamma_0 - f_{(0)} + \Phi'' = \Phi'', \quad \mathcal{R}_f = \Phi - \Phi''$$

hold true.

It remains to check inequalities (26). If $J^{\boxdot} \subset \mathfrak{Q}_{k+1} \setminus \mathfrak{Q}_{k-1}^{\circ}$, where $k \geqslant 0$, then $I^J = I^{(k)}$ or $I^J = I^{(k+1)}$, so that $\mathfrak{c}_k \in (I^J)^{\boxtimes}$. By analogy with (27) we have

$$\big\|a_{ij} - a_{ij}(\mathfrak{c}_k)\big\|_J \leqslant c(n) \sum_{H\in\{I^J\}\cup\overrightarrow{IJ}} |A|_H,$$
$$\big\|(A - A_k) f_{(k)}\big\|_J \leqslant c(n)\left(\sum_{H\in\{I^J\}\cup\overrightarrow{IJ}} |A|_H\right)\|D^2 f_{(k)}\|_J.$$

In view of (30) and by Theorem 1 we obtain

$$l_I^{-1}\|\Phi_k - \Phi_k'\|_I + \big\|D(\Phi_k - \Phi_k')\big\|_I \leqslant c(n,\lambda,\mu)\Theta_k,$$
$$l_I^{-1}\big\|D_n(\Phi_k - \Phi_k') - \mathbf{x}_n^{-1}(\Phi_k - \Phi_k')\big\|_I + \big\|D^2(\Phi_k - \Phi_k')\big\|_I \leqslant c(n,\lambda,\mu)\Theta_k^*,$$

where

$$\Theta_k = \sum_{J\colon J^{\boxdot}\subset\mathfrak{Q}_{k+1}\setminus\mathfrak{Q}_{k-1}^{\circ}} \Gamma_{IJ}^{(0,n)} l_J \left(\sum_{H\in\overleftarrow{IJ}\cup\overrightarrow{IJ}} |A|_H\right)\mathcal{F}_J,$$
$$\Theta_k^* = \sum_{J\colon J^{\boxdot}\subset\mathfrak{Q}_{k+1}\setminus\mathfrak{Q}_{k-1}^{\circ}} \Gamma_{IJ}^{(0,n+1)} \left(\sum_{H\in\overleftarrow{IJ}\cup\overrightarrow{IJ}} |A|_H\right)\mathcal{F}_J.$$

By this and the convergence of the series Φ and Φ' we conclude that

$$l_I^{-1}\|\Phi - \Phi'\|_I + \big\|D(\Phi - \Phi')\big\|_I \leqslant c(n,\lambda,\mu)\Theta, \tag{33a}$$
$$l_I^{-1}\big\|D_n(\Phi - \Phi') - \mathbf{x}_n^{-1}(\Phi - \Phi')\big\|_I + \big\|D^2(\Phi - \Phi')\big\|_I \leqslant c(n,\lambda,\mu)\Theta^*. \tag{33b}$$

We let

$$\mathcal{H}_k = \big\{K \in \mathcal{D}\colon \mathfrak{c}_K^{\boxdot} \in (I^{(j)})^{\boxtimes} \text{ for some } j \in \mathbb{N}_0,\ k - \log_2(6/\kappa) < j \leqslant k\big\}.$$

For each $(k, K) \in \mathbb{N}_0 \times \mathcal{D}$ we shall show that

$$\text{if } \psi_K\big(Z(x,y)\big) A_k f_{(k)}(y) \neq 0 \text{ for some } y \in \mathbb{R}_+^n, \text{ then } K \in \mathcal{H}_k. \tag{34}$$

Suppose the assumption in (34). Then $y \in \mathfrak{Q}_{k+1} \setminus \mathfrak{Q}_{k-1}$ by (30) and hence,

$$\begin{aligned} &|x - \widetilde{y}| > x_n \geqslant l_I, && k = 0, \\ &|x - \widetilde{y}| \geqslant \max\{|x' - y'|_\infty, y_n\} > l_{I^{(k-1)}}, && k \neq 0, \\ &|x - \widetilde{y}|^2 \leqslant (n-1)|x' - y'|^2_\infty + (x_n + y_n)^2 \\ &\qquad \leqslant 16(n-1)l^2_{I^{(k)}} + (2l_I + 4l_{I^{(k)}})^2 < \kappa^{-2} l^2_{I^{(k)}}, \\ &l_I < l_I + \frac{\kappa}{2} l_{I^{(k)}} < Z(x,y)_n = x_n + \kappa|x - \widetilde{y}| < 2l_I + l_{I^{(k)}} \leqslant 3l_{I^{(k)}}. \end{aligned}$$

This is why there exists j such that $0 \leqslant j \leqslant k$ and

$$Z(x,y) \in \Big[\overline{I^{(j)}} \times (l_{I^{(j)}}, 3l_{I^{(j)}})\Big] \cap \operatorname{supp} \psi_K.$$

We have

$$\frac{\kappa}{2} l_{I^{(k)}} < Z(x,y)_n < 3l_{I^{(j)}},$$

and hence, $k - \log_2(6/\kappa) < j$. By (7a)

$$\overline{I^{(j)}} \cap \overline{\frac{3}{2}K} \neq \varnothing \quad \& \quad \frac{1}{2} \leqslant \frac{l_{I^{(j)}}}{l_K} \leqslant 2.$$

It is easy to confirm that this implies the belonging $\mathfrak{c}^{\boxdot}_K \in (I^{(j)})^{\boxtimes}$. The proof of (34) is complete.

It follows from relations (7b), (30) and (34) that

$$\Phi'_k(x) - \Phi''_k(x) = \sum_{K \in \mathcal{H}_k} \int_{\mathfrak{Q}_{k+1} \setminus \mathfrak{Q}^\circ_{k-1}} \big(G_{A[\mathfrak{c}^{\boxdot}_K]}(x,y) - G_{A_k}(x,y)\big) \psi_K\big(Z(x,y)\big) A_k f_{(k)}(y)\, dy. \tag{35}$$

If $J^{\boxdot} \subset \mathfrak{Q}_{k+1} \setminus \mathfrak{Q}^\circ_{k-1}$, then $I^J = I^{(k)}$ or $I^J = I^{(k+1)}$ and therefore,

$$I \subset I^{(j)} \subset I^{(k)} \subset I^J \quad \& \quad l_{I^{(j)}} > \frac{\kappa}{6} l_{I^{(k)}} \geqslant \frac{\kappa}{12} l_{I^J} \quad \& \quad I^{(j)} \in \overleftarrow{IJ}$$

for each index j in the definition of the set $\mathcal{H}_k$. This is why

$$\max_{i,j=\overline{1,n}} \max_{K \in \mathcal{H}_k} \big|a_{ij}(\mathfrak{c}^{\boxdot}_K) - a_{ij}(\mathfrak{c}_k)\big| \leqslant c(n) \sum_{H \in \overleftarrow{IJ}} |A|_H, \quad J^{\boxdot} \subset \mathfrak{Q}_{k+1} \setminus \mathfrak{Q}^\circ_{k-1}, \tag{36a}$$

$$\|A_k f_{(k)}\|_J \leqslant c(n,\lambda) \mathcal{F}_J, \tag{36b}$$

where the second inequality is implied trivially by (30). By a simple modification of the constructions in work [1], from (35) and (36) we obtain the estimates

$$l_I^{-1} \|\Phi'_k - \Phi''_k\|_I + \big\|D(\Phi'_k - \Phi''_k)\big\|_I \leqslant c(n,\lambda,\mu)\Theta_k, \tag{37a}$$

$$\big\|D^2(\Phi'_k - \Phi''_k)\big\|_I \leqslant c(n,\lambda,\mu)\Theta^*_k. \tag{37b}$$

In [1, Subsect. 2.1], an estimate for the derivatives of the Green functions was proved (see (15)), which was applied for estimating the norms of $\|D^\alpha \Phi\|_I$ in [1, Subsect. 2.2], where the function Φ is similar to the potential Φ_f. At the same time, in [1, Subsect. 2.1], there was obtained an estimate for the derivatives of the difference $G_{B_1} - G_{B_2}$ applied then in [1, Subsect. 2.2] for estimating the norm $\|f - A\Phi\|_I$ of the error $f - A\Phi$. These two lines can be easily combined to obtain inequalities (37). By (12) and (37a)

$$l_I^{-1} \big\|D_n(\Phi'_0 - \Phi''_0) - \mathbf{x}_n^{-1}(\Phi'_0 - \Phi''_0)\big\|_I \leqslant c(n,\lambda,\mu)\Theta^*_0.$$

This is why, if we establish the inequality

$$l_I^{-1} \big\|D_n(\Phi'_k - \Phi''_k) - \mathbf{x}_n^{-1}(\Phi'_k - \Phi''_k)\big\|_I \leqslant c(n,\lambda)\Theta^*_k \quad (k \geqslant 1), \tag{38}$$

then the convergence of the series Φ' and Φ'' in $C^{2,\mu}(I^{\boxdot})$ and the relation $\mathcal{R}_f = \Phi - \Phi''$ (on $I^{\boxdot}$) and (33) will imply required estimates (26).

Let $k \geqslant 1$, $J^{\boxdot} \subset \mathfrak{Q}_{k+1} \setminus \mathfrak{Q}_{k-1}^{\circ}$, $\bar{x} = (x', \tau x_n)$ for $0 < \tau \leqslant 1$, $y \in J^{\boxdot} \setminus \mathfrak{P}_1$ and $K \in \mathcal{H}_k$. By equation (23') in [1], (16) and (36a) we get

$$\Big| D_{\bar{x}}^{\beta}\big(G_{A[\mathfrak{c}_K^{\boxdot}]}(\bar{x}, y) - G_{A_k}(\bar{x}, y)\big)\Big| \leqslant c(\beta, \lambda) y_n |\bar{x} - y|^{1-n-|\beta|} \sum_{H \in \overleftarrow{IJ}} |A|_H$$
$$\leqslant c(\beta, \lambda) l_J [I, J]^{1-n-|\beta|} \sum_{H \in \overleftarrow{IJ}} |A|_H, \quad |\beta| \leqslant 3,$$

which can be considered as an analogue of inequality (17d). Reproducing (17) for the functions

$$\delta_K(\bar{x}, y) = \big(G_{A[\mathfrak{c}_K^{\boxdot}]}(\bar{x}, y) - G_{A_k}(\bar{x}, y)\big)\psi_K\big(Z(\bar{x}, y)\big),$$
$$\delta_K^*(x, y) = D_{x_n}\delta_K(x, y) - x_n^{-1}\delta_K(x, y),$$

we arrive at the estimates

$$\big|D_{\bar{x}}^{\beta}\delta_K(\bar{x}, y)\big| \leqslant c(\beta, \lambda) l_J [I, J]^{1-n-|\beta|} \sum_{H \in \overleftarrow{IJ}} |A|_H, \tag{39}$$

$$\big|D_x^{\alpha}\delta_K^*(x, y)\big| \leqslant c(\alpha, \lambda) l_I^{1-|\alpha|} l_J [I, J]^{-n-1} \sum_{H \in \overleftarrow{IJ}} |A|_H, \quad \alpha \in \{0, e_1, \dots, e_{n-1}\}, \tag{40}$$

$$\big|D_{x_n}\delta_K^*(x, y)\big| \leqslant c(n, \lambda) l_J [I, J]^{-n-1} \sum_{H \in \overleftarrow{IJ}} |A|_H. \tag{41}$$

In view of (4a), (35), (39) and the belonging $A_k f_{(k)} \in \mathrm{VL}(0)$, the formula

$$D^{\alpha}\big(D_n(\Phi_k' - \Phi_k'') - \mathbf{x}_n^{-1}(\Phi_k' - \Phi_k'')\big)(x)$$
$$= \sum_{J,K\colon J^{\boxdot} \subset \mathfrak{Q}_{k+1} \setminus \mathfrak{Q}_{k-1}^{\circ} \text{ and } K \in \mathcal{H}_k} \int_{J^{\boxdot} \setminus \mathfrak{P}_1} D_x^{\alpha}\delta_K^*(x, y) A_k f_{(k)}(y)\, dy, \quad |\alpha| \leqslant 1$$

holds true, where the series converges absolutely. Hence, in view of (2), (36b), (40) and (41) we obtain estimate (38). This completes the proof of (26) and of the theorem. □

3. Standard set and calculations with the potential Φ_L

3.1. Standard set and potential Φ_{Aw}. To a Lipschitz function $\omega : \mathbb{R}^{n-1} \to \mathbb{R}$ we associate its overgraph Ω and the approximation numbers b_I:

$$\Omega = \big\{x = (x', x_n) \in \mathbb{R}^n \colon x_n > \omega(x')\big\},$$
$$b_I = l_I^{-\frac{n+1}{2}} \left(\min_{\gamma \in \mathbb{P}_1^{n-1}} \int_{5I} |\omega - \gamma|^2 \, d\xi\right)^{1/2}, \quad I \in \mathcal{D}.$$

We introduce a series of auxiliary notions needed for studying harmonic functions in the domain Ω by straightening this domain.

Theorem 3. *Given $K \in \mathcal{D}$, let $\gamma_K \in \mathbb{P}_1^{n-1}$ be a polynomial with the property*

$$\int_K |\omega - \gamma_K|^2 \, d\xi = \min_{\gamma \in \mathbb{P}_1^{n-1}} \int_K |\omega - \gamma|^2 \, d\xi.$$

For the partition of unity $\{\psi_K\}$ in (7) we let

$$w(x) = \sum_K \psi_K(x)\gamma_K(x'), \quad x \in \mathbb{R}_+^n.$$

Then the function w belongs to $C^{\infty}(\mathbb{R}_+^n)$, is Lipschitz and

$$w(\xi, 0+) = \omega(\xi), \quad \xi \in \mathbb{R}^{n-1}. \tag{42}$$

We choose a constant $\theta \geqslant \|\omega\|_{\mathrm{Lip}}$. Then for each $I \in \mathcal{D}$

$$|\nabla\gamma_I| \leqslant c(n)\theta, \tag{43}$$

$$b_I \leqslant c(n)\theta, \tag{44}$$

$$\|D^\alpha w\|_{L^\infty(I^\boxtimes)} \leqslant c(\alpha) l_I^{1-|\alpha|} b_I, \quad \alpha \notin \{0, e_1, \ldots, e_{n-1}\}, \tag{45}$$

$$\|D^\alpha w\|_{L^\infty(I^\boxtimes)} \leqslant c(\alpha) l_I^{1-|\alpha|} \theta, \quad \alpha \neq 0. \tag{46}$$

There exists $W = c(n,\theta)$ such that for the mapping $\mathbf{x}' : x \mapsto x'$

$$\|\omega - \gamma_I\|_{L^\infty(5I)} \leqslant W l_I/3, \qquad I \in \mathcal{D}, \tag{47}$$

$$\|w - \gamma_I \circ \mathbf{x}'\|_{L^\infty(I^\boxtimes)} \leqslant W l_I/3, \quad I \in \mathcal{D}, \tag{48}$$

the mapping $g : \mathbb{R}^n_+ \to \mathbb{R}^n$ of the form $g(x) = \big(x', g_n(x)\big)$ with the function

$$g_n = w + W\mathbf{x}_n$$

is a diffeomorphism of $\mathbb{R}^n_+$ onto Ω, while the inverse diffeomorphism $\mathfrak{g} = g^{-1}$ is represented by the formula

$$\mathfrak{g}(y) = \big(y', \mathfrak{G}(y)\big)$$

with a Lipschitz function $\mathfrak{G} \in C^\infty(\Omega)$ satisfying the inequalities

$$\big\|(D^\alpha \mathfrak{G}) \circ g\big\|_{L^\infty(I^\boxtimes)} \leqslant c(\alpha,\theta) l_I^{1-|\alpha|} b_I, \quad |\alpha| > 1, \tag{49}$$

$$\big\|(D^\alpha \mathfrak{G}) \circ g\big\|_{L^\infty(I^\boxtimes)} \leqslant c(\alpha,\theta) l_I^{1-|\alpha|}, \quad \alpha \neq 0. \tag{50}$$

The operator

$$A = \sum_{i=1}^{n-1} \left\{ D_{ii} + \frac{\partial \mathfrak{G}}{\partial y_i}(g) D_{in} + \frac{\partial \mathfrak{G}}{\partial y_i}(g) D_{ni} \right\} + \big|\nabla \mathfrak{G}(g)\big|^2 D_{nn}$$

belongs to $\mathcal{A}^\mu_\lambda$ for some $\lambda(n,\theta) \geqslant 1$ and each $0 < \mu < 1$. The inequalities

$$|A|_I \leqslant c(n,\theta) b_I, \tag{51}$$

$$\|L\|_I \leqslant c(n,\theta) l_I^{-1} b_I \quad \text{for } L = -(\Delta \mathfrak{G}) \circ g = -\sum_{i=1}^{n} (D_{ii}\mathfrak{G}) \circ g \tag{52}$$

hold true.

We call $\big(\{\gamma_K\}, w, W, g, \mathfrak{g}, \mathfrak{G}, A, \lambda, L\big)$ the standard set of the pair (ω,θ).

Proof. It is obvious that $w \in C^\infty(\mathbb{R}^n_+)$. It is elementary to check (it is sufficient to consider one-dimensional dyadic intervals) that

$$\text{if } (I^\times)^\circ \cap \overline{\tfrac{3}{2}K} \neq \varnothing \text{ and } l_K \in \{l_I/2, l_I, 2l_I\}, \text{ then } K \subset 5I.$$

By analogy with [10, Subsect. 2.7], now one can obtain properties (42)–(46) and the estimate

$$\|\omega - \gamma_I\|_{L^\infty(5I)} \leqslant c_1(n,\theta) l_I, \quad I \in \mathcal{D}.$$

The Lipschitz property for w is implied by inequalities (46) with $|\alpha| = 1$.

By (7a) and (7b), the function w coincides with the polynomial $\gamma_I \circ \mathbf{x}'$ in some neighbourhood of the point $(\mathfrak{c}_I, 11l_I/8) \in I^\boxtimes$. By (45), the Taylor formula, the convexity of the parallelepiped $I^\boxtimes$ and (44) we obtain

$$\|w - \gamma_I \circ \mathbf{x}'\|_{L^\infty(I^\boxtimes)} \leqslant c(n) l_I b_I, \tag{53}$$

$$\|w - \gamma_I \circ \mathbf{x}'\|_{L^\infty(I^\boxtimes)} \leqslant c_2(n,\theta) l_I.$$

It follows immediately from (46) that $\|D_n w\|_{L^\infty(\mathbb{R}^n_+)} \leqslant c_3(n,\theta)$. We let

$$W(n,\theta) = 3\max\{c_1, c_2, c_3\}.$$

Inequalities (47) and (48) are trivial. The required properties of the mappings g and $\mathfrak{g}$ including estimates (49) and (50) on $\mathfrak{G}$ are obtained due to Theorem 2.5 in [10].

The bi-Lipschitz constant of the mapping g is less than some number $c(n,\theta)$ due to (46) and (50). Hence, for some $\lambda(n,\theta) \geqslant 1$, one can easily obtain the uniform ellipticity condition $A[x] \in \mathcal{A}_\lambda$, see [9]. Hence, $A \in \mathcal{A}^\mu_\lambda$.

By inequalities (46), (49) and (50) we have

$$\|Da_{ij}\|_{L^\infty(I^\boxtimes)} + \|L\|_{L^\infty(I^\boxtimes)} + l_I\|DL\|_{L^\infty(I^\boxtimes)} \leqslant c(n,\theta)l_I^{-1}b_I.$$

Now (51) and (52) are obtained by the analogue of estimate (2) for the set $I^\boxtimes$. □

Remark. *The operator A and function L are such that each function U harmonic in the domain Ω solves the equation $A(U\circ g) = LD_n(U\circ g)$.*

In the rest of the section we restrict ourselves by the functions $\omega \in \mathrm{LI}\mathbb{P}$.

Definition 1. *The set* LIℙ *consists of Lipschitz functions*

$$\mathbb{R}^{n-1} \to \mathbb{R},$$

each of which coincides with some polynomial in $\mathbb{P}^{n-1}_1$ on the complement of some compact set.

Let us find out, what Theorem 2 gives once it is applied to the potential Φ_{Aw}.

Lemma 4. *Let $\omega \in \mathrm{LI}\mathbb{P}$ and $I \in \mathcal{D}$. Then*

$$\Theta_1 := \sum_J \Gamma^{(0,n)}_{IJ} b_J < \infty, \quad \Theta_2 := \sum_J \Gamma^{(0,n)}_{IJ} b_J^2 < \infty, \quad \Theta_2^* := \sum_J \Gamma^{(1,n)}_{IJ} b_J^2 < \infty.$$

Given a constant $\theta \geqslant \|\omega\|_{\mathrm{Lip}}$, let $\big(\{\gamma_K\}, w, W, g, \mathfrak{g}, \mathfrak{G}, A, \lambda, L\big)$ be a standard set of the pair (ω,θ). As $k \geqslant 0$, we denote

$$\gamma'_k = \gamma_{I^{(k)}}, \quad \tau_{1,k} = D_1\gamma'_k, \quad \ldots, \quad \tau_{n-1,k} = D_{n-1}\gamma'_k.$$

Then the inequalities

$$\|\omega - \gamma'_{k+1}\|_{L^2(5I^{(k)})} + \|\omega - \gamma'_k\|_{L^2(5I^{(k)})} \leqslant c(n) l_{I^{(k)}}^{\frac{n+1}{2}} b_{I^{(k)}}, \tag{54}$$

$$l^{-1}_{I^{(k)}} \big\|\gamma'_{k+1} - \gamma'_k\big\|_{L^\infty(5I^{(k)})} + \big|\nabla(\gamma'_{k+1} - \gamma'_k)\big| \leqslant c(n) b_{I^{(k)}} \tag{55}$$

hold true.

For each $\mu \in (0,1)$ and the function $f = w$ all assumptions of Theorem 2 hold and in terms of the notations of this theorem the relations

$$\Theta \leqslant c(n,\theta)\Theta_2, \tag{56}$$

$$\Theta^* \leqslant c(n,\theta) l_I^{-1}\Theta_2^*, \tag{57}$$

$$A_k = \sum_{i=1}^{n-1}\Big\{D_{ii} - \frac{\tau_{i,k}}{W}D_{in} - \frac{\tau_{i,k}}{W}D_{ni}\Big\} + \frac{1 + \sum_{s=1}^{n-1}\tau_{s,k}^2}{W^2}D_{nn}, \tag{58}$$

$$\gamma_k(x) = \gamma'_k(x'), \tag{59}$$

$$F_k(x) = W\frac{\Gamma(n/2)}{\pi^{n/2}} \lim_{r\to\infty} \int_{|\xi - \mathfrak{c}'_k| < r} \frac{\omega_{(k)}(\xi)}{\Big|\big(x', \gamma'_k(x') + Wx_n\big) - \big(\xi, \gamma'_k(\xi)\big)\Big|^n} d\xi \tag{60}$$

hold, where

$$\omega_{(k)} = \omega_{k+1} - \omega_k, \quad \omega_k = \varphi'_k\omega + (1 - \varphi'_k)\gamma'_k, \quad \varphi'_k(\xi) = \varphi_k(\xi, 0+).$$

Remark. *The limit $\varphi'_k(\xi)$ exists thanks to inequality* (4c).

Proof. Due to (44) we have $b_J \leqslant c_1(n,\theta)$. The definition of the set LIP and the definition of the numbers b_J show that

$$b_J \leqslant C_1(\omega) l_J^{-\frac{n+1}{2}} \leqslant C_2(\omega, l_I) l_I^{\frac{n+1}{2}} l_J^{-\frac{n+1}{2}},$$

that by Lemma 1 yields

$$b_J \leqslant \min\left\{c_1, C_2 l_I^{\frac{n+1}{2}} l_J^{-\frac{n+1}{2}}\right\} \leqslant c_1^{\frac{n}{n+1}} C_2^{\frac{1}{n+1}} l_I^{1/2} l_J^{-1/2}, \quad \Theta_1 \leqslant c_1^{\frac{n}{n+1}} C_2^{\frac{1}{n+1}} \sum_J \Gamma_{IJ}^{(1/2, n-1/2)} < \infty.$$

The relations $\Theta_2 < \infty$ and $\Theta_2^* < \infty$ are implied by the inequalities $b_J \leqslant c_1$ and $\Theta_1 < \infty$.

Estimates (54) and (55) are obtained by the embeddings $I^{(k)} \subset I^{(k+1)} \subset 5I^{(k)}$ and simple properties of the polynomials similarly to [10, Subsect. 2.7].

The conditions $\lambda \geqslant 1$ and $A \in \mathcal{A}_\lambda^\mu$ of Theorem 2 are implied by Theorem 3. In view of (2) and (45) we have

$$\|D^2 w\|_J \leqslant c_2(n) l_J^{-1} b_J, \qquad (61)$$

$$\sum_J \Gamma_{IJ}^{(0,n)} l_J \|D^2 w\|_J \leqslant c_2 \Theta_1 < \infty,$$

and hence, $D^2 w \in \mathrm{VL}(0)$. By (51), (61) and the Cauchy inequality we get

$$\mathcal{F}_J \leqslant c_2 l_J^{-1} b_J + c_2 [I,J]^{-1} \sum_{H \in \{I^J\} \cup \overrightarrow{IJ}} b_H \leqslant 2 c_2 l_J^{-1} \sum_{H \in \overleftarrow{IJ} \cup \overrightarrow{IJ}} b_H,$$

$$\left(\sum_{H \in \overleftarrow{IJ} \cup \overrightarrow{IJ}} |A|_H\right) \mathcal{F}_J \leqslant c(n,\theta) l_J^{-1} \left(\sum_{H \in \overleftarrow{IJ} \cup \overrightarrow{IJ}} b_H\right)^2 \leqslant c_3(n,\theta) l_J^{-3/2} \sum_{H \in \overleftarrow{IJ} \cup \overrightarrow{IJ}} l_H^{1/2} b_H^2.$$

By (32) we obtain

$$\sum_{J\colon H \in \overleftarrow{IJ} \cup \overrightarrow{IJ}} \Gamma_{IJ}^{(0,n)} l_J l_J^{-3/2} \leqslant c_4(n) \Gamma_{IH}^{(0,n)} l_H^{-1/2}, \qquad \sum_{J\colon H \in \overleftarrow{IJ} \cup \overrightarrow{IJ}} \Gamma_{IJ}^{(0,n+1)} l_J^{-3/2} \leqslant c_5(n) l_I^{-1} \Gamma_{IH}^{(1,n)} l_H^{-1/2}.$$

Therefore,

$$\Theta \leqslant c_3 \sum_H \left(\sum_{J\colon H \in \overleftarrow{IJ} \cup \overrightarrow{IJ}} \Gamma_{IJ}^{(0,n)} l_J l_J^{-3/2}\right) l_H^{1/2} b_H^2 \leqslant c_3 c_4 \Theta_2,$$

$$\Theta^* \leqslant c_3 \sum_H \left(\sum_{J\colon H \in \overleftarrow{IJ} \cup \overrightarrow{IJ}} \Gamma_{IJ}^{(0,n+1)} l_J^{-3/2}\right) l_H^{1/2} b_H^2 \leqslant c_3 c_5 l_I^{-1} \Theta_2^*.$$

We have obtained estimates (56) and (57), which imply $\Theta < \infty$. Hence, the function $f = w$ satisfies all assumptions of Theorem 2.

By (7a) the function $w(x)$ coincides with $\gamma_k'(x')$ in a "half-neighbourhood" of the point $\mathfrak{c}_k$, while the function $\mathfrak{G}(y)$ coincides with the function $\frac{y_n - \gamma_k'(y')}{W}$ in a "half-neighbourhood" of the point $g(\mathfrak{c}_k)$. This leads us to (58) and (59).

Let us prove identity (60). In view of (58) it is easy to confirm that

$$\det\nolimits_{A_k} = W^{-2}.$$

We omit the subscript k in notation of the numbers $\tau_{i,k}$ and the coefficients $a_{ij,k}$ of the operator A_k. Introducing the shorthand notation $\tau_n = W$, we can write (58) as

$$a_{ij} = \delta_{ij} - \delta_{in} \frac{\tau_j}{W} - \delta_{jn} \frac{\tau_i}{W} + \delta_{in} \delta_{jn} \frac{1 + \sum_{s=1}^n \tau_s^2}{W^2}.$$

The numbers

$$b_{ij} = \delta_{ij} - \delta_{in}\delta_{jn} + \tau_i\tau_j$$

satisfy the identities

$$\sum_{j=1}^{n} a_{ij}b_{jq} = \sum_{j} a_{ij}(\delta_{jq} - \delta_{jn}\delta_{qn} + \tau_j\tau_q) = a_{iq} - a_{in}\delta_{qn} + \left(\sum_{j} a_{ij}\tau_j\right)\tau_q,$$

$$a_{iq} - a_{in}\delta_{qn} = \delta_{iq} - \delta_{in}\frac{\tau_q}{W} - \delta_{qn}\frac{\tau_i}{W} + \delta_{in}\delta_{qn}\frac{1+\sum_{s=1}^{n}\tau_s^2}{W^2}$$
$$- \left(\delta_{in} - \delta_{in} - \frac{\tau_i}{W} + \delta_{in}\frac{1+\sum_{s=1}^{n}\tau_s^2}{W^2}\right)\delta_{qn} = \delta_{iq} - \delta_{in}\frac{\tau_q}{W},$$

$$\sum_{j} a_{ij}\tau_j = \tau_i - \frac{\delta_{in}}{W}\sum_{j}\tau_j^2 + \left(-\frac{\tau_i}{W} + \delta_{in}\frac{1+\sum_{s=1}^{n}\tau_s^2}{W^2}\right)\tau_n = \frac{\delta_{in}}{W},$$

$$\sum_{j=1}^{n} a_{ij}b_{jq} = \delta_{iq} - \delta_{in}\frac{\tau_q}{W} + \frac{\delta_{in}}{W}\tau_q = \delta_{iq}.$$

Hence, $(b_{ij}) = (a_{ij})^{-1}$. Letting $\xi_n = 0$, for $\xi \in \mathbb{R}^{n-1}$ we have

$$\begin{aligned} Q_{A_k}\big(x - (\xi, 0)\big) &= \sum_{i,j=1}^{n} b_{ij}(x_i - \xi_i)(x_j - \xi_j) = \sum_{i=1}^{n-1}(x_i - \xi_i)^2 + \sum_{i,j=1}^{n}\tau_i\tau_j(x_i - \xi_i)(x_j - \xi_j) \\ &= |x' - \xi|^2 + \big(\gamma_k'(x') - \gamma_k'(\xi) + \tau_n x_n - \tau_n\xi_n\big)^2 \\ &= \Big|\big(x', \gamma_k'(x') + Wx_n\big) - \big(\xi, \gamma_k'(\xi)\big)\Big|^2, \end{aligned}$$

which by (42) and (59) leads us to (60). The proof is complete. □

Under the assumptions of Lemma 4 we denote

$$H_0 = \big\{x \in \overline{I} \times \mathbb{R}\colon x_n \geqslant \gamma_0'(x') + 2Wl_I/3\big\}, \quad H_k = \overline{I} \times \mathbb{R} \quad \text{as } k \geqslant 1. \tag{62}$$

We let $\mathfrak{x} = \mathfrak{c}_{I^{(k)}}$. It is obvious that for $x \in H_k$ there exists the limit

$$F_{(k)}(x) = \frac{\Gamma(n/2)}{\pi^{n/2}}\lim_{r\to\infty}\int_{|\xi-\mathfrak{x}|<r}\omega_{(k)}(\xi)\Big|x - \big(\xi, \gamma_k'(\xi)\big)\Big|^{-n}d\xi. \tag{63}$$

We have $g(I^{\boxdot}) \subset H_k$ (in view of (48)) and $\big(x', \gamma_k'(x') + Wx_n\big) \in H_k$ for $x \in I^{\boxdot}$, see (60).

Lemma 5. *Under assumptions of Lemma 4 for $(x, \xi) \in H_k \times \mathbb{R}^{n-1}$ we let*

$$\xi^* = 2\mathfrak{x} - \xi, \quad M_k(x,\xi) = \frac{1}{2}\left\{\omega_{(k)}(\xi)\Big|x - \big(\xi, \gamma_k'(\xi)\big)\Big|^{-n} + \omega_{(k)}(\xi^*)\Big|x - \big(\xi^*, \gamma_k'(\xi^*)\big)\Big|^{-n}\right\}.$$

Then $F_{(k)} \in C^\infty(H_k)$ and for each $\alpha \in \mathbb{N}_0^n$

$$\int_{\mathbb{R}^{n-1}}\big|D_x^\alpha M_k(x,\xi)\big|\,d\xi \leqslant c(\alpha,\theta)\,l_{I^{(k)}}^{-|\alpha|}b_{I^{(k)}}, \tag{64a}$$

$$D^\alpha F_{(k)}(x) = \frac{\Gamma(n/2)}{\pi^{n/2}}\int_{\mathbb{R}^{n-1}}D_x^\alpha M_k(x,\xi)\,d\xi. \tag{64b}$$

Proof. We denote

$$\mathfrak{X} = \big(\mathfrak{x}, \gamma_k'(\mathfrak{x})\big), \quad \Xi = |x - \mathfrak{X}| + |\mathfrak{x} - \xi|.$$

Let $T_k(\xi)$ be the convex hull of the set $\big\{\gamma_k'(\xi), \omega(\xi), \omega_{k+1}(\xi)\big\} \subset \mathbb{R}$.

Let us prove that

$$\begin{aligned}&\text{if } (x,\xi)\in H_k\times\mathbb{R}^{n-1} \text{ with } t\in T_k(\xi), \text{ and } \xi\notin 3I^{(k-1)} \text{ for } k\geqslant 1,\\ &\text{then } \big|x-(\xi,t)\big|\geqslant c(n,\theta)\Xi\geqslant c(n,\theta)l_{I^{(k)}}.\end{aligned}\tag{65}$$

If the assumption in (65) holds, then $t=\zeta(\xi)$ for a convex linear combination

$$\zeta=\beta_1\gamma'_k+\beta_2\omega+\beta_3\gamma'_{k+1},$$

since $\omega_{k+1}=\varphi'_{k+1}\omega+(1-\varphi'_{k+1})\gamma'_{k+1}$. We denote

$$Z=\big(x',\zeta(x')\big),\quad R=|x-Z|+|x'-\xi|.$$

It follows from the inequalities $\theta\geqslant\|\omega\|_{\mathrm{Lip}}$ and (43) that $\|\zeta\|_{\mathrm{Lip}}\leqslant c(n)\theta$ and this is why by the triangle with the vertices x, Z and $(\xi,t)=(\xi,\zeta(\xi))$ we get that

$$\big|x-(\xi,t)\big|\geqslant c_1(n,\theta)\big\{|x-Z|+\big|Z-(\xi,t)\big|\big\}\geqslant c_1R.$$

Let us treat the cases $\xi\in 3I$ and $\xi\notin 3I$. If $\xi\in 3I$, then $k=0$ and $\omega_{k+1}(\xi)=\omega(\xi)$ due to (4a) and hence, $\beta_3=0$ without loss of generality. By $x\in H_0$ and (47) we have

$$\begin{aligned}&x_n-\gamma'_0(x')\geqslant 2Wl_I/3,\\ &x_n-\omega(x')\geqslant x_n-\gamma'_0(x')-Wl_I/3\geqslant Wl_I/3,\\ &R\geqslant|x-Z|=\beta_1\big[x_n-\gamma'_0(x')\big]+\beta_2\big[x_n-\omega(x')\big]\geqslant Wl_I/3.\end{aligned}$$

If $\xi\notin 3I$, then $|x'-\xi|\geqslant l_I$ as $k=0$ and $|x'-\xi|\geqslant l_{I^{(k-1)}}$ as $k\geqslant 1$ and hence,

$$R\geqslant\min\{W/3,1/2\}l_{I^{(k)}}\quad\text{for each }\xi.$$

By (43), (44), (47) and (55) we conclude that

$$\begin{aligned}&\big|\omega(x')-\gamma'_k(x')\big|\leqslant\|\omega-\gamma_{I^{(k)}}\|_{L^\infty(I^{(k)})}\leqslant Wl_{I^{(k)}}/3,\quad\big|\gamma'_{k+1}(x')-\gamma'_k(x')\big|\leqslant c(n,\theta)l_{I^{(k)}},\\ &|\mathfrak{X}-Z|\leqslant\Big|\big(\mathfrak{x},\gamma'_k(\mathfrak{x})\big)-\big(x',\gamma'_k(x')\big)\Big|+\big|\gamma'_k(x')-\zeta(x')\big|\leqslant c(n,\theta)l_{I^{(k)}},\\ &\Xi\leqslant R+|\mathfrak{X}-Z|+|\mathfrak{x}-x'|\leqslant R+c(n,\theta)l_{I^{(k)}}\leqslant c_2(n,\theta)R,\quad\big|x-(\xi,t)\big|\geqslant c_1c_2^{-1}\Xi.\end{aligned}$$

If $\xi\in 3I$, then $k=0$ and $\gamma'_0(\mathfrak{x})\in T_0(\mathfrak{x})$ and therefore,

$$\Xi\geqslant|x-\mathfrak{X}|\geqslant c_1R\big|_{\xi=\mathfrak{x}}\geqslant c_1\min\{W/3,1\}l_I.$$

If $\xi\notin 3I$, then $|\mathfrak{x}-\xi|\geqslant 3l_I/2$ as $k=0$, $|\mathfrak{x}-\xi|\geqslant l_{I^{(k-1)}}$ as $k\geqslant 1$ and hence, $\Xi\geqslant l_{I^{(k)}}/2$ for each k. Thus, $\Xi\geqslant c(n,\theta)l_{I^{(k)}}$ for each ξ and we complete the proof of implication (65).

If $\omega_{(k)}(\xi)\neq 0$, then $\xi\notin 3I^{(k-1)}$ as $k\geqslant 1$ due to (4a) and hence, by (65),

$$\begin{aligned}&\text{if } (x,\xi)\in H_k\times\mathbb{R}^{n-1} \text{ and either } \omega_{(k)}(\xi)\neq 0, \text{ or } \xi\notin 5I^{(k)},\\ &\text{then } \Big|D^\alpha_x\big|x-(\xi,t)\big|^{-n}\Big|\leqslant c(\alpha,\theta)\Xi^{-n-|\alpha|}\leqslant c(\alpha,\theta)l_{I^{(k)}}^{-n-|\alpha|}\quad\text{as } t\in T_k(\xi).\end{aligned}\tag{66}$$

By (54) and the Hölder inequality this follows that

$$\omega_{(k)}=(\varphi'_{k+1}-1)(\omega-\gamma'_{k+1})+(1-\varphi'_k)(\omega-\gamma'_k),\tag{67}$$

$$\|\omega_{(k)}\|_{L^1(5I^{(k)})}\leqslant c(n)l^n_{I^{(k)}}b_{I^{(k)}},$$

$$\big\|D^\alpha_xM_k(x,\cdot)\big\|_{L^1(5I^{(k)})}\leqslant c(\alpha,\theta)\|\omega_{(k)}\|_{L^1(5I^{(k)})}l_{I^{(k)}}^{-n-|\alpha|}\leqslant c(\alpha,\theta)l_{I^{(k)}}^{-|\alpha|}b_{I^{(k)}}.\tag{68}$$

Let $\xi \in \mathbb{R}^{n-1} \setminus \overline{5I^{(k)}}$ ($\Rightarrow \xi^* \in \mathbb{R}^{n-1} \setminus 5I^{(k)}$). Then $\omega_{(k)}(\xi) = \gamma'_{k+1}(\xi) - \gamma'_k(\xi)$ and $\omega_{(k)}(\xi^*) = \gamma'_{k+1}(\xi^*) - \gamma'_k(\xi^*)$ due to (4b). By (55) and (66) we obtain

$$\left|\frac{\omega_{(k)}(\xi)+\omega_{(k)}(\xi^*)}{2}\right| = \left|\gamma'_{k+1}(\mathfrak{x}) - \gamma'_k(\mathfrak{x})\right| \leqslant c(n) l_{I^{(k)}} b_{I^{(k)}}, \tag{69}$$

$$\left|\frac{\omega_{(k)}(\xi)+\omega_{(k)}(\xi^*)}{2} D^\alpha_x \Big|x - \big(\xi, \gamma'_k(\xi)\big)\Big|^{-n}\right| \leqslant c(\alpha,\theta) l_{I^{(k)}} b_{I^{(k)}} \Xi^{-n-|\alpha|},$$

$$\left|\omega_{(k)}(\xi^*)\right| \leqslant c(n)\big(l_{I^{(k)}} + |\mathfrak{x}-\xi^*|\big) b_{I^{(k)}} \leqslant c(n)|\mathfrak{x}-\xi| b_{I^{(k)}} \leqslant c(n)\Xi b_{I^{(k)}}, \tag{70}$$

$$\left|D^\alpha_x M_k(x,\xi)\right| \leqslant c(\alpha,\theta) l_{I^{(k)}} b_{I^{(k)}} \Xi^{-n-|\alpha|} + c(n)\Xi b_{I^{(k)}} |Y_\alpha|,$$

where

$$Y_\alpha = D^\alpha_x \Big|x - \big(\xi, \gamma'_k(\xi)\big)\Big|^{-n} - D^\alpha_x \Big|x - \big(\xi^*, \gamma'_k(\xi^*)\big)\Big|^{-n}.$$

Majorizing each term by (66), we get

$$|Y_\alpha| \leqslant c(\alpha,\theta)\Xi^{-n-|\alpha|}, \tag{71}$$

$$\begin{aligned}\left|D^\alpha_x M_k(x,\xi)\right| &\leqslant c_3(\alpha,\theta)(l_{I^{(k)}} + \Xi) b_{I^{(k)}} \Xi^{-n-|\alpha|} \\ &\leqslant \frac{7c_3}{5} b_{I^{(k)}} \Xi^{1-n-|\alpha|} \leqslant \frac{7c_3}{5} b_{I^{(k)}} |\mathfrak{x}-\xi|^{1-n-|\alpha|}.\end{aligned} \tag{72}$$

Relations $x' \in \overline{I} \subset \overline{I^{(k)}}$, $\mathfrak{X}' = \mathfrak{x} \in I^{(k)}$ and $\xi \notin 5I^{(k)}$ show that

$$\left|\tau x' + (1-\tau)\mathfrak{x} - \xi\right|_\infty \geqslant \frac{4}{5}|\mathfrak{x}-\xi|_\infty,$$

$$\left|\tau x + (1-\tau)\mathfrak{X} - (\xi,t)\right| \geqslant \frac{4}{5\sqrt{n-1}}|\mathfrak{x}-\xi| \quad \text{for each } \tau \in [0,1] \text{ and } t \in \mathbb{R}. \tag{73}$$

Hence, by the identity $\Big|\mathfrak{X} - \big(\xi, \gamma'_k(\xi)\big)\Big| = \Big|\mathfrak{X} - \big(\xi^*, \gamma'_k(\xi^*)\big)\Big|$ and by (71) we obtain

$$|Y_0| \leqslant c(n)|x-\mathfrak{X}||\mathfrak{x}-\xi|^{-n-1},$$

$$|Y_0| \leqslant \min\big\{c(n)|x-\mathfrak{X}||\mathfrak{x}-\xi|^{-n-1}, c(n,\theta)\Xi^{-n}\big\} \leqslant c(n,\theta)|x-\mathfrak{X}|\Xi^{-n-1},$$

$$\left|M_k(x,\xi)\right| \leqslant c(n,\theta)\big(l_{I^{(k)}} + |x-\mathfrak{X}|\big) b_{I^{(k)}} \Xi^{-n},$$

$$\int_{|\mathfrak{x}-\xi| \geqslant 5l_{I^{(k)}}/2} \big(|x-\mathfrak{X}| + |\mathfrak{x}-\xi|\big)^{-n} d\xi \leqslant c(n)\big(l_{I^{(k)}} + |x-\mathfrak{X}|\big)^{-1},$$

$$\left\|M_k(x,\cdot)\right\|_{L^1(\mathbb{R}^{n-1}\setminus 5I^{(k)})} \leqslant c(n,\theta) b_{I^{(k)}}. \tag{74}$$

Estimate (64a) as $\alpha = 0$ is implied by (68) and (74), while as $\alpha \neq 0$, it is due to (68) and (72). Identity (64b) as $\alpha = 0$ is yielded by (63) and the change of variable $\xi \to \xi^*$, while as $\alpha \neq 0$ (together with the statement $F_{(k)} \in C^\infty(H_k)$) it is implied by differentiating the integral formula (64b) that is possible thanks to (72). □

3.2. Function S and potential Φ_L. Let us give a "qualitative" analogue of Lemma 5 for the functions determined by the volume integrals.

Lemma 6. *Let $\omega_+, \omega_- \in \mathrm{LIP}$, $\Omega_\pm = \big\{x \in \mathbb{R}^n \colon x_n > \omega_\pm(x')\big\}$ and $\chi = \chi_+ - \chi_-$, where $\chi_\pm$ are the characteristic functions of the sets Ω_+ and Ω_-. We let*

$$\xi^* = 2\mathfrak{x} - \xi, \quad N_{\mathfrak{x}}(x,\xi) = \int_{\mathbb{R}} \frac{\chi(\xi,t)\big|x-(\xi,t)\big|^{-n} + \chi(\xi^*,t)\big|x-(\xi^*,t)\big|^{-n}}{2}\, dt$$

for $(\mathfrak{x}, x, \xi) \in \mathbb{R}^{n-1} \times (\mathbb{R}^n \setminus \operatorname{supp}\chi) \times \mathbb{R}^{n-1}$. Then the following statements hold true.

(i) The function $N_{x'}(x,\cdot)$ belongs to $L^1(\mathbb{R}^{n-1})$, there exists the limit

$$s(x) = \lim_{r\to\infty} \int_{|x-y|<r} \chi(y)|x-y|^{-n}\, dy,$$

and the identity $s(x) = \int_{\mathbb{R}^{n-1}} N_{x'}(x,\xi)\, d\xi$ holds true.

(ii) For each $(\mathfrak{x}, x, \alpha) \in \mathbb{R}^{n-1} \times (\mathbb{R}^n \setminus \operatorname{supp}\chi) \times \mathbb{N}_0^n$ the belonging

$$D_x^\alpha N_{\mathfrak{x}}(x,\cdot) \in L^1(\mathbb{R}^{n-1}) \tag{75a}$$

holds true, the function s is infinitely differentiable in $\mathbb{R}^n \setminus \operatorname{supp}\chi$ and

$$D^\alpha s(x) = \int_{\mathbb{R}^{n-1}} D_x^\alpha N_{\mathfrak{x}}(x,\xi)\, d\xi. \tag{75b}$$

Proof. (i) For $\omega \in \mathrm{LI}\mathbb{P}$ by $\chi[\omega]$ we denote the characteristic function of the overgraph of the function ω, while by $\gamma[\omega]$ we denote the polynomial in $\mathbb{P}_1^{n-1}$, with which ω coincides in the vicinity of infinity. As $x \notin \operatorname{supp}\chi$, we let

$$\gamma_\pm = \gamma[\omega_\pm] \quad \& \quad \gamma^\pm = \gamma_\pm - \gamma_\pm(x') + x_n.$$

The functions χ_+ and χ_- coincide in the vicinity of x and this is why there exist $\omega^\pm \in \mathrm{LI}\mathbb{P}$ such that

$$\chi_+ = \chi[\omega^+] = \chi[\omega^-] = \chi_- \text{ in the vicinity of } x \quad \& \quad \gamma^\pm = \gamma[\omega^\pm].$$

In view of the representation

$$\chi = \{\chi_+ - \chi[\omega^+]\} + \{\chi[\omega^+] - \chi[\omega^-]\} + \{\chi[\omega^-] - \chi_-\}$$

we see that in order to check Statement (i), it is sufficient to check (i) for the pairs (ω_+, ω^+), (ω^+, ω^-) and (ω^-, ω_-) instead of (ω_+, ω_-). Therefore, it is sufficient to check (i) in particular cases

(a) $\gamma_+ - \gamma_- = \text{const}$;

(b) $\gamma_+(x') = x_n = \gamma_-(x')$.

In Case (a), the function $y \mapsto \chi(y)|x-y|^{-n}$ belongs to $L^1(\mathbb{R}^n)$, which gives (i) by the Fubini theorem and the change of variables $\xi \to \xi^\star = 2x' - \xi$. In Case (b), the change of variables $y = (\xi, t) \to 2x - y$ and the Fubini theorem shows that

$$N_{x'}(x,\xi) = 0 \quad \text{for large } |x' - \xi|,$$
$$(\exists r_0 > 0)\ (\forall r > r_0) \quad \int_{|x-y|<r} \chi(y)|x-y|^{-n}\, dy = \int_{\mathbb{R}^{n-1}} N_{x'}(x,\xi)\, d\xi.$$

Thus, the proof of Statement (i) is complete.

(ii) For $\xi \in \mathbb{R}^{n-1}$ we let

$$\nu(\xi) = \int_{\mathbb{R}} \frac{\chi(\xi^\star, t)\big|x - (\xi^\star, t)\big|^{-n} - \chi(\xi^*, t)\big|x - (\xi^*, t)\big|^{-n}}{2}\, dt.$$

By the identity $\xi^\star - \xi^* = 2x' - 2\mathfrak{x}$ it is easy to get that

$$\sup_\xi \big|\nu(\xi)\big| |\xi|^n < \infty, \qquad \nu \in L^1(\mathbb{R}^{n-1})$$

and

$$\int_{\mathbb{R}^{n-1}} \nu(\xi)\, d\xi = 0.$$

This is why properties (75) with $\alpha = 0$ are implied by Statement (i). The case $\alpha \neq 0$ can be treated similarly to Lemma 5, via checking an analogue of estimate (72) for the function $N_{\mathfrak{x}}(x,\xi)$. □

Let $\omega \in \mathrm{LIP}$. To compare Lemmata 4, 5 and 6, we introduce the function

$$S(x) := S_\Omega(x) := \lim_{r\to\infty}\left\{\ln r - \frac{\Gamma(n/2)}{\pi^{n/2}}\int_{y\in\mathbb{R}^n\setminus\Omega:\ |x-y|<r}|x-y|^{-n}\,dy\right\},\quad x\in\Omega.$$

Here the limit exists since the area of the unit sphere $\mathbb{S}^{n-1}\subset\mathbb{R}^n$ is equal to $\frac{2\pi^{n/2}}{\Gamma(n/2)}$. The function S is invariant w.r.t. the shifts and rotations of the domain Ω in the obvious sense.

Lemma 7. *Under assumptions of Lemma 4 let* $\Omega_k=\{x\in\mathbb{R}^n\colon x_n>\omega_k(x')\}$ *and* $S_k=S_{\Omega_k}$. *Then the inequalities*

$$\|D^\alpha S - D^\alpha S_0\|_{L^\infty(g(I^\boxdot))}\leqslant c(\alpha,\theta)\sum_{k=0}^\infty l_{I^{(k)}}^{-|\alpha|}b_{I^{(k)}},\quad \alpha\in\mathbb{N}_0^n, \tag{76}$$

$$\big\|D^\alpha(S\circ g - S_0\circ g)\big\|_I\leqslant c(\alpha,\theta)\sum_{k=0}^\infty l_{I^{(k)}}^{-|\alpha|}b_{I^{(k)}},\quad |\alpha|\leqslant 1, \tag{77}$$

$$\big\|D^\alpha(S\circ g - S_0\circ g)\big\|_I\leqslant c(\alpha,\theta)\sum_{j=0}^1 l_I^{(1-|\alpha|)j}b_I^j\sum_{k=0}^\infty l_{I^{(k)}}^{(j-1)|\alpha|-j}b_{I^{(k)}},\quad |\alpha|\geqslant 2 \tag{78}$$

hold true. If $\varepsilon=0$ *as* $|\alpha|\leqslant 1$ *and* $0<\varepsilon\leqslant 2$ *as* $|\alpha|=2$, *then the sum* F *of the series* $\sum_{k=0}^\infty F_k$ *satisfies*

$$\big\|D^\alpha(F+WS\circ g - WS_0\circ g)\big\|_I\leqslant c(\alpha,\theta,\varepsilon)l_I^{-\varepsilon}\sum_{k=0}^\infty l_{I^{(k)}}^{\varepsilon-|\alpha|}b_{I^{(k)}}^2,\quad |\alpha|\leqslant 2. \tag{79}$$

Proof. By the embedding $g(I^\boxdot)\subset H_k$ (see (62)) and by Lemma 5 we have

$$\|D^\alpha F_{(k)}\|_{L^\infty(g(I^\boxdot))}\leqslant c(\alpha,\theta)l_{I^{(k)}}^{-|\alpha|}b_{I^{(k)}},\quad \alpha\in\mathbb{N}_0^n. \tag{80}$$

Let us establish the second main inequality

$$\|D^\alpha F_{(k)} + D^\alpha S_{k+1} - D^\alpha S_k\|_{L^\infty(g(I^\boxdot))}\leqslant c(\alpha,\theta)l_{I^{(k)}}^{-|\alpha|}b_{I^{(k)}}^2,\quad \alpha\in\mathbb{N}_0^n. \tag{81}$$

For $x\in g(I^\boxdot)\subset H_k$ we let $\mathfrak{x}$, $F_{(k)}$, ξ^*, M_k, $\mathfrak{X}$, Ξ and T_k to have the same meaning as in Lemma 5 and in its proof, while the functions χ, $N_\mathfrak{x}$ and s are defined by Lemma 6 for the pair of the functions $(\omega_+,\omega_-)=(\omega_{k+1},\omega_k)$. We let

$$U(\xi)=D_x^\alpha\Big|x-\big(\xi,\gamma_k'(\xi)\big)\Big|^{-n} - D_x^\alpha\Big|x-\big(\xi,\omega_k(\xi)\big)\Big|^{-n},$$
$$V(\xi,\tau)=D_x^\alpha\Big|x-\big(\xi,\omega_k(\xi)\big)\Big|^{-n} - D_x^\alpha\Big|x-\big(\xi,\omega_k(\xi)+\omega_{(k)}(\xi)\tau\big)\Big|^{-n}.$$

In view of the belongings $\gamma_k'(\xi),\omega_k(\xi)\in T_k(\xi)$, the relations (66) (for $\alpha+e_n$), (67), (54) and the Hölder inequality

$$\big|U(\xi)\big|\leqslant c_1(\alpha,\theta)\big|\gamma_k'(\xi)-\omega_k(\xi)\big|l_{I^{(k)}}^{-n-|\alpha|-1}\quad \text{as } \omega_{(k)}(\xi)\neq 0,$$
$$|\omega_{(k)}U|\leqslant c_1\big[|\omega-\gamma_{k+1}'|+|\omega-\gamma_k'|\big]|\omega-\gamma_k'|l_{I^{(k)}}^{-n-|\alpha|-1}\quad \text{in } \mathbb{R}^{n-1},$$
$$\|\omega_{(k)}U\|_{L^1(5I^{(k)})}\leqslant c(\alpha,\theta)l_{I^{(k)}}^{-|\alpha|}b_{I^{(k)}}^2.$$

In the same way,

$$\big|V(\xi,\tau)\big|\leqslant c(\alpha,\theta)\big|\omega_{(k)}(\xi)\big|l_{I^{(k)}}^{-n-|\alpha|-1}\quad \text{as } \omega_{(k)}(\xi)\neq 0 \text{ and } 0\leqslant\tau\leqslant 1,$$
$$\bigg\|\omega_{(k)}\int_0^1 V(\cdot,\tau)\,d\tau\bigg\|_{L^1(5I^{(k)})}\leqslant c(\alpha,\theta)l_{I^{(k)}}^{-|\alpha|}b_{I^{(k)}}^2.$$

Hence, in view of the identity $U\big|_{\mathbb{R}^{n-1}\setminus 5I^{(k)}} \equiv 0$ we obtain

$$\omega_{(k)}(\xi)\left\{U(\xi)+\int_0^1 V(\xi,\tau)\,d\tau\right\} = \omega_{(k)}(\xi)D_x^\alpha\Big|x-\big(\xi,\gamma_k'(\xi)\big)\Big|^{-n} + \int_{\mathbb{R}}\chi(\xi,t)D_x^\alpha\big|x-(\xi,t)\big|^{-n}\,dt,$$

$$D_x^\alpha M_k(x,\xi)+D_x^\alpha N_{\mathfrak{x}}(x,\xi) = \frac{\omega_{(k)}(\xi)}{2}\left\{U(\xi)+\int_0^1 V(\xi,\tau)\,d\tau\right\} + \frac{\omega_{(k)}(\xi^*)}{2}\left\{U(\xi^*)+\int_0^1 V(\xi^*,\tau)\,d\tau\right\},$$

$$\big\|D_x^\alpha M_k(x,\cdot)+D_x^\alpha N_{\mathfrak{x}}(x,\cdot)\big\|_{L^1(\mathbb{R}^{n-1})} \leqslant c(\alpha,\theta)l_{I^{(k)}}^{-|\alpha|}b_{I^{(k)}}^2+\frac{1}{2}\|\Theta\|_{L^1(\mathbb{R}^{n-1}\setminus 5I^{(k)})},$$

where

$$\Theta(\xi)=\omega_{(k)}(\xi)\int_0^1 V(\xi,\tau)\,d\tau+\omega_{(k)}(\xi^*)\int_0^1 V(\xi^*,\tau)\,d\tau.$$

Let $\xi\in\mathbb{R}^{n-1}\setminus\overline{5I^{(k)}}$. By (66), (69) and (70) we get

$$\left|\int_0^1 V(\xi,\tau)\,d\tau\right| \leqslant \frac{c(\alpha,\theta)\big|\omega_{(k)}(\xi)\big|}{|\mathfrak{x}-\xi|^{n+|\alpha|+1}} \leqslant c(\alpha,\theta)b_{I^{(k)}}|\mathfrak{x}-\xi|^{-n-|\alpha|},$$

$$\big|\Theta(\xi)\big| \leqslant \frac{c(\alpha,\theta)l_{I^{(k)}}b_{I^{(k)}}^2}{|\mathfrak{x}-\xi|^{n+|\alpha|}}+c(n)|\mathfrak{x}-\xi|b_{I^{(k)}}\left|\int_0^1\big[V(\xi^*,\tau)-V(\xi,\tau)\big]\,d\tau\right|.$$

Hence, as $\alpha\neq 0$,

$$\|\Theta\|_{L^1(\mathbb{R}^{n-1}\setminus 5I^{(k)})}\leqslant c(\alpha,\theta)l_{I^{(k)}}^{-|\alpha|}b_{I^{(k)}}^2. \tag{82}$$

Let $\alpha=0$ and therefore,

$$V(\xi,\tau)=\omega_{(k)}(\xi)\tau\int_0^1 D_{x_n}\Big|x-\big(\xi,\omega_k(\xi)+\omega_{(k)}(\xi)\tau\sigma\big)\Big|^{-n}\,d\sigma,$$

$$V(\xi^*,\tau)=\omega_{(k)}(\xi^*)\tau\int_0^1 D_{x_n}\Big|x-\big(\xi^*,\omega_k(\xi^*)+\omega_{(k)}(\xi^*)\tau\sigma\big)\Big|^{-n}\,d\sigma.$$

Now in view of (69), (70) and (73)

$$\big|V(\xi,\tau)-V(\xi^*,\tau)\big| \leqslant \frac{c(n)l_{I^{(k)}}b_{I^{(k)}}}{|\mathfrak{x}-\xi|^{n+1}}+c(n)|\mathfrak{x}-\xi|b_{I^{(k)}}\max_{0\leqslant\rho\leqslant 1}\big|\Theta(\xi,\rho)\big|,$$

$$\Theta(\xi,\rho)\equiv D_{x_n}\Big|x-\big(\xi,\omega_k(\xi)+\omega_{(k)}(\xi)\rho\big)\Big|^{-n}+D_{x_n}\Big|x-\big(\xi^*,\omega_k(\xi^*)+\omega_{(k)}(\xi^*)\rho\big)\Big|^{-n}.$$

We consider the center X of the corresponding segment:

$$2X=\big(\xi,\omega_k(\xi)+\omega_{(k)}(\xi)\rho\big)+\big(\xi^*,\omega_k(\xi^*)+\omega_{(k)}(\xi^*)\rho\big).$$

By relations (46), $\mathfrak{x}=\mathfrak{c}_{I^{(k)}}$, $\mathfrak{X}=\big(\mathfrak{x},\gamma_k'(\mathfrak{x})\big)$, $\gamma_k'=\gamma_{I^{(k)}}$, (48), $\omega_k(\xi)=\gamma_k'(\xi)$, $\omega_k(\xi^*)=\gamma_k'(\xi^*)$, $\omega_{k+1}(\xi)=\gamma_{k+1}'(\xi)$, $\omega_{k+1}(\xi^*)=\gamma_{k+1}'(\xi^*)$, (44) and (55) we obtain

$$\big|x-g(\mathfrak{c}_{I^{(k)}}^{\boxdot})\big|\leqslant c(n,\theta)\big|\mathfrak{g}(x)-\mathfrak{c}_{I^{(k)}}^{\boxdot}\big|\leqslant c(n,\theta)l_{I^{(k)}},$$

$$\big|g(\mathfrak{c}_{I^{(k)}}^{\boxdot})-\mathfrak{X}\big|=\big|w(\mathfrak{c}_{I^{(k)}}^{\boxdot})+3Wl_{I^{(k)}}/2-\gamma_k'(\mathfrak{x})\big|<2Wl_{I^{(k)}},$$

$$X=\mathfrak{X}+\big(0,(\gamma_{k+1}'(\mathfrak{x})-\gamma_k'(\mathfrak{x}))\rho\big),\quad |x-X|\leqslant c(n,\theta)l_{I^{(k)}}.$$

Similarly to (73) we have

$$\begin{aligned}
&D_{X_n}\Big|X-\big(\xi,\omega_k(\xi)+\omega_{(k)}(\xi)\rho\big)\Big|^{-n}+D_{X_n}\Big|X-\big(\xi^*,\omega_k(\xi^*)+\omega_{(k)}(\xi^*)\rho\big)\Big|^{-n}=0,\\
&\big|\tau x+(1-\tau)X-(\xi,t)\big|\geqslant\frac{4}{5\sqrt{n-1}}|\mathfrak{x}-\xi|\quad\text{for all }\tau\in[0,1]\text{ and }t\in\mathbb{R},\\
&\big|\Theta(\xi,\rho)\big|\leqslant c(n)|x-X||\mathfrak{x}-\xi|^{-n-2}\leqslant c(n,\theta)l_{I^{(k)}}|\mathfrak{x}-\xi|^{-n-2},\\
&\big|V(\xi,\tau)-V(\xi^*,\tau)\big|\leqslant c(n,\theta)l_{I^{(k)}}b_{I^{(k)}}|\mathfrak{x}-\xi|^{-n-1},\\
&\big|\Theta(\xi)\big|\leqslant c(n,\theta)l_{I^{(k)}}b^2_{I^{(k)}}|\mathfrak{x}-\xi|^{-n}.
\end{aligned}$$

This implies (82) as $\alpha=0$.

By inequality (82) we get that

$$\big\|D_x^\alpha M_k(x,\cdot)+D_x^\alpha N_{\mathfrak{x}}(x,\cdot)\big\|_{L^1(\mathbb{R}^{n-1})}\leqslant c(\alpha,\theta)l_{I^{(k)}}^{-|\alpha|}b^2_{I^{(k)}}.$$

We have $g(I^{\boxdot})\subset H_0\subset\Omega_0$ and $g(I^{\boxdot})\subset\bigcap_{j=1}^\infty\Omega_j$ (see (4a)) and hence, the condition $x\notin\operatorname{supp}\chi$ of Lemma 6 holds true. This is why

$$\begin{aligned}
&S_{k+1}-S_k=\frac{\Gamma(n/2)}{\pi^{n/2}}s\quad\text{in the vicinity of each point }x\in g(I^{\boxdot}),\\
&D^\alpha(F_{(k)}+S_{k+1}-S_k)(x)=\frac{\Gamma(n/2)}{\pi^{n/2}}\int_{\mathbb{R}^{n-1}}\big[D_x^\alpha M_k(x,\xi)+D_x^\alpha N_{\mathfrak{x}}(x,\xi)\big]\,d\xi
\end{aligned}$$

due to the definition of the function $s(x)$ and identities (64b) and (75b). This leads us to (81).

By (44), (80) and (81) we conclude that

$$\|D^\alpha S_{k+1}-D^\alpha S_k\|_{L^\infty(g(I^{\boxdot}))}\leqslant c(\alpha,\theta)l_{I^{(k)}}^{-|\alpha|}b_{I^{(k)}},\quad\alpha\in\mathbb{N}_0^n.$$

The series of the right hand sides converges thanks to $\Theta_1<\infty$, this is why the limit $\lim_{k\to\infty}(S_k-S_0)$ exists in $C^\infty\big(g(I^{\boxdot})\big)$ and is obviously equal to $S-S_0$. This yields (76). Differentiating the composition and applying (45), (46) and (76), we have

$$\begin{aligned}
&\big\|D^\alpha(S\circ g-S_0\circ g)\big\|_{L^\infty(I^{\boxdot})}\leqslant c(\alpha,\theta)\sum_{k=0}^\infty l_{I^{(k)}}^{-1}b_{I^{(k)}},\quad|\alpha|=1,\\
&\big\|D^\alpha(S\circ g-S_0\circ g)\big\|_{L^\infty(I^{\boxdot})}\leqslant c(\alpha,\theta)\sum_{j=0}^1 l_I^{(1-|\alpha|)j}b_I^j\sum_{k=0}^\infty l_{I^{(k)}}^{(j-1)|\alpha|-j}b_{I^{(k)}},\ |\alpha|\geqslant2.
\end{aligned}$$

In view of (2), (44) and (76), we obtain estimates (77) and (78).

By (60) and (63) we have

$$F_k=WF_{(k)}\circ h^k$$

on the cube $I^{\boxdot}$, where

$$h^k(x)=\big(x',\gamma_k'(x')+Wx_n\big),\quad x_n>0.$$

By (2), (43) and Lemma 5 we get the estimate

$$\big\|(D^\alpha F_{(k)})\circ h^k\big\|_I\leqslant c(\alpha,\theta)l_{I^{(k)}}^{-|\alpha|}b_{I^{(k)}},\quad\alpha\in\mathbb{N}_0^n.\tag{83}$$

For $x\in I^{\boxdot}$, by (53) and (55),

$$\begin{aligned}
\big|g(x)-h^k(x)\big|&=\big|g_n(x)-h_n^k(x)\big|=\big|w(x)-\gamma_k'(x')\big|\\
&\leqslant\big|w(x)-\gamma_0'(x')\big|+\big|\gamma_0'(x')-\gamma_k'(x')\big|\leqslant c(n)\sum_{j=0}^k l_{I^{(j)}}b_{I^{(j)}}.
\end{aligned}\tag{84}$$

The points $g(x)$ and $h^k(x)$ belong to the convex set H_k and hence, by Lemma 5 and inequality (81),

$$\Big|D^\alpha F_{(k)}\big|_{g(x)} - D^\alpha F_{(k)}\big|_{h^k(x)}\Big| \leqslant \big|g(x)-h^k(x)\big| \sup |D^{\alpha+e_n} F_{(k)}| \leqslant c(\alpha,\theta) l_{I^{(k)}}^{-|\alpha|} b_{I^{(k)}} \sum_{j=0}^{k} 2^{j-k} b_{I^{(j)}},$$

$$\|f_{k,\alpha}\|_{L^\infty(I^\boxdot)} \leqslant c(\alpha,\theta) l_{I^{(k)}}^{-|\alpha|} b_{I^{(k)}} \sum_{j=0}^{k} 2^{j-k} b_{I^{(j)}}, \tag{85}$$

where

$$f_{k,\alpha} = (D^\alpha F_{(k)}) \circ h^k + (D^\alpha S_{k+1} - D^\alpha S_k) \circ g.$$

It is obvious that on the cube $I^\boxdot$,

$$D_i f_{k,\alpha} = \sum_{p=1}^{n} \Big\{ \big[(D^{\alpha+e_p} F_{(k)}) \circ h^k\big] [D_i h_p^k - D_i g_p] + f_{k,\alpha+e_p} D_i g_p \Big\}. \tag{86}$$

Hence, in view of (2), (46), (83) and (85), for each $\alpha \in \mathbb{N}_0^n$ we conclude that

$$\|D_i g_p - D_i h_p^k\|_I \leqslant c(n) \sum_{j=0}^{k} b_{I^{(j)}} \quad \text{(by analogy with (84))}, \tag{87a}$$

$$\|D_i g_p\|_I \leqslant \|D_i g_p\|_{L^\infty(I^\boxdot)} + n l_I \big\|D(D_i g_p)\big\|_{L^\infty(I^\boxdot)} \leqslant c(n,\theta), \tag{87b}$$

$$\|D_i f_{k,\alpha}\|_{L^\infty(I^\boxdot)} \leqslant c(\alpha,\theta) l_{I^{(k)}}^{-|\alpha|-1} b_{I^{(k)}} \sum_{j=0}^{k} b_{I^{(j)}},$$

$$\|f_{k,\alpha}\|_I \leqslant c(\alpha,\theta) l_{I^{(k)}}^{-|\alpha|} b_{I^{(k)}} \sum_{j=0}^{k} 2^{j-k} b_{I^{(j)}}, \tag{87c}$$

$$\|D_i f_{k,\alpha}\|_I \leqslant c(\alpha,\theta) l_{I^{(k)}}^{-|\alpha|-1} b_{I^{(k)}} \sum_{j=0}^{k} b_{I^{(j)}}. \tag{87d}$$

Let us check the inequality

$$\|f'_{k,\alpha}\|_I \leqslant c(\alpha,\theta) l_{I^{(k)}}^{-|\alpha|} b_{I^{(k)}} \sum_{j=0}^{k} 2^{(j-k)(1-|\alpha|)} b_{I^{(j)}}, \quad |\alpha| \leqslant 2, \tag{88}$$

for the function $f'_{k,\alpha} = D^\alpha f_{k,0}$. As $\alpha = 0$, it is identical to (87c), while as $|\alpha| = 1$, it coincides with estimate (87d) for $\alpha = 0$. Differentiating formula (86), we get the identity

$$\begin{aligned} D_{ij} f_{k,0} = &\sum_{p=1}^{n} \Big\{ -\big[(D_p F_{(k)}) \circ h^k\big] D_{ij} g_p + f_{k,e_p} D_{ij} g_p + (D_j f_{k,e_p}) D_i g_p \Big\} \\ &+ \sum_{p,q=1}^{n} \big[(D_{pq} F_{(k)}) \circ h^k\big] [D_i h_p^k - D_i g_p] D_j h_q^k. \end{aligned}$$

Applying (83), (87) and the inequalities

$$\|D_{ij} g_p\|_I \leqslant c(n) l_I^{-1} b_I, \quad \|D_j h_q^k\|_I \leqslant c(n,\theta), \quad \sum_{j=0}^{k} 2^{j-k} b_{I^{(j)}} \leqslant c(n,\theta)$$

implied by (2) and (43)–(45), we arrive at estimate (88) with $|\alpha| = 2$.

For each $\delta \in \mathbb{R}$, by (88) and the Cauchy inequality we obtain

$$\sum_{k=0}^{\infty} \|f'_{k,\alpha}\|_I \leqslant c_2(\alpha,\theta)\Lambda^{1/2} \left(\sum_{k=0}^{\infty} l_{I(k)}^{-\varepsilon-|\alpha|} \left(\sum_{j=0}^{k} 2^{(j-k)(1-|\alpha|)} b_{I(j)} \right)^2 \right)^{1/2},$$

$$\left(\sum_{j=0}^{k} 2^{(j-k)(1-|\alpha|)} b_{I(j)} \right)^2 \leqslant \left(\sum_{j=0}^{k} 2^{-2j\varepsilon+2(j-k)\delta} \right) \sum_{j=0}^{k} 2^{2j\varepsilon+2(j-k)(1-|\alpha|-\delta)} b_{I(j)}^2,$$

where $\Lambda = \sum_{k=0}^{\infty} l_{I(k)}^{\varepsilon-|\alpha|} b_{I(k)}^2$. Let $\delta = \frac{1}{4}$ as $|\alpha| \leqslant 1$ and $\delta = 0$ as $|\alpha| = 2$, so that

$$\sum_{j=0}^{k} 2^{-2j\varepsilon+2(j-k)\delta} \leqslant c_3(\alpha,\varepsilon),$$

$$\begin{aligned} \sum_{k=0}^{\infty} \|f'_{k,\alpha}\|_I \leqslant & c_2 c_3^{1/2} l_I^{-\frac{\varepsilon+|\alpha|}{2}} \Lambda^{1/2} \left(\sum_{j=0}^{\infty} 2^{2j(\varepsilon+1-|\alpha|-\delta)} b_{I(j)}^2 \sum_{k=j}^{\infty} 2^{k(-\varepsilon+|\alpha|-2+2\delta)} \right)^{1/2} \\ \leqslant & c_4(\alpha,\theta,\varepsilon) l_I^{-\frac{\varepsilon+|\alpha|}{2}} \Lambda^{1/2} \left(\sum_{j=0}^{\infty} 2^{j(\varepsilon-|\alpha|)} b_{I(j)}^2 \right)^{1/2} = c_4 l_I^{-\varepsilon} \Lambda. \end{aligned} \tag{89}$$

But $\Lambda < \infty$ due to $\Theta_2 < \infty$, and this is why the series

$$\sum_{k=0}^{\infty} f_{k,0} = \sum_{k=0}^{\infty} \left\{ F_{(k)} \circ h^k + S_{k+1} \circ g - S_k \circ g \right\}$$

converges absolutely in $C^{2,\mu}(I^{\boxdot})$. By the same convergence of the series $F = \sum_{k=0}^{\infty} W F_{(k)} \circ h^k$ (Theorem 2) and the aforementioned relation $C^\infty\big(g(I^{\boxdot})\big)\text{-}\lim_{k\to\infty}(S_k - S_0) = S - S_0$ we have the identity

$$W \sum_{k=0}^{\infty} f_{k,0} = F + WS \circ g - WS_0 \circ g.$$

Together with (89) it proves (79). □

Let us calculate S_Ω, when Ω is a half-space. We introduce the distance function

$$\varrho_\omega(x) = \min_{\xi \in \mathbb{R}^{n-1}} \big| x - (\xi, \omega(\xi)) \big|, \quad x \in \mathbb{R}^n. \tag{90}$$

Theorem 4. *If $\omega \in \mathbb{P}_1^{n-1}$, then*

$$S \equiv \ln \varrho_\omega \big|_\Omega + \sigma_n, \tag{91}$$

where

$$\sigma_n = \begin{cases} \ln 2 + \sum_{k=1}^{\frac{n-2}{2}} \frac{1}{2k}, & n \text{ is even}, \\ \sum_{k=0}^{\frac{n-3}{2}} \frac{1}{2k+1}, & n \text{ is odd}. \end{cases} \tag{92}$$

Proof. While checking (91), we can assume that $\Omega = \mathbb{R}^n_+$ and $x = (0, x_n)$. We introduce the spherical coordinates

$$\begin{aligned} & y_1 = \rho \cos\phi_2 \cos\phi_3 \ldots \cos\phi_n, \\ & y_2 = \rho \sin\phi_2 \cos\phi_3 \ldots \cos\phi_n, \\ & \ldots \\ & y_{n-1} = \rho \sin\phi_{n-1} \cos\phi_n, \\ & x_n - y_n = \rho \sin\phi_n. \end{aligned}$$

The set $\{y \in \mathbb{R}^n\colon y_1 \dots y_{n-1}(x_n - y_n) \neq 0\}$ is described as

$$\rho > 0, \quad |\phi_2| \in (0, \pi/2) \cup (\pi/2, \pi), \quad 0 < |\phi_3|, \dots, |\phi_n| < \pi/2.$$

By the formula for the change of variables

$$\Theta(r) := \int_{\substack{y\in\mathbb{R}^n\setminus\mathbb{R}^n_+\colon\\ |x-y|<r}} |x-y|^{-n}\, dy = \int \frac{\cos\phi_3 \cos^2\phi_4 \dots \cos^{n-2}\phi_n}{\rho}\, d\rho\, d\phi_2 \dots d\phi_n,$$

where the right integral is taken under the restrictions $x_n \leqslant \rho\sin\phi_n$ and $\rho < r$. We have

$$\int_{(-\pi/2,\pi/2)^{n-2}} \cos\phi_3 \dots \cos^{n-3}\phi_{n-1}\, d\phi_2 \dots d\phi_{n-1} = \frac{\operatorname{vol}\mathbb{S}^{n-2}}{2} = \frac{\pi^{\frac{n-1}{2}}}{\Gamma\left(\frac{n-1}{2}\right)},$$

$$\Theta(r) = \operatorname{vol}\mathbb{S}^{n-2} \int_{\rho>0\ \&\ -\pi/2<\phi<\pi/2\colon\, x_n \leqslant \rho\sin\phi < r\sin\phi} \frac{\cos^{n-2}\phi}{\rho}\, d\rho\, d\phi.$$

The cases $n = 2$ and $n > 2$ should be studied independently. If $r > x_n$, then

$$\begin{aligned}
\Theta(r) &= \operatorname{vol}\mathbb{S}^{n-2} \int_{\arcsin\frac{x_n}{r}}^{\pi/2} \cos^{n-2}\phi\, d\phi \int_{\frac{x_n}{\sin\phi}}^{r} \frac{d\rho}{\rho} \\
&= \operatorname{vol}\mathbb{S}^{n-2} \int_{\arcsin\frac{x_n}{r}}^{\pi/2} \left(\ln\frac{r}{x_n} + \ln\sin\phi\right) \cos^{n-2}\phi\, d\phi \\
&= \frac{\operatorname{vol}\mathbb{S}^{n-1}}{2} \ln\frac{r}{x_n} + O\left(\frac{x_n}{r}\ln\frac{r}{x_n}\right) \\
&\quad + \operatorname{vol}\mathbb{S}^{n-2} \int_0^{\pi/2} (\ln\sin\phi) \cos^{n-2}\phi\, d\phi + O\left(\frac{x_n}{r}\ln\frac{r}{x_n}\right) \quad \text{as } r \to \infty,
\end{aligned}$$

since

$$\int_{-\pi/2}^{\pi/2} \cos^{n-2}\phi\, d\phi = \operatorname{vol}\mathbb{S}^{n-1} / \operatorname{vol}\mathbb{S}^{n-2}.$$

This gives (91) with the constant

$$\sigma_n = -\frac{2\operatorname{vol}\mathbb{S}^{n-2}}{\operatorname{vol}\mathbb{S}^{n-1}} \int_0^{\pi/2} (\ln\sin\phi) \cos^{n-2}\phi\, d\phi.$$

It is easy to see that $\sigma_2 = \ln 2$ and $\sigma_3 = 1$. We integrate by parts:

$$n\int_0^{\pi/2} (\ln\sin\phi)\cos^n\phi\, d\phi = (n-1)\int_0^{\pi/2} (\ln\sin\phi)\cos^{n-2}\phi\, d\phi - \int_0^{\pi/2} \cos^n\phi\, d\phi.$$

Hence, $\sigma_{n+2} = \sigma_n + \frac{1}{n}$, which proves (92). □

The next theorem is the main result of the paper. Together with Theorems 1, 3, 4 and Lemma 7 it is aimed for proving (1) and related formulae.

Theorem 5. *Given $\omega \in \mathrm{LI\mathbb{P}}$ and $\theta \geqslant \|\omega\|_{\mathrm{Lip}}$, let $\big(\{\gamma_K\}, w, W, g, \mathfrak{g}, \mathfrak{G}, A, \lambda, L\big)$ be a standard set of the pair (ω, θ). Then $L \in \mathrm{VL}(0)$ and the potential*

$$\Phi_L(x) = \int_{y_n>0} E(A; x, y) L(y)\, dy$$

and the function $S \equiv S_\Omega$ satisfy the inequality

$$\big\| D_n\Phi_L - \mathbf{x}_n^{-1}\Phi_L + 1 - \mathbf{x}_n D_n(S\circ g)\big\|_I \leqslant c(n,\theta,\mu) \sum_J \Gamma_{IJ}^{(1,n)} b_J^2 \tag{93a}$$

for each $I \in \mathcal{D}$. The function $\varrho_{\gamma_I} \circ g$ is positive on $I^{\boxdot}$ (see (90)) and the estimate

$$\Big\| D_{ij}\big\{\Phi_L - W^{-1}w + \mathbf{x}_n[\ln \varrho_{\gamma_I} \circ g - S \circ g]\big\}\Big\|_I \leqslant \frac{c(n,\theta,\mu)}{l_I} \sum_J \Gamma^{(1,n)}_{IJ} b_J^2 \tag{93b}$$

holds true.

Proof. Lemma 4 allows us to apply Theorem 2 to the function $f = w$ and to apply Theorem 7. In this way, the notations f, Θ, $\overleftarrow{IJ}$, $\overrightarrow{IJ}$, $\mathcal{F}_J$, $\mathfrak{c}_k$, A_k, γ_k, w_k, $w_{(k)}$, F_k, F, γ', $\mathcal{R}_w$, Ψ, Θ^*, Θ_1, Θ_2, Θ_2^*, γ_k', $\tau_{i,k}$, $\omega_{(k)}$, ω_k, φ_k', $\tau_{s,\infty}$, Ω_k and S_k make sense. The belonging $L \in \mathrm{VL}(0)$ is implied by (52) and the relation $\Theta_1 < \infty$. The integral $\Phi_L(x)$ is well-defined by Theorem 1. The function $\varrho_{\gamma_I} \circ g$ is positive on $I^{\boxdot}$ due to the identity $\gamma_I = \gamma_0'$ and the belonging $g(I^{\boxdot}) \subset H_0$ (see the statement after (63)).

The function $U(x) = x_n$ is harmonic in the domain Ω. Therefore, by the remark after Theorem 3 and by inequalities (2), (45) and (52) we have

$$\begin{aligned} &Aw = Ag_n = LD_n g_n = LD_n w + WL, \\ &\|Aw - WL\|_J \leqslant \|D_n w\|_J \|L\|_J \leqslant c(n,\theta) l_J^{-1} b_J^2, \quad J \in \mathcal{D}. \end{aligned}$$

By inequality (9) in Theorem 1 we obtain

$$\begin{aligned} &\|WD_n\Phi_L - W\mathbf{x}_n^{-1}\Phi_L - D_n\Phi_{Aw} + \mathbf{x}_n^{-1}\Phi_{Aw}\|_I \leqslant c(n,\theta,\mu)\Theta_2^*, \\ &\|WD_{ij}\Phi_L - D_{ij}\Phi_{Aw}\|_I \leqslant c(n,\theta,\mu) l_I^{-1}\Theta_2^*. \end{aligned}$$

At the same time,

$$\begin{aligned} &\|D_n\Phi_{Aw} - \mathbf{x}_n^{-1}\Phi_{Aw} - D_n\Psi + \mathbf{x}_n^{-1}\Psi\|_I \leqslant c(n,\theta,\mu)\Theta_2^*, \\ &\|D_{ij}\Phi_{Aw} - D_{ij}\Psi\|_I \leqslant c(n,\theta,\mu) l_I^{-1}\Theta_2^* \end{aligned}$$

due to estimates (26b) and (57) in Theorem 2 and to Lemma 4. By (59),

$$D_n\Psi - \mathbf{x}_n^{-1}\Psi = D_n w - \mathbf{x}_n^{-1} w + \mathbf{x}_n^{-1}\gamma_0 - \mathbf{x}_n D_n F, \quad D_{ij}\Psi = D_{ij} w - D_{ij}(\mathbf{x}_n F).$$

By inequality (79) in Lemma 7 with $\varepsilon = |\alpha| - 1 \in \{0,1\}$ we obtain

$$\begin{aligned} &\big\|\mathbf{x}_n D_n F + \mathbf{x}_n D_n[WS \circ g - WS_0 \circ g]\big\|_I \leqslant c(n,\theta)\Theta_2^*, \\ &\big\|D_{ij}(\mathbf{x}_n F) + D_{ij}(\mathbf{x}_n[WS \circ g - WS_0 \circ g])\big\|_I \leqslant c(n,\theta) l_I^{-1}\Theta_2^*. \end{aligned}$$

At that, $S_0 \circ g = \ln \varrho_{\gamma_I} \circ g + \sigma_n$ on $I^{\boxdot}$ by Theorem 4. The identity

$$\begin{aligned} &\Big\{WD_n\Phi_L - W\frac{\Phi_L}{\mathbf{x}_n} - D_n\Phi_{Aw} + \frac{\Phi_{Aw}}{\mathbf{x}_n}\Big\} + \Big\{D_n\Phi_{Aw} - \frac{\Phi_{Aw}}{\mathbf{x}_n} - D_n\Psi + \frac{\Psi}{\mathbf{x}_n}\Big\} \\ &+ D_n w - \mathbf{x}_n^{-1} w + \mathbf{x}_n^{-1}\gamma_0 - \big\{\mathbf{x}_n D_n F + \mathbf{x}_n D_n[WS \circ g - WS_0 \circ g]\big\} \\ &= W\big[D_n\Phi_L - \mathbf{x}_n^{-1}\Phi_L + \mathbf{x}_n D_n[\ln \varrho_{\gamma_I} \circ g - S \circ g]\big] \end{aligned}$$

and the identity

$$\begin{aligned} &\{WD_{ij}\Phi_L - D_{ij}\Phi_{Aw}\} + \{D_{ij}\Phi_{Aw} - D_{ij}\Psi\} + D_{ij} w \\ &\qquad - \big\{D_{ij}(\mathbf{x}_n F) + D_{ij}(\mathbf{x}_n[WS \circ g - WS_0 \circ g])\big\} = WD_{ij}\big[\Phi_L + \mathbf{x}_n[\ln \varrho_{\gamma_I} \circ g - S \circ g]\big] \end{aligned}$$

show that checking inequalities (93) is reduced to checking the estimate

$$\|u\|_I \leqslant c(n,\theta,\mu)\Theta_2^*,$$

where

$$u = D_n w - \mathbf{x}_n^{-1} w + \mathbf{x}_n^{-1}\gamma_0 + W - W\mathbf{x}_n D_n[\ln \varrho_{\gamma_I} \circ g].$$

In view of the formula $\gamma_I \circ \mathbf{x}' = \gamma_0$ we write

$$\varrho_{\gamma_I} \circ g = C[g_n - \gamma_I \circ \mathbf{x}'] = C[w + W\mathbf{x}_n - \gamma_0], \quad C = C(\nabla\gamma_I) > 0,$$
$$W\mathbf{x}_n D_n[\ln \varrho_{\gamma_I} \circ g] = W\mathbf{x}_n \frac{D_n w + W}{w + W\mathbf{x}_n - \gamma_0} = \frac{D_n w + W}{1 + \frac{w-\gamma_0}{W\mathbf{x}_n}},$$
$$u = \frac{\gamma_0 - w}{\mathbf{x}_n} + D_n w + W - \frac{D_n w + W}{1 + \frac{w-\gamma_0}{W\mathbf{x}_n}} = \frac{\gamma_0 - w}{\mathbf{x}_n} \frac{\frac{w-\gamma_0}{\mathbf{x}_n} - D_n w}{1 + \frac{w-\gamma_0}{W\mathbf{x}_n}} W^{-1}.$$

Due to (2), the Taylor formula, (45), (48) and (44) we have

$$\left\| \frac{\gamma_0 - w}{\mathbf{x}_n} \right\|_I = \left\| \frac{w - \gamma_0}{\mathbf{x}_n} \right\|_I \leqslant \|w - \gamma_0\|_I \|\mathbf{x}_n^{-1}\|_I \leqslant c_1(n) b_I,$$
$$\|D_n w\|_I \leqslant c_2(n) b_I, \quad \left\| \frac{w - \gamma_0}{W\mathbf{x}_n} \right\|_{L^\infty(I^\boxdot)} \leqslant \frac{1}{3},$$
$$\left\| \frac{1}{1 + \frac{w-\gamma_0}{W\mathbf{x}_n}} \right\|_I \leqslant \frac{3}{2} + l_I^\mu \frac{9}{4} \left| \frac{w - \gamma_0}{W\mathbf{x}_n} \right|_{C^\mu(I^\boxdot)} \leqslant \frac{3}{2} + \frac{9 c_1 b_I}{4W} \leqslant c_3(n, \theta),$$
$$\|u\|_I \leqslant [c_1 b_I][c_1 b_I + c_2 b_I] c_3 W^{-1} \leqslant c(n, \theta)\Theta_2^*.$$

This completes the proof. □

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