LOWER BOUNDS FOR THE AREA OF THE IMAGE OF A CIRCLE

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Abstract. In the work we consider \( Q \)-homeomorphisms w.r.t \( p \)-modulus on the complex plane as \( p > 2 \). We obtain a lower bound for the area of the image of a circle under such mappings. We solve the extremal problem on minimizing the functional of the area of the image of a circle.

Keywords: \( p \)-modulus of a family of curves, \( p \)-capacity of condenser, quasiconformal mappings, \( Q \)-homeomorphisms w.r.t. \( p \)-modulus.

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1. Introduction

The problem on area deformations under quasi-conformal mappings originates from work by B. Bojarskii [1]. A series of results in this direction were obtained in works [2]–[4].

First an upper bound for the area of the image of a circle under quasi-conformal mappings was provided in monograph by M.A. Lavrent’ev, see [5]. In [6, Prop. 3.7], the Lavrentiev’s inequality was specified in terms of the angular dilatation. Also earlier in works [7]–[8] there were obtained the upper bounds for the area deformation for annular and lower and \( Q \)-homeomorphisms. In the present work we obtain lower bounds for the area of the image of a circle under \( Q \)-homeomorphisms w.r.t. \( p \)-modulus as \( p > 2 \).

To simplify the presentation, we restrict ourselves by the planar case. We recall some definitions. Assume that we are given a family \( \Gamma \) of curves \( \gamma \) in the complex plane \( \mathbb{C} \). A Borel function \( \varrho : \mathbb{C} \to [0, \infty) \) is called admissible for \( \Gamma \), which is written as \( \varrho \in \text{adm} \Gamma \), if

\[
\int_{\gamma} \varrho(z) \, |dz| \geq 1 \quad \forall \, \gamma \in \Gamma.
\]  

(1)

Let \( p \in (1, \infty) \). Then a \( p \)-modulus of the family \( \Gamma \) is the quantity

\[
\mathcal{M}_p(\Gamma) = \inf_{\varrho \in \text{adm} \Gamma} \int_{\mathbb{C}} \varrho^p(z) \, dm(z).
\]  

(2)

Assume that \( D \) is a domain in the complex plane \( \mathbb{C} \), that is, an open connected subset \( \mathbb{C} \) and \( Q : D \to [0, \infty] \) is a measurable function. A homeomorphism \( f : D \to \mathbb{C} \) is called a \( Q \)-homeomorphism w.r.t. \( p \)-modulus if

\[
\mathcal{M}_p(f\Gamma) \leq \int_D Q(z) \, \varrho^p(z) \, dm(z)
\]  

(3)

for each family \( \Gamma \) of curves in \( D \) and each admissible function \( \varrho \) for \( \Gamma \).

We also note that if the function \( Q \) in (3) is bounded almost everywhere by some constant \( K \in [1, \infty) \) and \( p = 2 \), then we arrive at classical quasi-conformal mappings introduced originally in works by Grötzsch, Lavrentiev and Morrey.

Let \( Q : D \to [0, \infty] \) be a measurable function. For each number \( r > 0 \) we denote by

\[
q_{z_0}(r) = \frac{1}{2\pi} \int_{S(z_0,r)} Q(z) \left| dz \right|
\]

the integral mean of the function \( Q \) over the circle \( S(z_0, r) = \{ z \in \mathbb{C} : |z - z_0| = r \} \).

**Theorem 1.** Let \( D \) and \( D' \) be bounded domains in \( \mathbb{C} \) and \( f : D \to D' \) be a \( Q \) \-homeomorphism w.r.t. \( p \)-modulus, \( p > 2 \), \( Q \in L^1_{\text{loc}}(D \setminus \{ z_0 \}) \). Then for all \( r \in (0, \delta_0) \), \( \delta_0 = \text{dist}(z_0, \partial D) \) the estimate

\[
|fB(z_0,r)| \geq \pi \left( \frac{p-2}{p-1} \right)^{\frac{2(p-1)}{p-2}} \left( \int_0^r \frac{dt}{t^{\frac{1}{p-1}} q_{z_0}^{\frac{1}{p-2}}(t)} \right)^{\frac{2(p-1)}{p-2}}
\]

(4)

holds true, where \( B(z_0,r) = \{ z \in \mathbb{C} : |z - z_0| \leq r \} \).

We note that as \( p > 2 \) and \( Q(z) \leq K \), by Theorem 1 we arrive to the result for a circle in [12 Lm. 7].

2. **Proof of main theorem**

We provide some auxiliary information about the capacity of a condenser. Following work [13], the pair \( \mathcal{E} = (A, C) \), where \( A \subset \mathbb{C} \) is an open set and \( C \) is a non-empty compact set contained in \( A \) is called **condenser**. A condenser \( \mathcal{E} \) is called an **annular condenser** if \( \mathcal{R} = A \setminus C \) is an annulus, that is, if \( \mathcal{R} \) is a domain whose complement \( \overline{\mathbb{C}} \setminus \mathcal{R} \) consists exactly of two components. A condenser \( \mathcal{E} \) is called a **bounded condenser** if the set \( A \) is bounded. We also say that a condenser \( \mathcal{E} = (A, C) \) lies in the domain \( D \) if \( A \subset D \). It is obvious that if \( f : D \to \mathbb{C} \) is a continuous open mapping and \( \mathcal{E} = (A, C) \) is a condenser in \( D \), then \((fA, fC)\) is also a condenser in \( fD \). We also have \( f\mathcal{E} = (fA, fC) \).

Let \( \mathcal{E} = (A, C) \) be a condenser. By \( \mathcal{C}_0(A) \) we denote the set of continuous compactly supported functions \( u : A \to \mathbb{R}^1 \), by \( \mathcal{W}_0(\mathcal{E}) = \mathcal{W}_0(A, C) \) we denote the family of non-negative functions \( u : A \to \mathbb{R}^1 \) such that

1) \( u \in \mathcal{C}_0(A) \),
2) \( u(x) \geq 1 \) for \( x \in C \),
3) \( u \) belongs to the class ACL.

As \( p \geq 1 \), the quantity

\[
\text{cap}_p \mathcal{E} = \text{cap}_p (A, C) = \inf_{u \in \mathcal{W}_0(\mathcal{E})} \int_A |\nabla u|^p \, dm(z),
\]

(5)

where

\[
|\nabla u| = \sqrt{\left( \frac{\partial u}{\partial x} \right)^2 + \left( \frac{\partial u}{\partial y} \right)^2}
\]

(6)
is called a $p$-capacity of the condenser $\mathcal{E}$. In what follows we shall make use the identity
\[ \text{cap}_p \mathcal{E} = \mathcal{M}_p(\Delta(\partial A, \partial C; A \setminus C)) \] (7)
eq

established in work [14], where for the sets $\mathcal{F}_1$, $\mathcal{F}_2$ and $\mathcal{F}$ in $C$, the symbol $\Delta(\mathcal{F}_1, \mathcal{F}_2; \mathcal{F})$ stands for the family of all continuous curves connecting $\mathcal{F}_1$ and $\mathcal{F}_2$ in $\mathcal{F}$.

It is known [15, Prop. 5] that as $p \geq 1$,
\[ \text{cap}_p \mathcal{E} \geq \left[ \inf l(\sigma) \right]^p_{|A \setminus C|^{p-1}}. \] (8)

Here $l(\sigma)$ is the length of a smooth (infinitely differentiable) curve $\sigma$ being the boundary $\sigma = \partial U$ of a bounded open set $U$ containing $C$ and contained together with its closure $\overline{U}$ in $A$ and the infimum is taken over all such $\sigma$.

**Proof of Theorem 1.** Let $\mathcal{E} = (A, C)$ be a condenser, where $A = \{ z \in D : |z - z_0| < t + \Delta t \}$, $C = \{ z \in D : |z - z_0| \leq t \}$, $t + \Delta t < d_0$. Then $f\mathcal{E} = (fA, fC)$ is an annular condenser in $D'$ and according to (7) we have the identity
\[ \text{cap}_p f\mathcal{E} = \mathcal{M}_p(\Delta(\partial fA, \partial fC; f(A \setminus C))). \] (9)

By inequality (8) we obtain
\[ \text{cap}_p f\mathcal{E} \geq \left[ \inf l(\sigma) \right]^p_{|fA \setminus fC|^{p-1}}. \] (10)

Here $l(\sigma)$ is the length of a smooth (infinitely differentiable) curve $\sigma$ being the boundary $\sigma = \partial U$ of a bounded open set $U$ containing $C$ and contained together with its closure $\overline{U}$ in $A$ and the infimum is taken over all such $\sigma$.

On the other hand, by the definition of $Q$-homeomorphism w.r.t. $p$-modulus we have
\[ \text{cap}_p f\mathcal{E} \leq \int_D Q(z) \varphi^p(z) \, dm(z) \] (11)
for each $\varphi \in \text{adm} \Delta(\partial A, \partial C; A \setminus C)$.

It is easy to check that the function
\[ \varphi(z) = \begin{cases} \frac{1}{|z - z_0| \ln \frac{t + \Delta t}{t}}, & z \in A \setminus C \\ 0, & z \notin A \setminus C \end{cases} \]
is admissible for the family $\Delta(\partial A, \partial C; A \setminus C)$ and hence,
\[ \text{cap}_p f\mathcal{E} \leq \frac{1}{\ln^p \left( \frac{t + \Delta t}{t} \right)} \int_R \frac{Q(z)}{|z - z_0|^p} \, dm(z), \] (12)
where $R = \{ z \in D : t \leq |z - z_0| \leq t + \Delta t \}$.

Combining inequalities (10) and (12), we get
\[ \frac{[\inf l(\sigma)]^p}{|fA \setminus fC|^{p-1}} \leq \frac{1}{\ln^p \left( \frac{t + \Delta t}{t} \right)} \int_R \frac{Q(z)}{|z - z_0|^p} \, dm(z). \] (13)

By the Fubini theorem we have
\[ \int_R \frac{Q(z)}{|z - z_0|^p} \, dm(z) = \int_t^{t + \Delta t} \frac{1}{\tau^p} \int_{S(z_0, \tau)} Q(z) \, |dz| = 2\pi \int_t^{t + \Delta t} \tau^{1-p} q_{z_0}(\tau) \, d\tau, \] (14)
where \( q_{z_0}(\tau) = \frac{1}{2\pi^2} \int_{S(z_0, \tau)} Q(z) |dz| \) and \( S(z_0, \tau) = \{ z \in \mathbb{C} : |z - z_0| = \tau \} \). Thus,

\[
\inf \ l(\sigma) \leq (2\pi)^\frac{1}{p} \left| \frac{|fA \setminus fC|}{\ln(t + \Delta t)} \right|^{\frac{p-1}{p}} \left[ \int_t^{t+\Delta t} \tau^{-p} q_{z_0}(\tau) \, d\tau \right]^{\frac{1}{p}}. \tag{15}
\]

Employing the isoperimetric inequality

\[
\inf \ l(\sigma) \geq 2\sqrt{\pi |fC|}, \tag{16}
\]

we obtain

\[
2\sqrt{\pi} |fC| \leq (2\pi)^\frac{1}{p} \left| \frac{|fA \setminus fC|}{\ln(t + \Delta t)} \right|^{\frac{p-1}{p}} \left[ \int_t^{t+\Delta t} \tau^{-p} q_{z_0}(\tau) \, d\tau \right]^{\frac{1}{p}}. \tag{17}
\]

We introduce a function \( \Phi(t) \) for this homeomorphism \( f \) as follows:

\[
\Phi(t) = |fB(z_0, t)|, \tag{18}
\]

where \( B(z_0, t) = \{ z \in \mathbb{C} : |z - z_0| \leq t \} \). Then it follows from (17) that

\[
2\sqrt{\pi} \frac{\Phi(t)}{t^{\frac{1}{p-1}} q_{z_0}^{-\frac{1}{p}}(t)} \leq \frac{2\pi |fC|}{t^{\frac{1}{p-1}} q_{z_0}^{-\frac{1}{p}}(t)}. \tag{19}
\]

Letting \( \Delta t \to 0 \) in inequality (19) and taking into consideration a monotonous increasing of the function \( \Phi \) in \( t \in (0, d_0) \), for almost all \( t \) we have:

\[
\frac{2\pi |fC|}{t^{\frac{1}{p-1}} q_{z_0}^{-\frac{1}{p}}(t)} \leq \frac{\Phi(t)}{t^{\frac{1}{p-1}} q_{z_0}^{-\frac{1}{p}}(t)}. \tag{20}
\]

This implies easily the following inequality:

\[
\frac{2\pi |fC|}{t^{\frac{1}{p-1}} q_{z_0}^{-\frac{1}{p}}(t)} \leq \left( \frac{\Phi(t)}{t^{\frac{1}{p-1}} q_{z_0}^{-\frac{1}{p}}(t)} \right)'. \tag{21}
\]

Since \( p > 2 \), the function

\[
g(t) = \frac{\Phi(t)}{t^{\frac{1}{p-1}} q_{z_0}^{-\frac{1}{p}}(t)}
\]

is non-decreasing on \( (0, d_0) \), where \( d_0 = \text{dist}(z_0, \partial D) \). Integrating both sides of the inequality in \( t \in [\varepsilon, r] \) and taking into consideration that

\[
\int_{\varepsilon}^{r} \left( \frac{\Phi(t)}{t^{\frac{1}{p-1}} q_{z_0}^{-\frac{1}{p}}(t)} \right)' \, dt = \int_{\varepsilon}^{r} g'(t) \, dt \leq g(r) - g(\varepsilon) \leq \frac{\Phi^{p-2}(r)}{2(p-1)} - \frac{\Phi^{p-2}(\varepsilon)}{2(p-1)}, \tag{22}
\]

see, for instance, [16] Thm. IV.7.4], we obtain

\[
2\pi |fC| \int_{\varepsilon}^{r} \frac{dt}{t^{\frac{1}{p-1}} q_{z_0}^{-\frac{1}{p}}(t)} \leq \frac{\Phi^{p-2}(r)}{2(p-1)} - \frac{\Phi^{p-2}(\varepsilon)}{2(p-1)}. \tag{23}
\]

Letting \( \varepsilon \to 0 \) in inequality (23), we arrive at the estimate

\[
\Phi(r) \geq \pi \left( \frac{p - 2}{p - 1} \right)^{\frac{2(p-1)}{p-2}} \left( \int_{0}^{r} \frac{dt}{t^{\frac{1}{p-1}} q_{z_0}^{-\frac{1}{p}}(t)} \right)^{\frac{2(p-1)}{p-2}}. \tag{24}
\]
Finally, denoting $\Phi(r) = |fB(z_0, r)|$ in the latter inequality, we get

$$|fB(z_0, r)| \geq \pi \left( \frac{p - 2}{p - 1} \right)^\frac{2(p-1)}{p-2} \left( \int_0^r \frac{1}{t^{p-1} q_0^{\frac{p}{p-2}}(t)} \right)^{\frac{2(p-1)}{p-2}}$$

and this completes the proof of Theorem 1.

$$\square$$

### 3. Corollaries of Theorem 1

Theorem 1 implies the following statements.

Employing the condition $q_{z_0}(t) \leq q_0 t^{-\alpha}$, we estimate the right hand side of inequality (4) and after elementary transformations we arrive at the following result.

**Corollary 1.** Let $D$ and $D'$ be bounded domains in $\mathbb{C}$ and $f : D \to D'$ be a $Q$-homeomorphism w.r.t. $p$-modulus as $p > 2$. Assume that the function $Q$ satisfies the condition

$$q_{z_0}(t) \leq q_0 t^{-\alpha}, \quad q_0 \in (0, \infty), \quad \alpha \in [0, \infty)$$

for $z_0 \in D$ and almost all $t \in (0, d_0)$, $d_0 = \text{dist}(z_0, \partial D)$. Then for each $r \in (0, d_0)$ the estimate

$$|fB(z_0, r)| \geq \pi^{-\frac{\alpha}{p-2}} \left( \frac{p - 2}{\alpha + p - 2} \right)^{\frac{2(p-1)}{p-2}} q_0^{2-p} |B(z_0, r)|^{1+\frac{\alpha}{p-2}}$$

holds true.

In particular, letting here $\alpha = 0$, we obtain the following conclusion.

**Corollary 2.** Let $D$ and $D'$ be bounded domains in $\mathbb{C}$ and $f : D \to D'$ be a $Q$-homeomorphism w.r.t. $p$-modulus as $p > 2$ and $q_{z_0}(t) \leq q_0 < \infty$ for almost each $t \in (0, d_0)$, $d_0 = \text{dist}(z_0, \partial D)$. Then the estimate

$$|fB(z_0, r)| \geq q_0^{2-p} |B(z_0, r)|$$

holds true for each $r \in (0, d_0)$.

**Corollary 3.** Suppose that the assumptions of Theorem 1 are satisfied and $Q(z) \leq K < \infty$ for almost each $z \in D$. Then the estimate

$$|fB(z_0, r)| \geq K^{\frac{2}{p-2}} |B(z_0, r)|$$

holds true for each $r \in (0, d_0)$.

**Remark 1.** Corollary 3 is a particular result by Gehring for $E = B(z_0, r)$, see [12] Lm. 7.

**Corollary 4.** Let $f : \mathbb{B} \to \mathbb{B}$ be a $Q$-homeomorphism w.r.t. $p$-modulus as $p > 2$. Assume that the function $Q(z)$ satisfies the condition

$$q(t) \leq \frac{q_0}{t \ln^{p-1} t}, \quad q_0 \in (0, \infty),$$

for almost each $t \in (0, 1)$, where $q(t) = \frac{1}{2\pi t} \int_{S_t} Q(z) |dz|$ is the integral mean over the circumference $S_t = \{ z \in \mathbb{C} : |z| = t \}$. Then for each $r \in (0, 1)$ the estimate

$$|fB_r| \geq \pi \left( \frac{p - 2}{p - 1} \right)^{\frac{2(p-1)}{p-2}} q_0^{2-p} \left( r \ln \frac{e}{r} \right)^{\frac{2(p-1)}{p-2}}$$

holds true, where $B_r = \{ z \in \mathbb{C} : |z| \leq r \}$. 
4. Extremal problems for area functional

Let $Q : \mathbb{B} \to [0, \infty]$ be a measurable function satisfying the condition

$$q(t) \leq q_0, \quad q_0 \in (0, \infty)$$

(32)

for almost each $t \in (0, 1)$, where $q(t) = \frac{1}{2\pi t} \int_{S_t} Q(z) |dz|$ is the integral mean over the circumference $S_t = \{z \in \mathbb{C} : |z| = t\}$.

Let $\mathcal{H} = \mathcal{H}(q_0, p, \mathbb{B})$ be the set of all $Q$-homeomorphisms $f : \mathbb{B} \to \mathbb{C}$ w.r.t. $p$-modulus as $p > 2$ obeying condition (32). On the class $\mathcal{H}$ we consider the area functional

$$S_r(f) = |fB_r|.$$  

(33)

**Theorem 2.** For each $r \in [0, 1]$ the identity

$$\min_{f \in \mathcal{H}} S_r(f) = \pi q_0^{\frac{2}{2-p}} r^2$$

(34)

holds true.

**Proof.** Corollary 2 implies immediately the estimate

$$S_r(f) \geq \pi q_0^{\frac{2}{2-p}} r^2.$$  

(35)

Let us specify a homeomorphism $f \in \mathcal{H}$, at which the minimum of the functional $S_r(f)$ is attained. Let $f_0 : \mathbb{B} \to \mathbb{C}$, where

$$f_0(z) = q_0^{\frac{1}{2-p}} z.$$  

(36)

It is obvious that (35) becomes the identity at the mapping $f_0$. It remains to show that the mapping defined in such way is a $Q$-homeomorphism w.r.t. $p$-modulus with $Q(z) = q_0$. Indeed,

$$l(z, f_0) = L(z, f_0) = q_0^{\frac{1}{2-p}} , \quad J(z, f_0) = q_0^{\frac{2}{2-p}}$$

(37)

and

$$K_{l,p}(z, f_0) = \frac{J(z, f_0)}{l^p(z, f_0)} = q_0.$$  

(38)

By Theorem 1.1 in work [17], the mapping $f_0$ is a $Q$-homeomorphism w.r.t. $p$-modulus with $Q(z) = K_{l,p}(z, f_0) = q_0$. \hfill \Box

**BIBLIOGRAPHY**


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